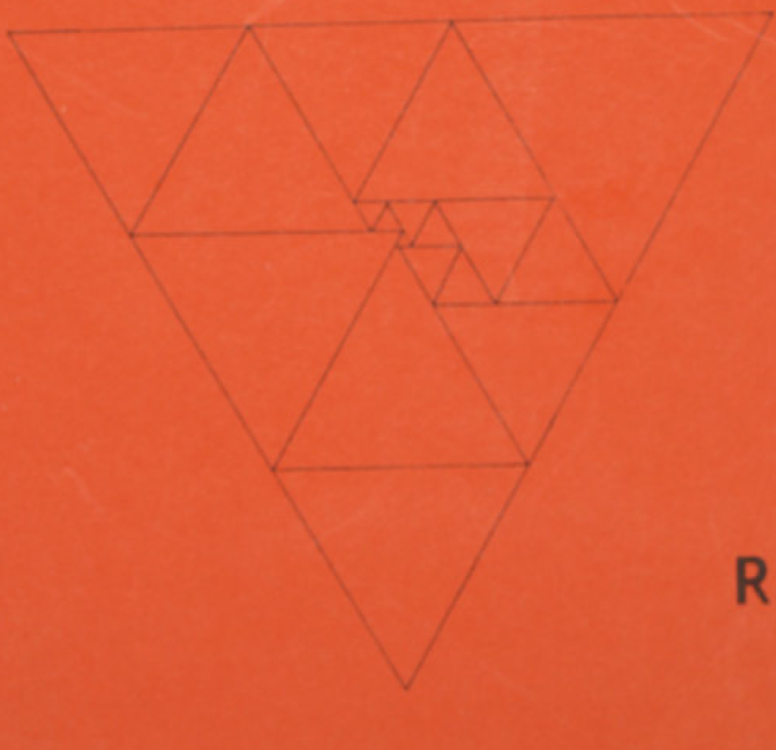
TECHNISCHE HOGESCHOOL TWENTE

TRIANGULATIONS

A. AUGUSTEIJN



REPORT

T R I A N G U L A T I O N S

Bachelor's thesis

by Lex Augusteijn.

Thesis advisers: Prof.Dr. Ir. A.J.W. Duijvestijn

Dr. F. Göbel

Ir. J. Smit

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Introduction.

The problem of the dissection of parallellograms and triangles into equilateral triangles has got less attention than the dissection of rectangles into unequal squares.
On the squaring problem many papers have been published. Computer programs have been written to find the lowest order simple perfect squared square (found in 1978 by Duijvestijn). However, the problem of triangulation seems to be studied only by Tutte. His work (and especially the representation by digraphs and his "electrical model") has been very helpful to us in undertaking the computersearch for the lowest order triangulated parallellogram, cylinder, triangle and rhombus, here described.

Lex Augusteijn.

1. Triangulations.

1.1. Squared rectangles.

This paper deals with the dissection of figures into equilateral triangles. Such a dissection is called an "equilateral triangulation" and the figure is called "triangulated". In order to describe the problem of triangulation I will first show the relationship to "squared rectangles".

A squared rectangle is a rectangle dissected into squares. The squares are called the elements of the dissection. The term "elements" is also used for the lengths of the sides of the elements.

A perfect squared rectangle is a rectangle dissected into unequal squares.

A simple squared rectangle is a squared rectangle which does not contain any sub-squared rectangles.

A squared rectangle which is not simple is called compound[4].

The order of the dissection is the number of the elements of the dissection.

1.2. Triangulated parallellograms.

When we sheare a squared rectangle R to a parallellogram P, we can call P "rhombussed". By dividing each rhombus into equilateral triangles we obtain a triangulated parallellogram. Therefore we can consider the problem of squared rectangles as a special case of that of triangulated parallellograms.

The question arises if P will be perfect if R is a perfect squared rectangle. For each triangle in P occurs twice, once with its apex upwards and once with its apex downwards. It can be shown by using Eulers theorem about polyhedra that it is not possible to dissect a parallellogram into all different triangles. See page 11.
However, when we define the "size" of a triangle as plus the length of its side when it has its apex upwards oriented and as minus the length of its side when it has its apex downwards oriented, we are able to construct perfect triangulated parallellograms, e.g. by shearing a perfect squared rectangle.

A perfect triangulated parallellogram is a triangulated parallellogram dissected into triangles of unequal "sizes". Where "size" should be interpreted as explained above.

A *simple* triangulated parallellogram is a triangulated parallellogram which does not contain any sub-triangulated triangle, trapezium or parallellogram.

A triangulated parallellogram which is not simple is called *compound*.

The *order* of a triangulation is the number of constituent triangles of that triangulation, divided by 2. It is defined in this way for easy comparison with squared rectangles.

Since a triangulated triangle always contains a trapezium, a it is defined as perfect if it does not contain any sub-triangulated triangle or parallellogram. The other definitions do also applie to triangulated triangles.

In this paper we will restrict ourselves to perfect, simple, equilateral triangulations.

2. Graphs and currents.

2.1. Discription of graphs corresponding to triangulated parallellograms.

Brooks, Smith, Stone and Tutte [4] proposed to associate a graph with a squared rectangle.
Brooks c.s. [2] made a proposal to associate a directed graph, or digraph for short, with a triangulated parallellogram. He showed how this could be done. See figure 1.

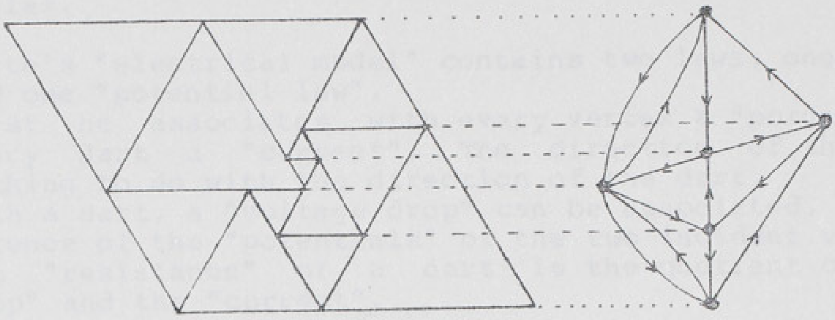


figure 1. A triangulated parallellogram P and its corresponding graph G.

- Each vertex in the graph G corresponds to a maximal horizontal segment in the parallellogram P.
- Each dart (or directed edge) in G corresponds to a triangle in P. (We use the word dart instead of directed edge in order to conform ourselves to Tutte).
- A triangle with its base incident with a horizontal segment corresponds to an outgoing dart.
- A triangle with its apex incident with a horizontal segment corresponds to an incoming dart.

The thus obtained digraph will be "balanced",which means that at each vertex the number of incoming darts is equal to the number of outgoing darts.
The degree of each vertex (the total number of incoming and outgoing darts incident with that vertex) will therefore be even.
It can easily be shown that the graph will be planar and that it can be drawn in the plane in such a way that the ordering of the darts around each vertex will be the same as the ordering of the corresponding triangles around a horizontal segment.
The incoming and outgoing darts around each vertex will then alternate. The drawing of such a planar graph in the plane is called an "alternating map".

2.2. An "electrical model" to calculate the sizes of the triangles.

Brooks c.s. [4] also showed how a graph, corresponding to a squared rectangle, could ba associated with an electrical network. This network can be used to calculate the elements of the rectangle.
Brooks c.s. [2] proposed an "electrical model" to calculate the sizes of the triangles in a triangulated parallellogram.This model, however, is somewhat extraordinary because it has no relationship to any physical electrical model. But it proved to be very useful in calculating the sizes of the constituent triangles.

Tutte's "electrical model" contains two laws, one "current law" and one "potential law".
First he associates with every vertex a "potential" and with every dart a "current". The direction of this current has nothing to do with the direction of the dart.
With a dart, a "voltage drop" can be associated, being the difference of the "potentials" of the two incident vertices.
The "resistance" of a dart is the quotient of the "voltage drop" and the "current".

The "current law" will then be:
The total sum of the "currents" in all outgoing darts incident with a vertex will be zero. This applies to every vertex of the graph.

The "potential law" will be:
The total "voltage drop" around any circuit in the graph is zero (without regard to the directions of the darts).

When we translate these two laws to the geometric aspect of the problem of triangulated parallellograms, we can say that:

- the "current" in a dart equals the size of the corresponding triangle.
- the "voltage drop" between two vertices is equal to the distance (along a slanting side) between the corresponding segments.

We can describe the current law in another way: the sum of the sizes of the triangles with their base incident with a certain horizontal segment is zero.
This means that the sum of the sizes of the triangles base-incident with and laying above a segment is minus the sum of the sizes of the triangles base-incident with and laying below that segment.

We can notice that the "resistance" of a dart is equal to the quotient of the lengths of the slanting side and the base of the corresponding triangle. In the case of equilateral triangles, it will be unity.

Using these two laws and the fact that all resistances are 1, we can deduce a set of linear equations from a certain graph after choosing one vertex as a source and another vertex as a sink for the "current" flowing through the network associated with the graph.
These equations can be solved using Cramers rule. In our case however, this would result in far too much computertime. We therefore have derived another method to calculate the "currents" from the set of linear equations.

3. Triality

In the case of squared rectangles we can rotate a rectangle through 90 degrees.We then obtain the same squaring but not necessarily the same associated graph.The two graphs related to a squared rectangle are called "duals".
In the case of triangulated parallellograms we can observe a similiar problem. To exhibit this we must pass from triangulated parallellograms to triangulated triangles.

Consider the parallellogram of figure 2a and its corresponding graph.

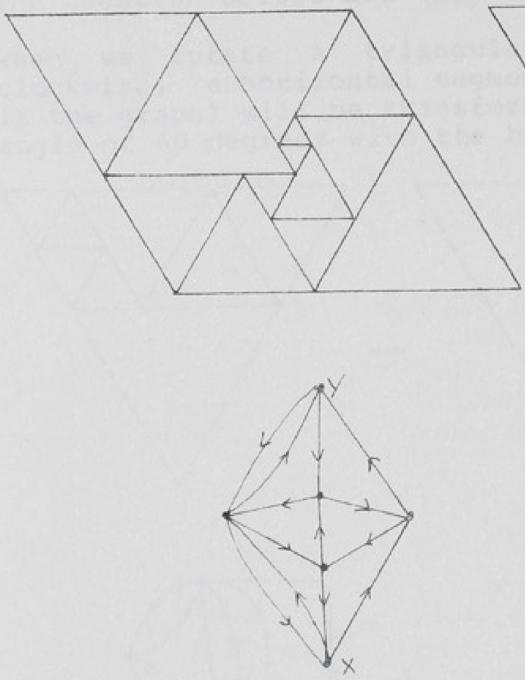


Fig. 2a.

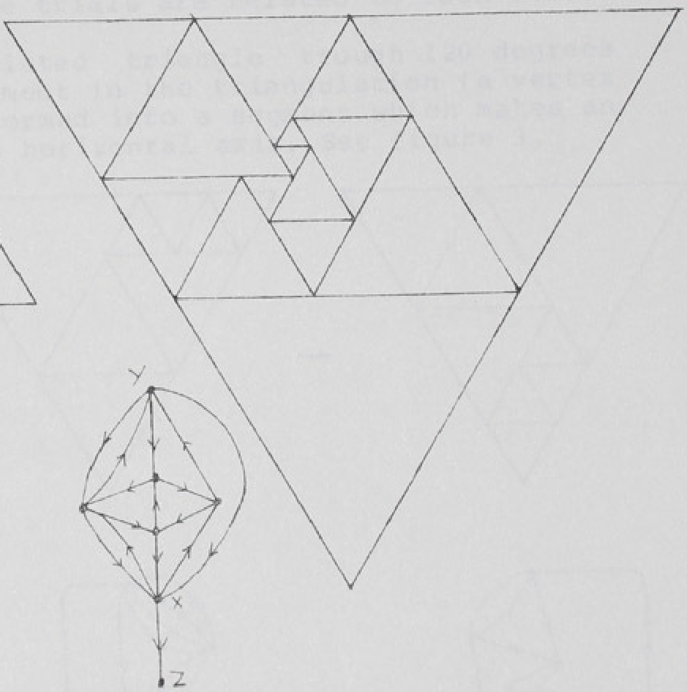


Fig. 2b.

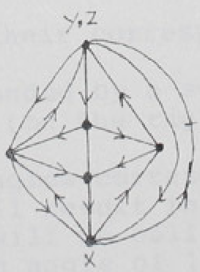


Fig. 2c.

figure 2a,b,c. Transforming a parallellogram into a triangle

We can transform this parallellogram into a triangle by adding two triangles (figure 2b.). The modification in the graph is obvious. We should add a dart from the old negative pole Y to the old postive pole X and one from X to a new vertex Z. It is in some ways convenient to make Z coincide with Y in the graph, to obtain a new balanced digraph. The dart between X and Z is then called the "polar dart", and is distinguished by a cross-bar. See figure 2c.

We observe that we can rotate the triangulated triangle through 120 degrees, to obtain the same triangle, but in another position. The corresponding balanced digraph will not necessarily be the same.
The triangle can be rotated once more, resulting in a third graph. These three graphs are called "trials".
The question arises how these trials are related to each other.

When we rotate a triangulated triangle trough 120 degrees clockwise, a horizontal segment in the triangulation (a vertex in the graph) will be transformed into a segment which makes an angle of 60 degrees with the horizontal axis. See figure 3.

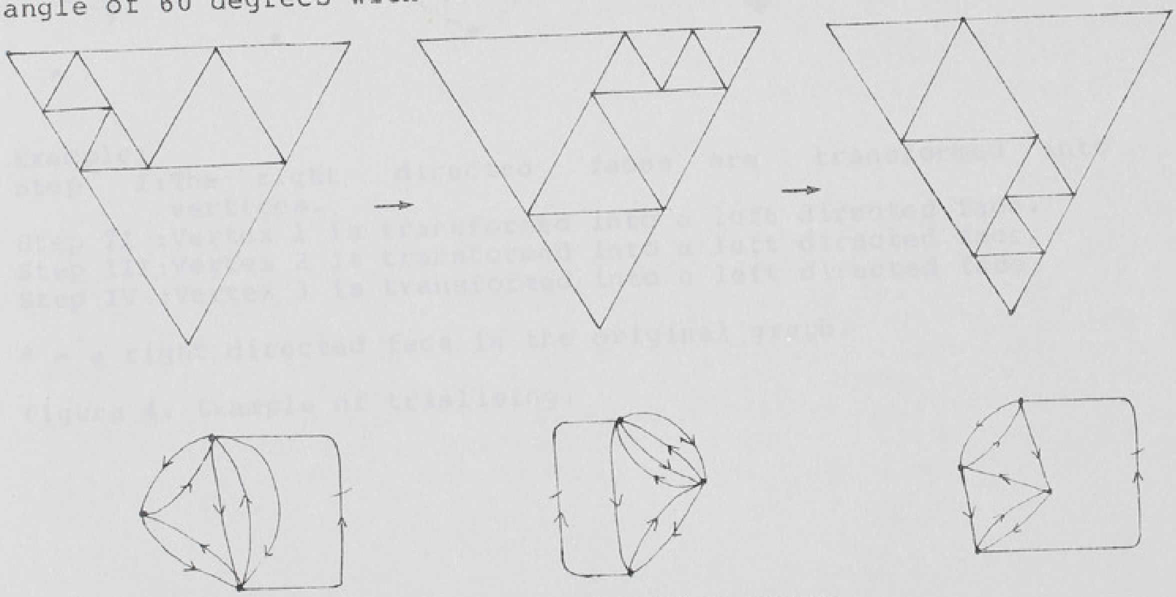


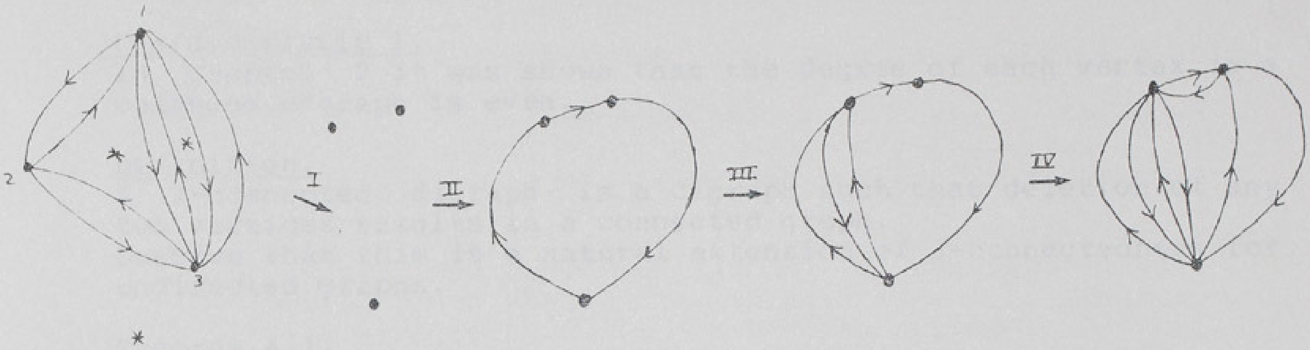
figure 3. Trials and their corresponding graphs.

This segment is bounded by a set of triangles which surround it, pointing upwards (on the righthand side) and downwards (on the lefthand side).
Each of these triangles corresponds to a dart in the map and drawing these darts wil result in a face. This face is directed counterclockwise and will be called a left directed face.
A segment making an angle of 120 degrees with the horizontal axis corresponds to a clockwise, or right directed face in the map.

Hence we can derive a trial from a balanced digraph, applying the following procedure to it:

- A vertex is transformed into a left directed face.
- A left directed face is transformed into a right directed face.
- A right directed face is transformed into a vertex.

Repeating this procedure will result in a third trial and another repetition will return us the original graph. See figure 4.



Example:
Step I:The right directed faces are transformed into vertices.
Step II :Vertex 1 is transformed into a left directed face.
Step III:Vertex 2 is transformed into a left directed face.
Step IV :Vertex 3 is transformed into a left directed face.

* = a right directed face in the original graph.

figure 4. Example of trialising.

4. Production of the balanced digraphs.

4.1. Relationship to c-nets.

Because the c-nets (3-connected planar graphs) up to and including order 22 were available on magnetic tape we thought it would be a very efficient way to produce the balanced digraphs using the c-nets as base nets.

4.2. Characteristics of the balanced digraphs.

Characteristic 1.
In chapter 2 it was shown that the degree of each vertex in a balanced digraph is even.

Definition.
A 3-connected digraph is a digraph such that deletion of any two vertices results in a connected graph.
Observe that this is a natural extension of 3-connectedness for undirected graphs.

Theorem 4.1.
If a triangulation is compound, than the associated digraph is not 3-connected.

Proof.
If the triangulation is compound, it contains a sub-triangulated triangle, trapezium or parallellogram.
This sub-figure consists of two horizontal segments, connected by two slanting sides. Hence its equivalent in the digraph is a sub-digraph, separated from the remainder of the digraph by two faces f1 and f2 (corresponding to the slanting sides) and connected to it by two vertices n1 and n2 (corresponding to the horizontal segments). f1 and f2 will both be incident with n1 and n2. The sub-net is therefore only connected by n1 and n2 to the remainder of the net and hence the digraph is not 3-connected. See figure 5.

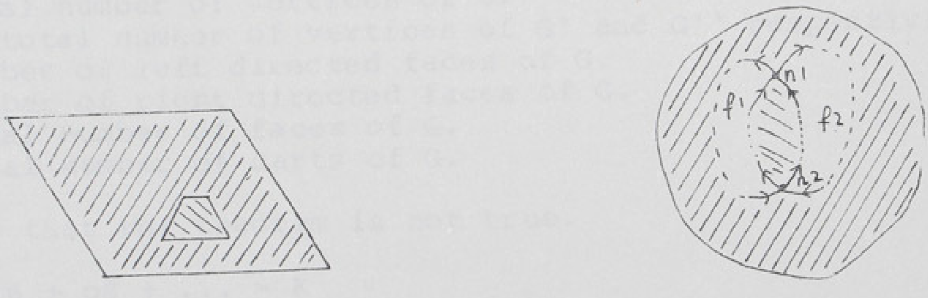


Figure 5. Compound triangulated parallellogram and its corresponding digraph.

<u>Theorem 4.2.</u>
If a balanced digraph is not 3-connected, the corresponding triangulation will be compound.

Proof.

Case 1: deletion of only one vertex will result in a not-connected graph.
The graph will then excist of two parts, connected by only one vertex. Both parts are balanced digraphs and will have their associated triangulated parallellograms. The associated triangulation of the total graph will therefore excist of two parallellograms and will be compound.

Case 2: deletion of exactly two vertices will result in a not-connected graph.
The graph will then contain a part which is connected by two vertices n1 and n2 to the remainder of the net. It is separated from the net by two faces f1 and f2.
n1 and n2 will correspond to horizontal segments in the triangulation, whereas f1 and f2 correspond to two slanting sides which connect these segments. These four lines will either form a triangle, trapezium or parallellogram. Therefore the triangulation is compound.

<u>Characteristic 2.</u>
From theorem 4.1 and 4.2 it follows that a simple triangulation corresponds to a 3-connected balanced digraph.

<u>Theorem 4.3.</u>
A balanced digraph G, containing only vertices of degree ≥ 6 can not have trials G' and G'', also containg only vertices of degree ≥ 6.

Proof.
We define:

pn =number of vertices of degree n of G.
K =total number of vertices of G.
K',K''=total number of vertices of G' and G'' respectivily.
Ml =number of left directed faces of G.
Mr =number of right directed faces of G.
M =total number of faces of G.
B =total number of darts of G.

Suppose that the theorem is not true.

$$pn = p6 + p8 + \ldots = K \quad (1)$$

$$n.pn = 6.p6 + 8.p8 \ldots = 2B \quad (2)$$

Elimination of p6 results in:

$$2.p8 + 4.p10 \ldots = 2B - 6K \geq 0 \quad (3)$$

$$K \leq \frac{B}{3} \quad (4)$$

Trialising results in:

$$K' \leq \frac{B}{3} \quad (5)$$

$$K'' \leq \frac{B}{3} \quad (6)$$

Since K' = Mr and K''= Ml, we can write:

$$K + K' + K'' = K + Mr + Ml = K + M \leq B \quad (7)$$

This is in contradiction with Eulers Polyhedran Theorem:

$$K + M = B + 2$$

Hence, at least one of the three trials should contain a vertex of degree 4.

Theorem 4.4.
A triangulation can never contain triangles which have all different lenghts of their sides.

Proof.
Using theorem 4.3 we can say that one of the trials will contain a vertex nl of degree 4. This vertex is incident with two left directed faces fl1 and fl2 and with two right directed faces fr1 and fr2.
Trialising will transform nl into a left directed face, incident with the vertices obtained from fr1 and fr2. This face consists of two antiparallel darts, since fr1 and fr2 are the only two right directed faces incident with nl.
It can easily be shown from the "electrical model" the "currents" through these two darts will be each others opposites. Hence, the lengths of the sides of the corresponding triangles are equal.

4.3. How the balanced digraphs are obtained from the c-nets.

When we omit from a balanced digraph the directions from the darts and undouble all branches, we will obtain an undirected graph in which two vertices are connected by at most one branch.

Since the digraph should be 3-connected when we want to obtain simple triangulations, the corresponding undirected graph will be 3-connected too.
A balanced digraph is planar, therefore the undirected graph will be planar.
Hence, the undirected graph will be a c-net.

It is our purpose to obtain from a c-net a balanced digraph. Therefore we should redouble branches in a systematical way until we reach a graph in which the degrees of all vertices are even. In directing the edges we obtain a balanced digraph.
In order to generate all balanced digraphs, the following procedure is used (in pseudo PASCAL):

```
procedure DOUBLE(i:integer);
comment: i is a branch of a c-net which is used as input data.
         The branches are arbitrary numbered from 1 to B;
var j:integer;
begin if there is no parallel branch to branch i
      then
      begin add one branch parallel to i;
            if the degrees of all vertices are even
            then output(graph)
      end;
      for j:=i+1 to B do DOUBLE(j)
end;
```

This procedure is called in the statement:

```
for n:=1 to B do DOUBLE(i);
```

The thus obtained graphs are directed to get the balanced digraphs.

Example:

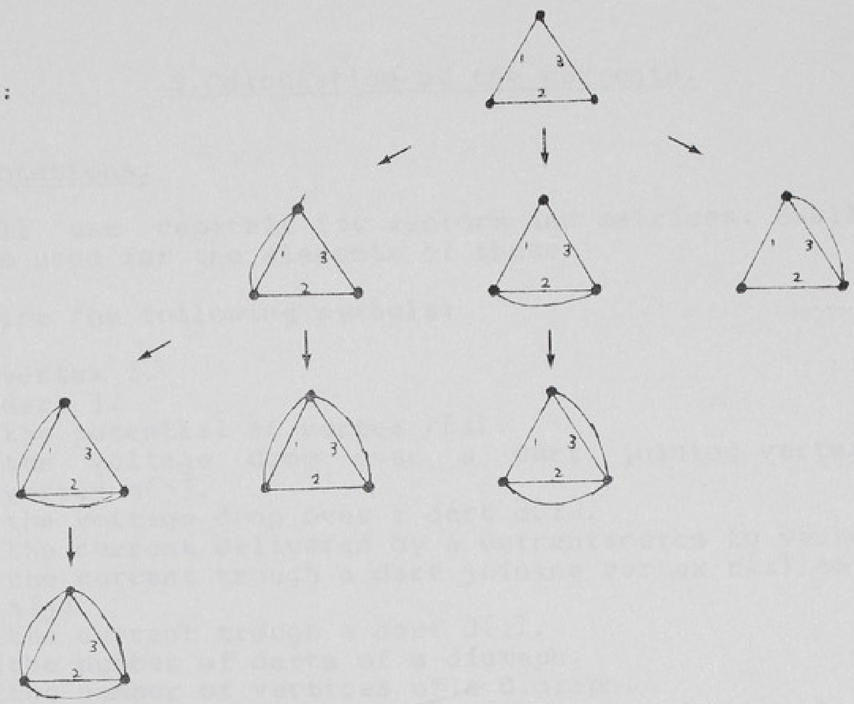


figure 6.

5.Calculation of the currents.

5.1. Notations.

We will use capitals for vectors and matrices. Small letters will be used for the elements of these.

We define the following symbols:

n[i] :vertex i.
d[j] :dart j.
v[j] :the potential of vertex n[i].
u[ij]:the voltage drop over a dart joining vertex n[i] to vertex n[j].
u[i] :the voltage drop over a dart d[i].
j[i] :the current delivered by a currentsource to vertex n[i].
i[ij] :the current trough a dart joining vertex n[i] to vertex n[j].
i[i] :the current trough a dart d[i].
B :the number of darts of a digraph.
K :the number of vertices of a digraph.
U :the vector of B elements, composed of the voltage drops u[i] over the darts.
V :the vector of K elements, composed of the potentials v[i] of the vertices.
I :the vector of B elements, composed of the currents i[i] through the darts.
J :the vector of K elements, composed of the currents j[i] of the currentsources at the vertices.

5.2. Relationships between the vectors.

Because all resistances are equal to one in the case of equilateral triangles, we can write:

$$U = I \qquad (1)$$

The relationship between u[ij] , v[i] and v[j] is:

$$u[ij] = v[i] - v[j] \qquad (2)$$

or, in vector notation:

$$CV = U \qquad (3)$$

where C is a B x K matrix, composed of the following elements:

$$c[i,j] = \begin{cases} 1 & \text{if } d[i] \text{ is positive incident to } n[j] \quad (4a) \\ -1 & \text{if } d[i] \text{ is negative incident to } n[j] \quad (4b) \\ 0 & \text{if } d[i] \text{ is not incident with } n[j] \quad (4c) \end{cases}$$

The dart d[i] is positive incident to n[j] when d[i] is an outgoing dart incident with n[j];

d[i] is negative incident to n[j] when d[i] is an incoming dart incident with n[j].

The relationship between i[ij] and j[i] is:

$$\sum_j i[ij] = j[i] \qquad (5)$$

or, in vector notation:

$$AI = J \qquad (6)$$

where A is a K x B matrix, composed of the following elements:

$$a[i,j] = \begin{cases} 1 & \text{if } d[j] \text{ is positive incident to } n[i] \quad (7a) \\ 0 & \text{otherwise} \quad (7b) \end{cases}$$

Using (1), (3) and (6) we can write:

$$AU = J \qquad (8)$$

$$A(CV) = J \qquad (9)$$

$$(AC)V = J \qquad (10)$$

We define the K x K matrix Y as

$$Y = AC \qquad (11)$$

The diagonal element y i,i of Y is:

$$y[i,i] = \sum_j a[i,j].c[j,i] = \sum_j a[i,j] \qquad (12)$$

So y[i,i] is the number of outgoing darts of n[i].

The off-diagonal element y[i,j] $(i \neq j)$ of Y is:

$$y[i,j] = \sum_k a[i,k].c[k,j] \qquad (13)$$

This element will be non-zero if a dart d[k] joins vertex n[i] to vertex n[j]. The value of y[i,j] is minus the number of darts joining n[i] to n[j]. In our case it will be zero or minus one.

The matrix Y is called the "Kirchhoff-matrix" K(G) of a graph G, mentioned by Tutte 1 , who directly derived it from the graph, not using the A and C matrices. This is, of course a very direct way and, in a computerprogram, very efficient. We also derived it in Tutte's way when calculating the currents, but it is necessary to show the relationship to the A and C matrices in order to derive a suitable model for calculating the currents.

Because $y[i,i] = -\sum_{k \neq i} y[i,k]$, the matrix Y is singular, but when we strike out the jth row and column of it, it will become regular. The determinant of the resulting matrix Yj will be the complexity of the digraph [2].

Striking out the jth element of V (obtaining Vj), of J (obtaining Jj), the jth column of C (obtaining Cj) and the jth row of A (obtaining Aj), we can rewrite (3), (10) and (11):

$$Yj.Vj = Jj \tag{14}$$

$$Cj.Vj = U \tag{15}$$

From (14) we derive:

$$(Yj)^{-1}.Jj = Vj \tag{16}$$

combined to (15) and (1) gives:

$$Cj.(Yj)^{-1}.Jj = U = I \tag{17}$$

This equation can be used to calculate the currents trough the darts.

5.3. Example of current calculation.

See figure 7.

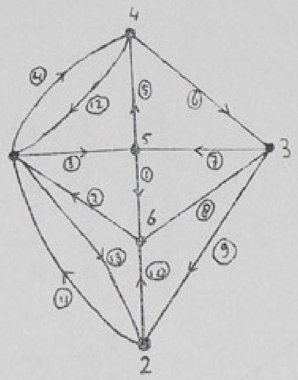


figure 7. Balanced digraph.

$$A = \begin{bmatrix} 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$J6 = \begin{bmatrix} 0 \\ 20 \\ 0 \\ -20 \\ 0 \end{bmatrix}$$

$$Y = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 & 0 \\ -1 & 2 & 0 & 0 & 0 & -1 \\ 0 & -1 & 2 & 0 & -1 & 0 \\ -1 & 0 & -1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 & 2 & -1 \\ -1 & 0 & -1 & 0 & 0 & 2 \end{bmatrix}$$

$$Y6 = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 2 & 0 & 0 & 0 \\ 0 & -1 & 2 & 0 & -1 \\ -1 & 0 & -1 & 2 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}$$

$$Y6^{-1} = \frac{1}{20} \begin{bmatrix} 14 & 10 & 6 & 12 & 10 \\ 7 & 15 & 3 & 6 & 5 \\ 6 & 10 & 14 & 8 & 10 \\ 10 & 10 & 10 & 20 & 10 \\ 5 & 5 & 5 & 10 & 15 \end{bmatrix}$$

$$(Y6)^{-1} . J6 = \begin{bmatrix} -2 \\ 9 \\ 2 \\ -10 \\ -5 \end{bmatrix}$$

$$C6.(Y6)^{-1} . J6 = \begin{bmatrix} -5 \\ 2 \\ 3 \\ 8 \\ 5 \\ -12 \\ 7 \\ -2 \\ -7 \\ 9 \\ 11 \\ -8 \\ -11 \end{bmatrix}$$

And this will result in the parallellogram of figure 1.

6.The various figures that could be triangulated.

Having created a set of graphs, we should choose in a certain graph a set of vertices as a source (so) and a sink (si) for the currentcalculation in that graph. The relationship between so and si determines the kind of figure we calculate.

a) If so and si are incidental, there will be a triangle between the upper and lower side of the triangulated parallellogram. See figure 8a.

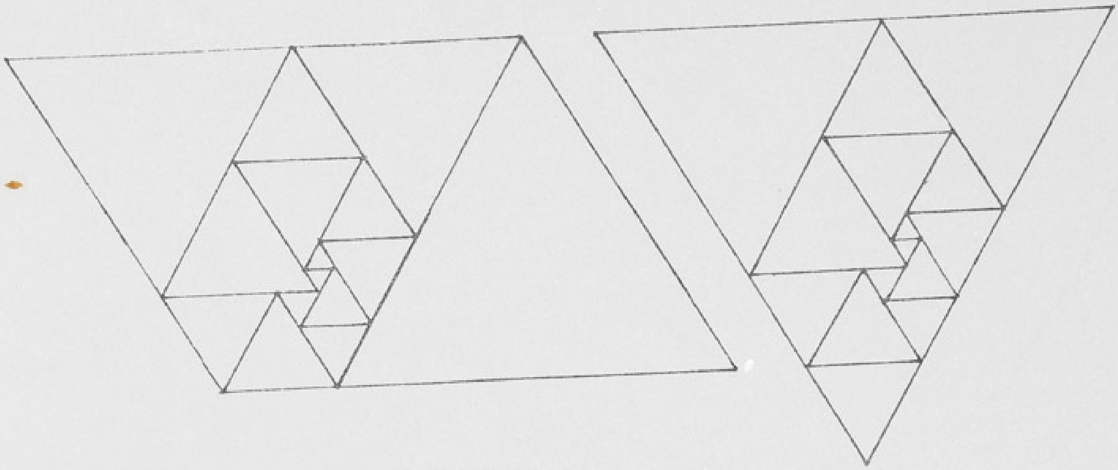


figure 8a and 8b. Transforming a compound parallellogram into a triangle.

This parallellogram, however, will be compound,since it contains a triangulated trapezium. But by deleting the triangle between the upper and lower side, and adding one to the small horizontal side off the trapezium we obtain a triangulated triangle (figure 8b).
Because every triangulated triangle can be transformed into a triangulated parallellogram, using this method the other way around, we can make all triangulated triangles.

b) If so and si are not incidental, we will not get necessarily a triangulated parallellogram. This will only be the case if so and si are incident with a common face, corresponding to a slanting side of the parallellogram.

c) If so and si are not incident with a common face, we will obtain a triangulated cilinder.

d) If all sides of a parallellogram are equal, we found a triangulated rhombus.

e) Other figures, e.g. a thorus or a moebiusstrip could also be triangulated, but this would require non-planar graphs, which are not generated.

7. The programs.

7.1. CNET.

This program is mainly used to change the representation of the c-nets. These were represented by a code proposed by Duijvestijn [3]. See figure 9.

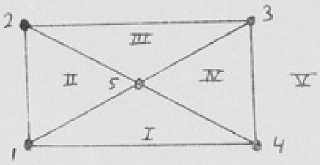


figure 9. c-net of order 8.

Face I could be represented by the code 1451. The first vertex is repeated for practical reasons. Other posible codes are 4514 or 5145 (using the left orientation).
A code of the c-net is the sequence of codes of its faces, separated by zeros and completed by two zeros.
A possible code for the net of figure 8 is:
14510152102532043540123410O

The outputcode for the c-nets, produced by CNET is just a table of the branches. They are represented by their from-vertex, their to-vertex, their left face and their right face.
A 'd' is added if the branch is doubled. This, of course will never be the case with c-nets, but it will occur if branches are doubled.
This code is completed by an integer, indicating the number of the last doubled branch + 1. (This is also done for practical reasons).
See figure 10.

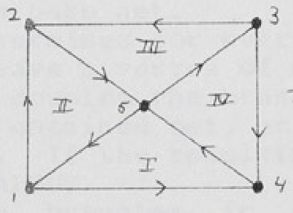


figure 10.

code:

n1	n2	f1	fr
1	2	5	2
2	5	3	2
5	1	1	2
5	3	3	4
3	2	3	5
3	4	5	4
4	5	1	4
1	4	1	5

1

n1 = from-vertex
n2 = to-vertex
f1 = lefthand face
fr = righthand face

The '1' at the end of the code indicates that no branch is doubled.

CNET also selects the c-nets for having at least one vertex of degree 3 or 4. The c-nets are written to the file NETOUT, the balanced c-nets to the file BALOUT.

7.2. DUAAL.

This program first dualizes a c-net, which is handled thereafter as in CNET.

7.3. GRAAF.

This program uses the c-nets produced by CNET and DUAAL as input data. It generates all graphes of a desired order which have these c-nets as a base net.
It reads a net, determines for every branch if doubling that branch would still leave a vertex of degree 3 or 4 and (if not allready doubled) it doubles the branch. The same procedure is applied to the now obtained net, until the desired number of branches is reached. If the resulting net is balanced, it is written to the file BALOUT.
The doubling of the branches is handled by the recursive procedure VERDUBBEL.

7.4. CUR.

This is the main program to calculate the currents in a certain balanced di graph. At first,the graph is read. Then it is di-

rected by the recur sive procedure RICHT. The incidence matrix YK is made, inversed and the complexity is calculated.
Then, taking each pair of vertices in succession as source or sink, the currents are calculated. If the triangulation doesn't contain zerocurrents and if it is perfect, the graph and the currents are written to the file RESULT. This is repeated for the reversed graph. This is the digraph in which the directions of all darts are reversed.
This program also determines whether a cylinder, parallelogram, triangle or rhombus is calculated.

7.5. CUR2.

This program is very similiar to CUR, but doesn't produce cylinders in order to keep control of the amount of output for higher orders ($B>16$).

7.6. RUIT.

This program also resembles CUR, but it is designed to produce rhombusses.

8. Results.

The results of the search are listed in table I and drawn in figure 11 to 19.

Tabel I.

order	parallellogram	cylinder	triangle	rhombus
6 1/2	20 x 19	21 x 20	-	-
7	-	28 x 29 28 x 23	-	-
7 1/2	-	36 x 38 33 x 35 39 x 32 37 x 41 37 x 41 36 x 35 35 x 40 36 x 29	39 (c)	-
8	48 x 42 47 x 43 47 x 44	17 cylinders	48 (c)	-
8 1/2	16 ones	not calculated	all compound	-
9	many	not calculated	45 90 48 48 91 many compound	-

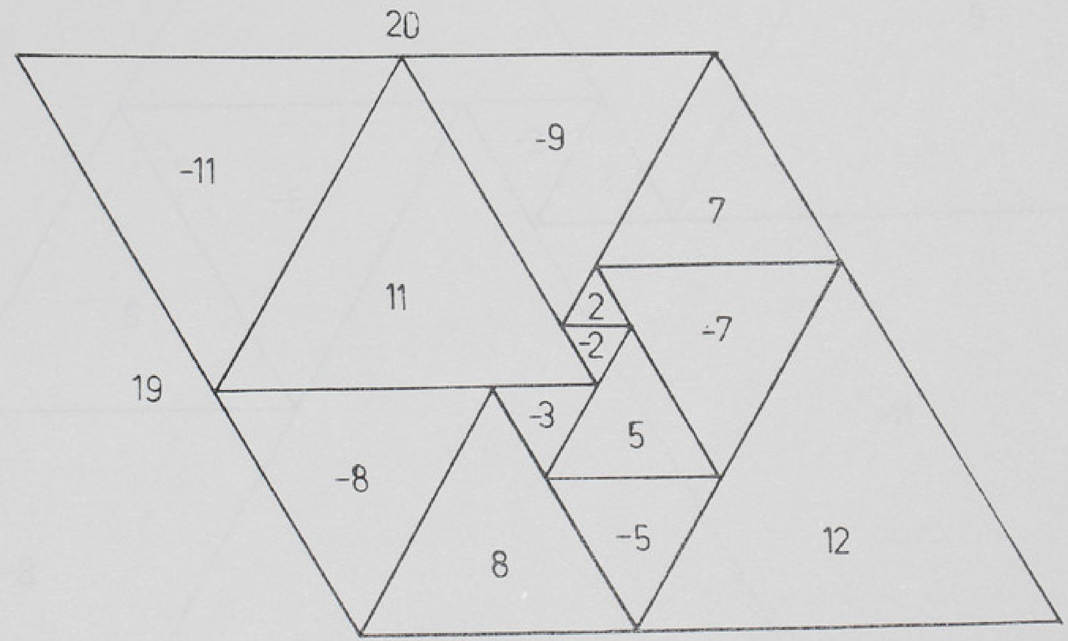


Figure 11. Perfect, simple triangulated parallellogram of order $6\frac{1}{2}$

(Tutte)

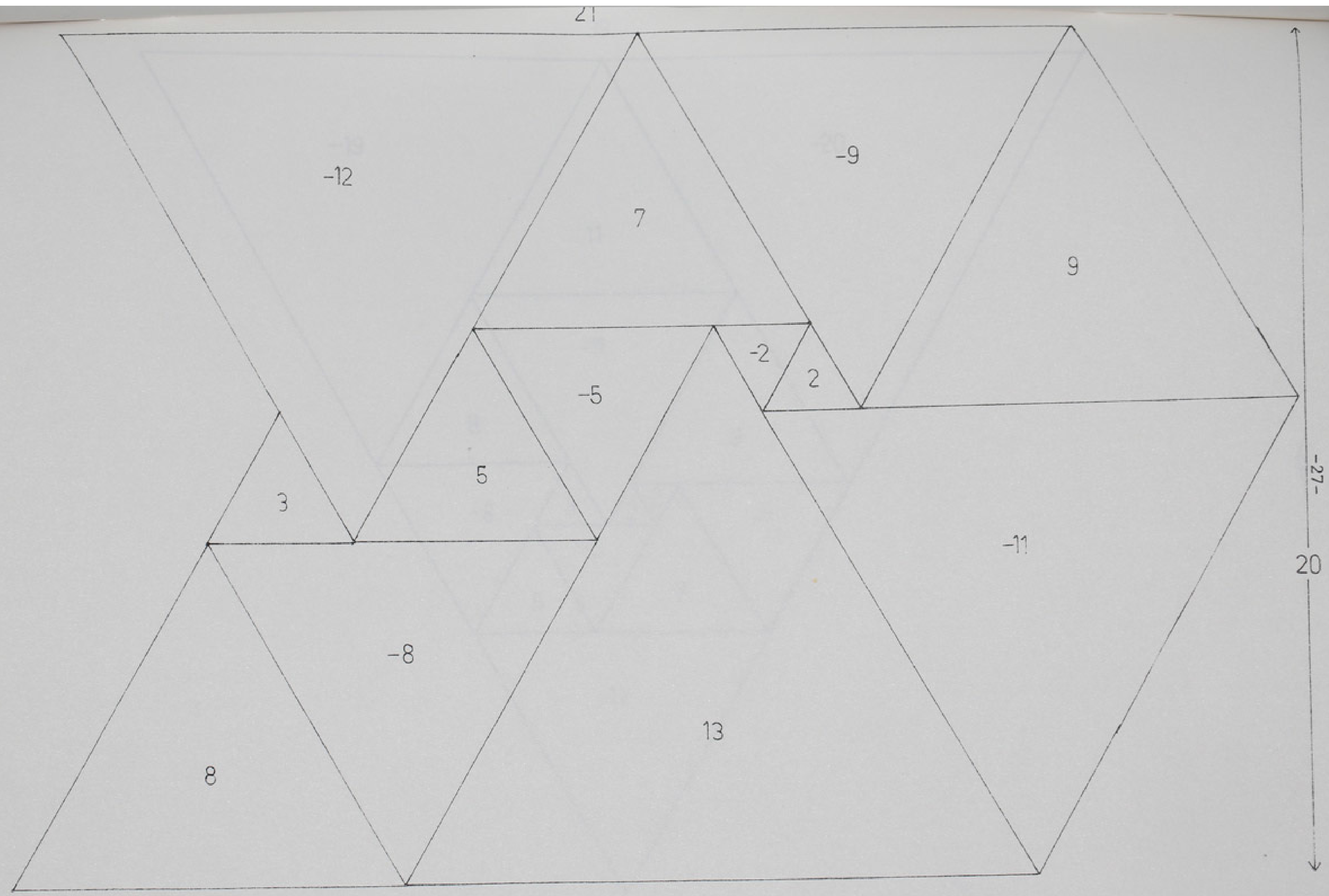


Figure 12. Perfect, simple triangulated cylinder of order 6½

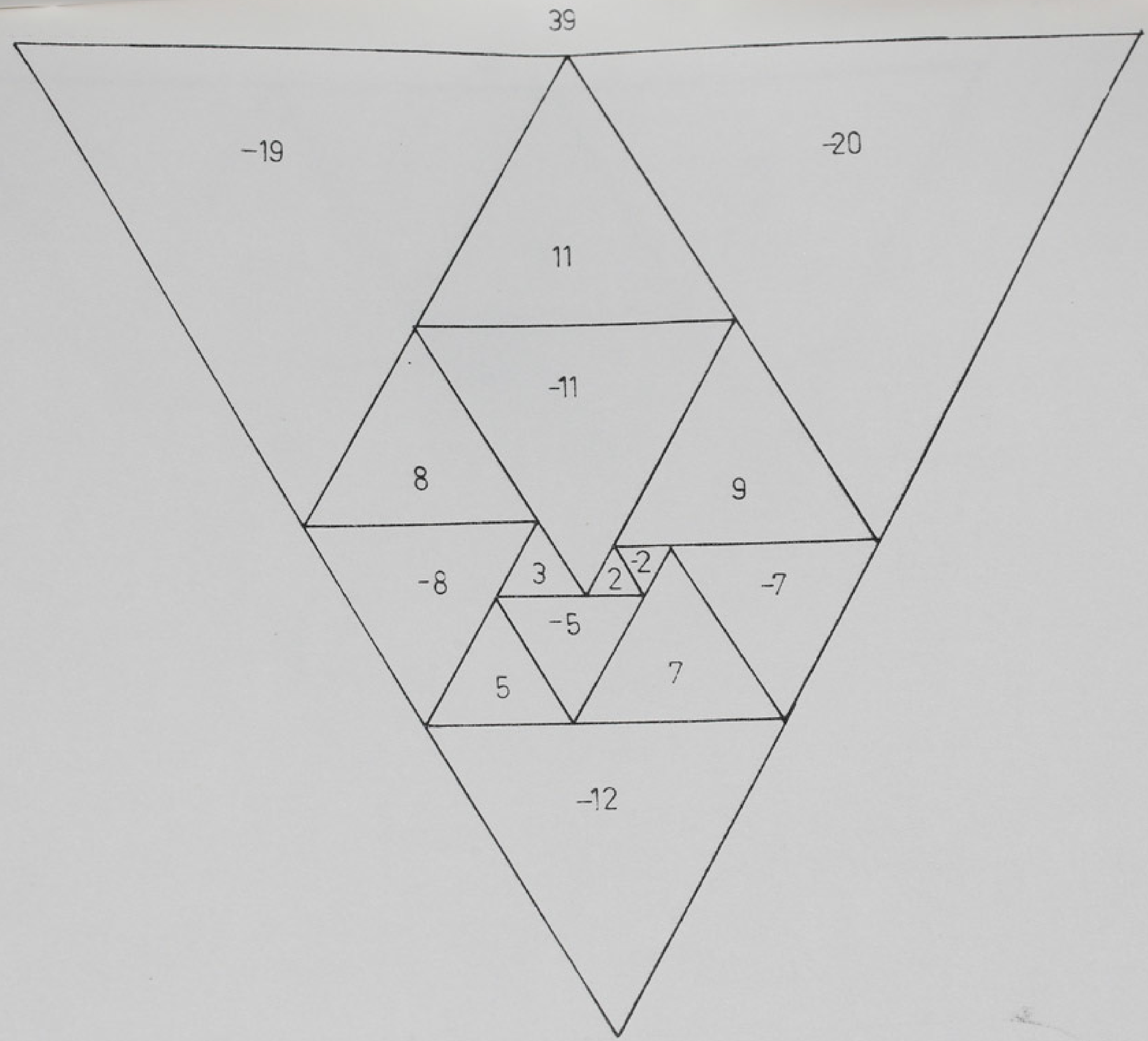


Figure 13. Perfect, compound triangulated triangle of order 7½ (Tutte)

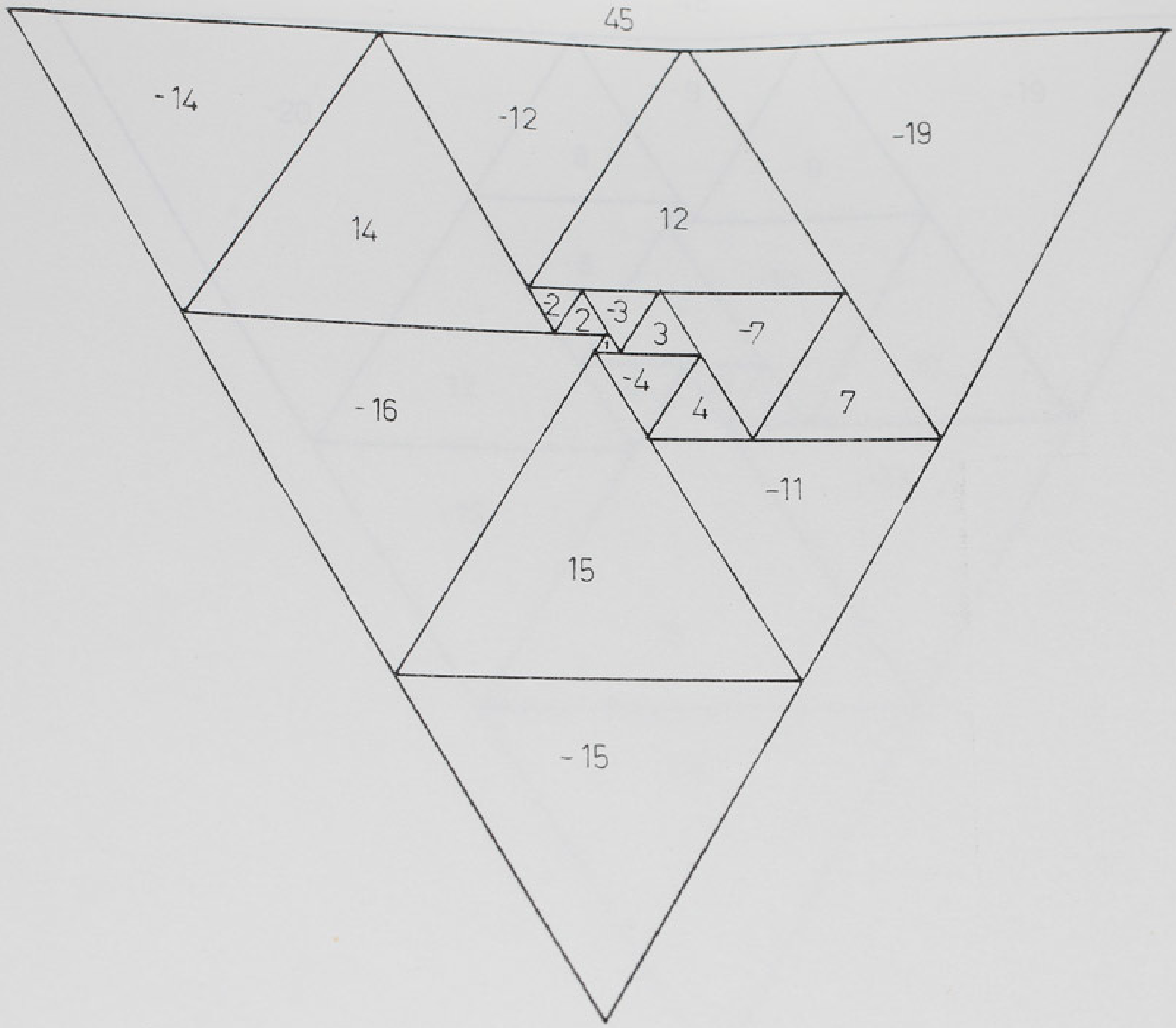


Figure 14. Perfect, simple triangulated triangle of order 9.

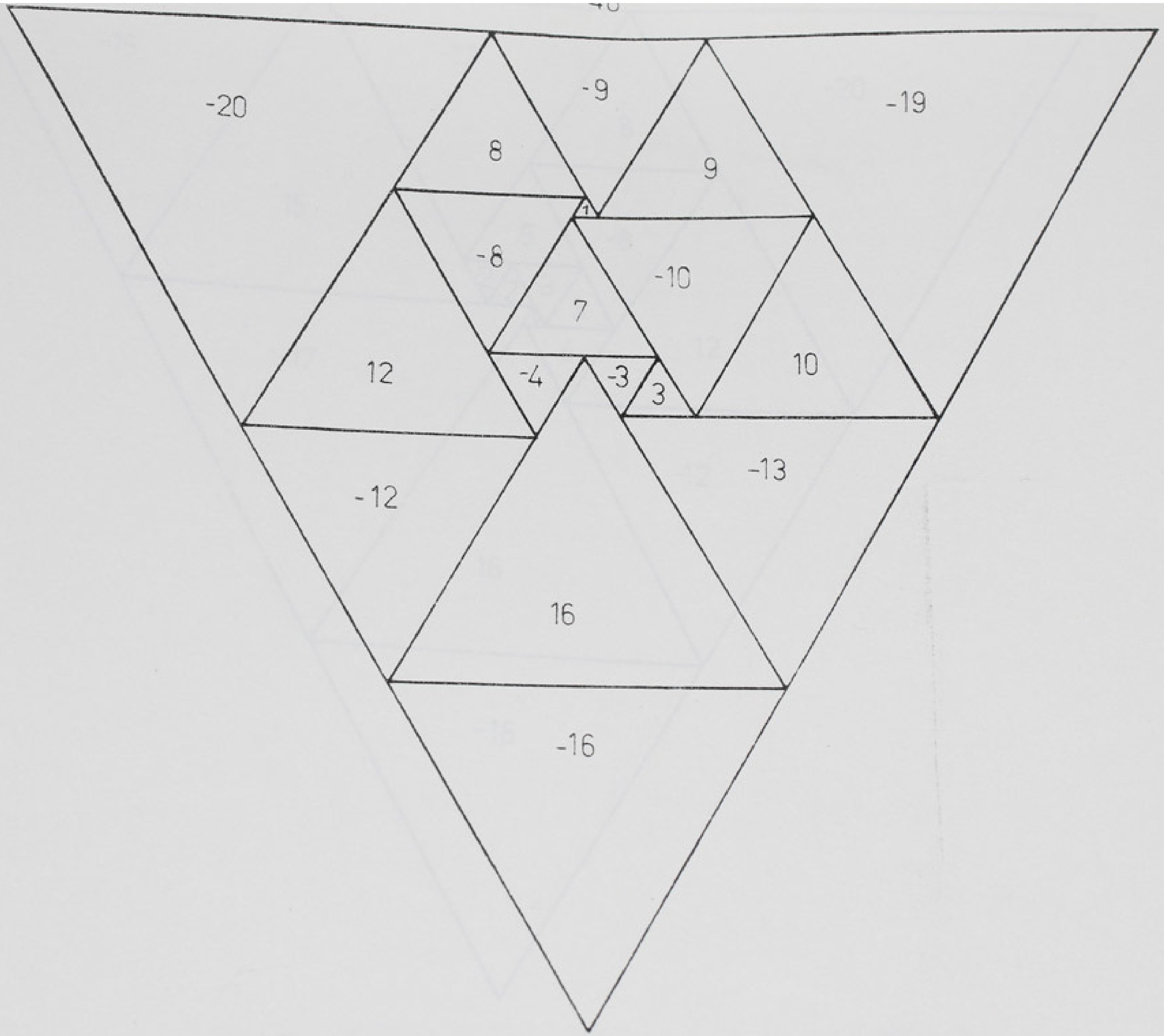


Figure 15. Perfect simple triangulated triangle of order 9.

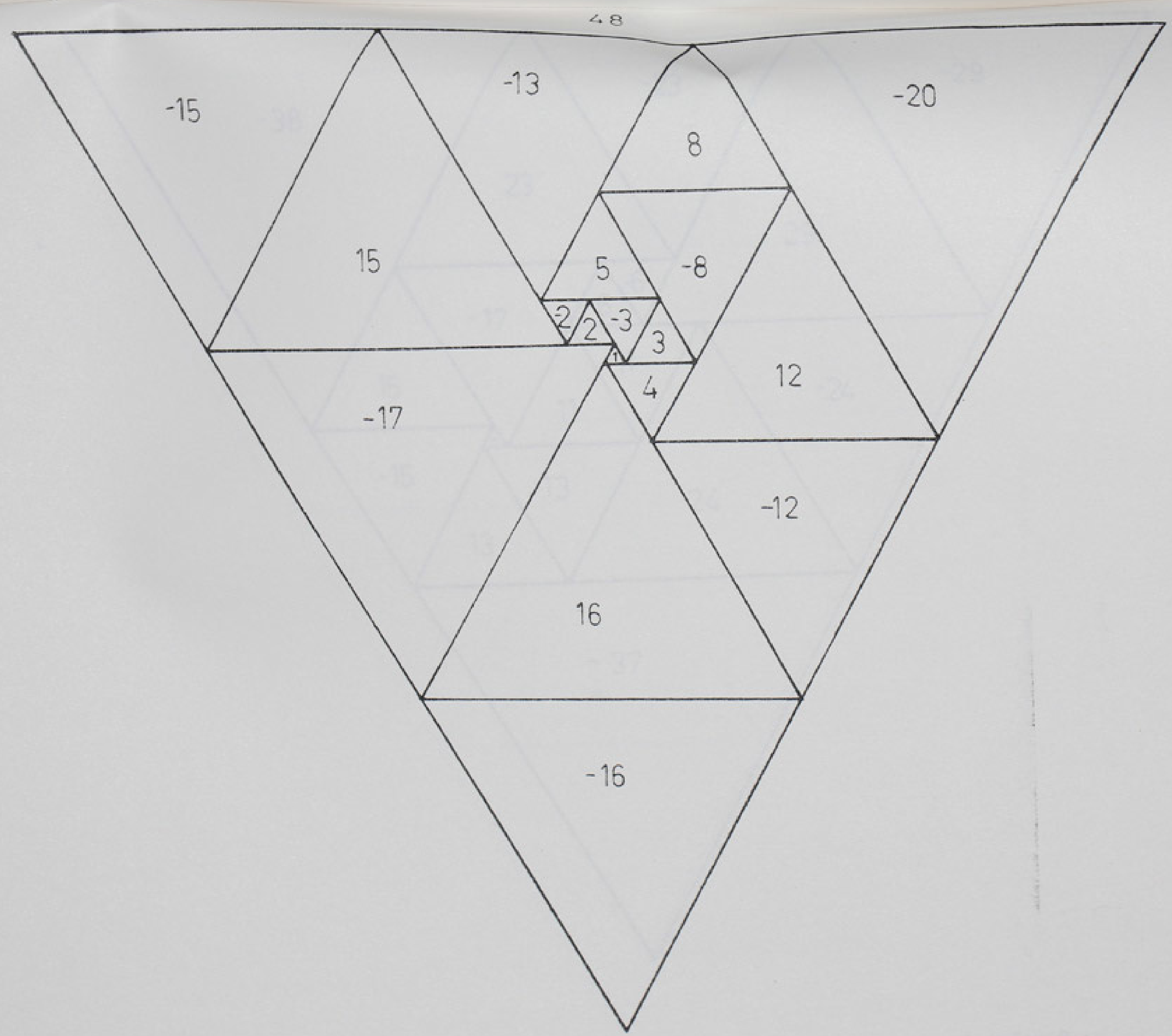


Figure 16. Perfect, simple triangulated triangle of order 9.

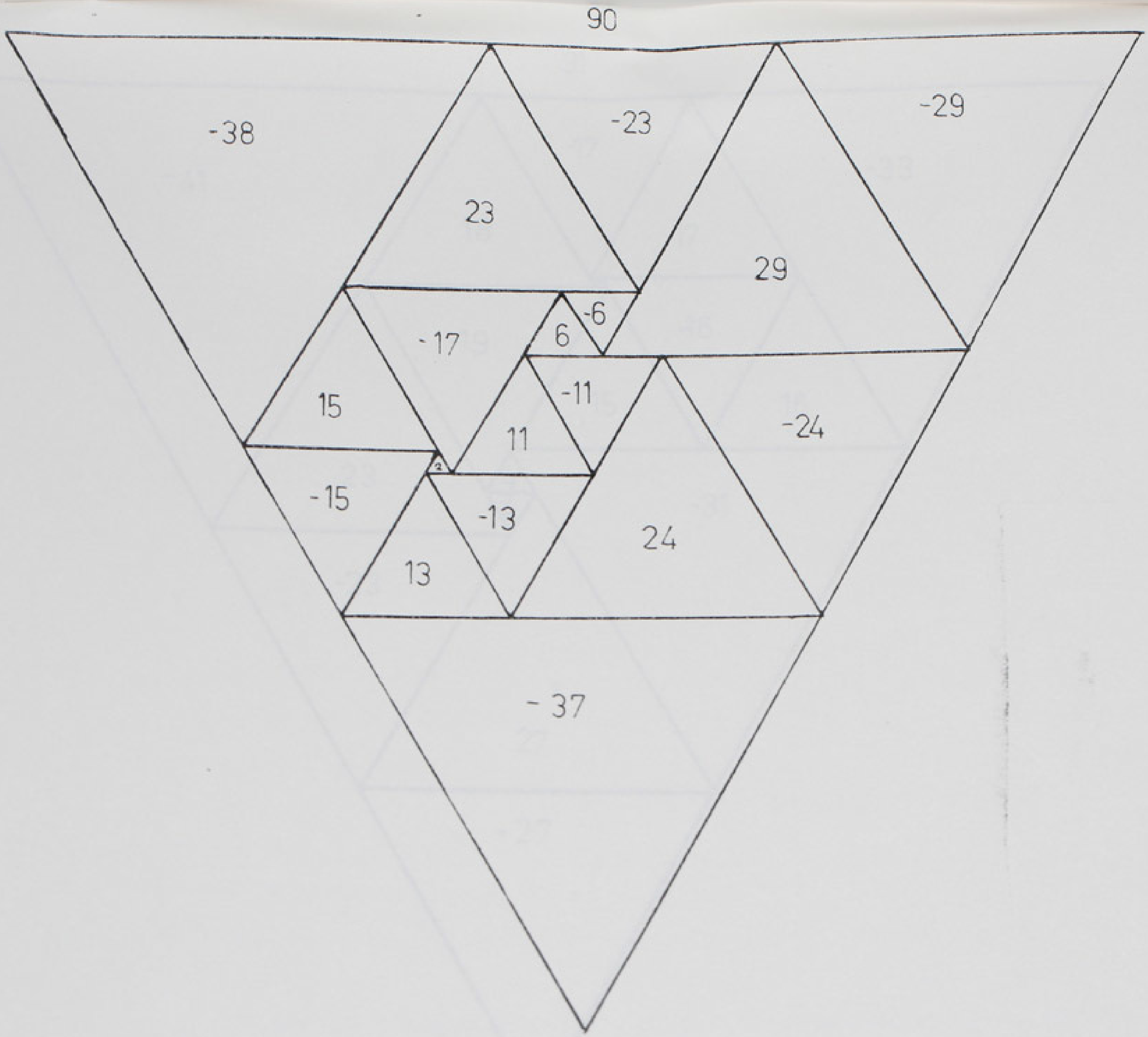


Figure 17. Perfect, simple triangulated triangle of order 9.

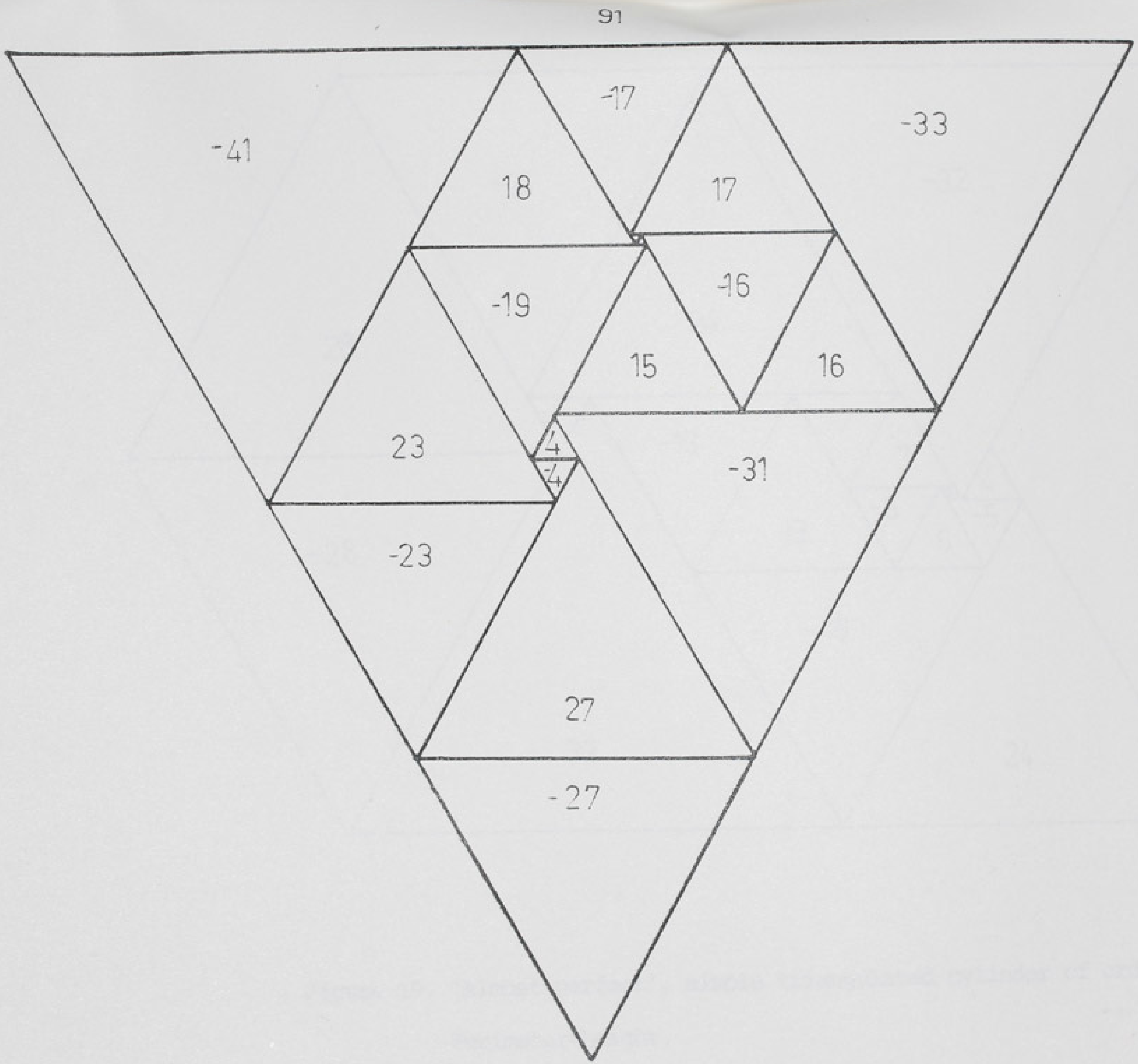


Figure 18. Perfect, simple triangulated triangle of order 9.

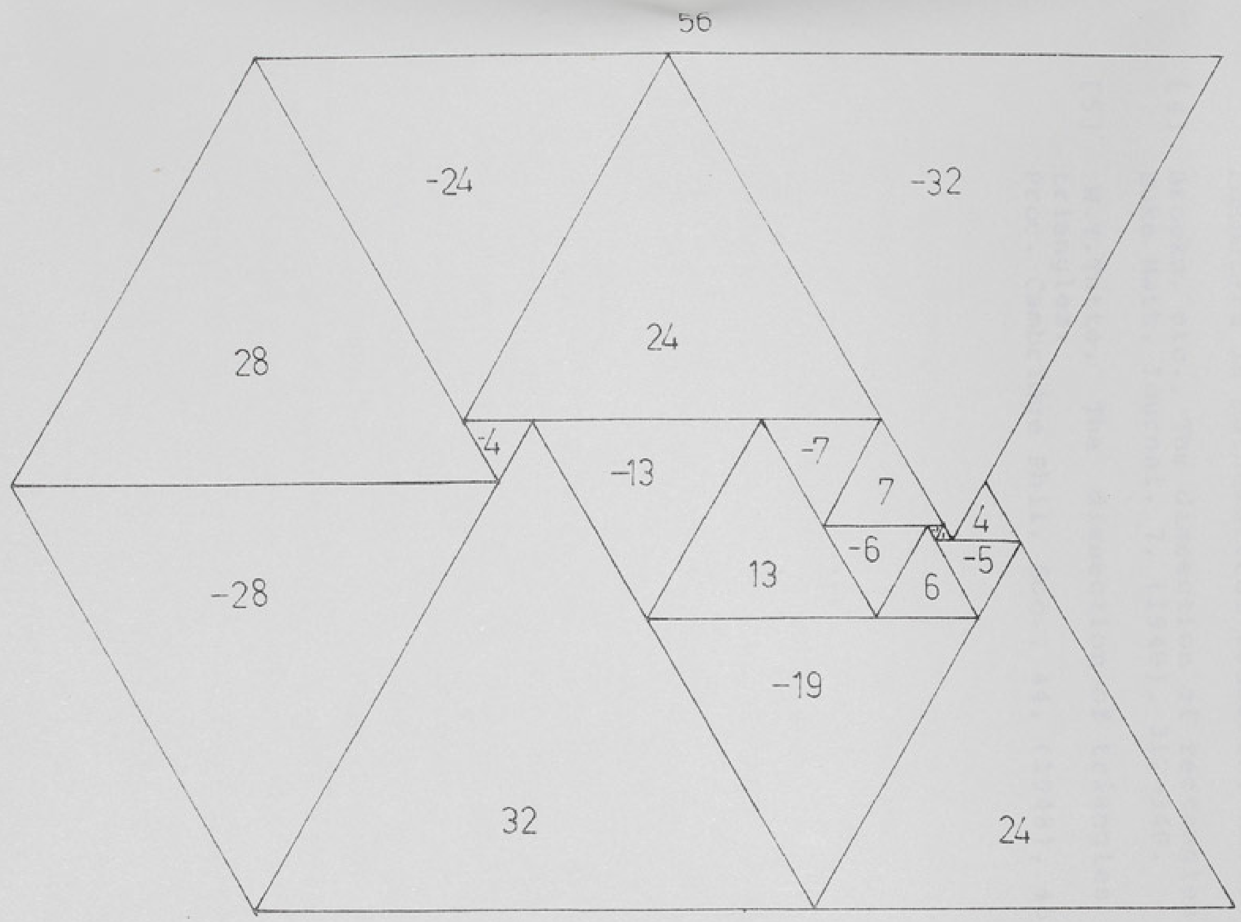


Figure 19. 'Almost perfect', simple triangulated cylinder of order 9½.

Perimeter=height.

Literature.

[1] W.T.Tutte, dissections into equilateral triangles.
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[4] Brooks, etc., The dissection of rectangles into squares.
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AANTALGRAAD4...................... Number of degree 4
PEPAALGRAAD4...................... Determine degree 4
DUBBEL........................... Double
GERICHT.......................... Directed
GESCHREVEN....................... Written
GEVONDEN......................... Found
GRAAD............................ Degree
GRAAF............................ Graph
HOOFDSTROOM...................... Main current
KEERGRAAF........................ Reverse graph
MAAKGRAAD........................ Make degree
MAAS............................. Face
MAASINCIDENT..................... Face incident
MAASTAB.......................... Face tabel
PRINTCODEENTAKKEN................ Print code and branches
RESTSTROOM....................... Rest current
RICHT............................ Direct
RUIT............................. Rhombus
VERDUBBEL........................ Redouble
WISSEL........................... Swap

'Het gekeerde net'............... The reversed net
'Inverse klopt niet'............. Inversed matrix is not allright
'Driehoek'....................... Triangle
'Compound, bevat een paral-...... Compound, contains a parallello-
lellogram van orde' gram of order
'Afmetingen van het paral-....... Demensions of the parallellogram are
lellogram zijn'
'Knooppunten van de hoofdstroom'.. Vertices of the main current