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POSSIBILITIES AND IMPOSSIBILITIES IN SQUARE-TILING

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ABSTRACT

A set of natural numbers **tiles the plane** if a square-tiling of the plane exists using exactly one square of sidelength n for every n in the set. From Ref. 8 we know that $\mathbb{N}$ itself tiles the plane. From that and Ref. 9 we know that the set of even numbers tiles the plane while the set of odd numbers doesn't. In this paper we explore the nature of this property. We show, for example, that neither tiling nor non-tiling is preserved by superset. We show that a set with one or three odd numbers may tile the plane—but a set with two odd numbers can't.

We find examples of both tiling and non-tiling sets that can be partitioned into tiling sets, non-tiling sets or a combination. We show that any set growing faster than the Fibonacci numbers cannot tile the plane.

Keywords: Tiling.

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1. Introduction

In 1903 M. Dehn[1] asked: Can a square be tiled with smaller squares, no two of the same size? In 1925 Z. Moroń found several rectangles that could be tiled with squares.[10] Dehn's question was answered affirmatively in 1938 by R. Sprague.[12] The problem and its solution were the subject of a memorable paper, "Squaring the Square" by Tutte,[13] reprinted in Martin Gardner's column in *Scientific American* (see Ref. 5). Papers continue to appear on the subject ever since (see for example, Refs. 2–4).

In 1975 S. Golomb[6] asked if the infinite plane can be tiled by different squares with every side-length represented. In 1997, Karl Scherer[11] succeeded in tiling the plane using squares of all integral sides, but each size is used multiple times. The number of squares of side n used, $t(n)$, is finite but the function t is not bounded. Golomb's question was answered affirmatively in "Squaring the Plane." (2008, Ref. 8). The solution opened a host of questions, for example, Which sets tile the plane? Is there a tiling free of squared rectangles (a "perfect tiling" in the language of Ref. 13)? Is there a three-colorable tiling? Can the half-plane be tiled?

A second paper[9] showed that neither the set of odd numbers nor the set of primes tiles the plane. It found a tiling free of squared rectangles. It showed that the set of natural numbers can tile many, even infinitely many planes. But it raised further questions. Does a superset of a tiling set tile the plane? Can $\mathbb{N}$ be partitioned into two tiling or two non-tiling sets? Can a Riemann surface be tiled?

There are connections between squaring planes and squaring squares (see for example the proof of Proposition 2). There are also curious disconnects. There is a clever proof that a cube cannot be cubed.[13] But the technique has not yet shown us that space cannot be cubed.

In this paper, we address and answer some of these questions, focusing on the structure of tiling sets. In Sec. 1 we show that a set with exactly two odd numbers cannot tile the plane. This provides an example of sets $A \subseteq B$ where A tiles the plane and B does not.

In Sec. 2 we find sets with exactly one odd number which tile the plane. We also find sets with exactly three odd numbers which tile the plane. This shows that all combinations of sets A, B, $A \cup B$, tiling and not tiling are possible.

The simplest tiling of the plane uses the set of Fibonacci numbers (with two squares of side-length 1). The Fibonacci numbers grow exponentially as ϕ^n, $\phi = \frac{1+\sqrt{5}}{2}$. Before it was shown that $\mathbb{N}$ tiles the plane, it was conjectured that all tiling sets had to grow exponentially.[7] In Sec. 3 we show that any set growing faster than ϕ^n cannot tile the plane.

Additional questions are posed in Sec. 4.

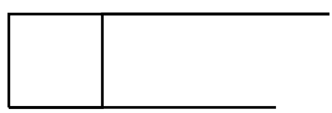
For simplicity, in this paper we will denote the square of side n with the boldface letter $\mathbf{n}$.

2. Two Odds

Proposition 1. *Let X be a subset of $\mathbb{N}$ containing exactly two odd numbers. Then X does not tile the plane.*

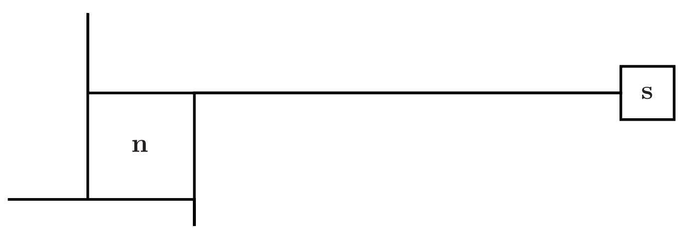
Proof. Let n, m be the odd numbers in X and suppose that there is a tiling $\mathcal{X}$ of the plane using exactly one square of sidelength k for every $k \in X$.

Definition 1. At every corner of a square **s** in $\mathcal{X}$ there is an edge extending away from **s**. We'll call such a line a **spoke** of **s**. We will say that **s** has **an integral side** if it has two spokes extending in parallel from adjacent corners.



If **s** has no integral side we say it is a **pinwheel**.

If a spoke of **m** is not also a spoke of **n**, then it must extend forever, for if, it ended at a square **s**,



it would mean that a sum of side-lengths plus n is equal to a different sum of side-lengths. This is not possible since all the side-lengths would be even and n is odd.

If a square **s** has an integral side, then along that side there must be squares whose side-lengths sum to s. Since n is odd, **n** can only have an integral side if it is adjacent to **m**. Thus, **n** and **m** must each have spokes in at least three different directions. Then however they are situated in $\mathcal{X}$, **n** and **m** must have a pair of parallel and separate spokes. But these spokes would form an infinite corridor in the tiling. This is not possible, however, since only a finite number of squares can fit in a space of finite width. □

Proposition 1 yields

Corollary 1. *There are sets $A \subseteq B \subseteq \mathbb{N}$ such that A tiles the plane and B does not.*

As noted in Ref. 8, the even numbers tile the plane. We have just seen, however, that the addition of two odd numbers forms a set that doesn't tile the plane.

Corollary 2. *There are sets $A, B \subseteq \mathbb{N}$ such that neither A nor B tiles the plane but $A \cup B$ does.*

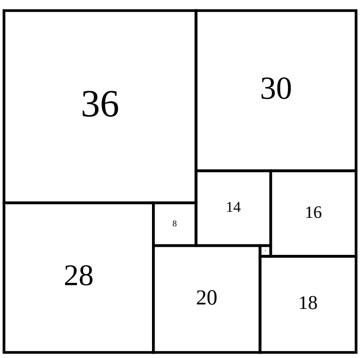
An example of such sets are the evens plus two odds and the rest of the odds (shown not to tile the plane in Ref. 9).

3. One or Three Odds

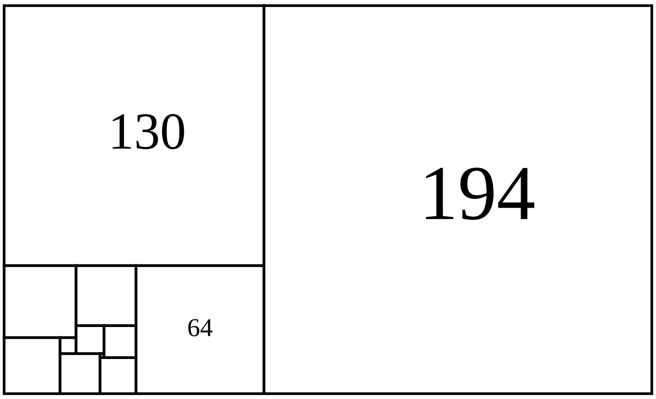
Proposition 2. *It is possible to tile the plane using only one odd square.*

Proof. As the proof of Proposition 1 shows, a single odd square in a tiling must be a pinwheel whose spokes continue forever. Thus to prove the proposition it is necessary and sufficient that we show it is possible to tile a quarter-plane in four ways using four disjoint sets of even tiles.

It is certainly possible to tile one quarter-plane. We can start with a 64×66 squared rectangle composed of nine squares of sides 2, 8, 14, 16, 18, 20, 28, 30, and 36 (one of Moroń's rectangles, doubled),



and add squares in a generalized Fibonacci sequence, 64, 130, 194, 324, ...



We can tile all four quadrants with multiples of these squares if we can find sets of multiples that don't intersect. We claim that multiples of the above sequence of squares,

$$\mathbf{a}\colon 2, 8, 14, 16, 18, 20, 28, 30, 36, 64, 130, 194, \ldots$$

by the factors 23, 24, 25, and 26 are disjoint. To check this, first verify that none of the multiples of the original 9 squares appear in any of the other sequences.

	23	24	25	26
2	46	48	50	52
8	184	192	200	208
14	322	336	350	364
16	368	384	400	416
18	414	432	450	468
20	460	480	500	520
28	644	672	700	728
30	690	720	750	780
36	828	864	900	936

If we continue the sequences,

	23	24	25	26
64	1472	1536	1600	1664
130	2990	3120	3250	3380

we see that $26a_n < 23a_{n+1}$ for $n = 10, 11$. This pattern must persist since $26a_{n+1} = 26a_n + 26a_{n-1} < 23a_{n+1} + 23a_n = 23a_{n+2}$. This shows that the sequences are disjoint. We can see, for example, that $24a_j > 25a_k$ for $j > k \geq 10$ since $25a_k < 26a_k < 23a_{k+1} < 24a_{k+1} \leq 24a_j$. Consequently the plane can be tiled with one odd square and these four multiples of sequence **a**. $\square$

Corollary 3. *There are sets $A, B \subseteq \mathbb{N}$ such that A and B tile the plane but $A \cup B$ does not.*

Proof. We need only show that it is possible to tile a quarter-plane in eight ways using eight different sets of even tiles. Then we can form A from the side-lengths of four of the eight sets plus one odd number and form B from the side-lengths of the other four sets and another odd number. By Proposition 2, A and B each tile the plane, but by Proposition 1, $A \cup B$ does not tile the plane.

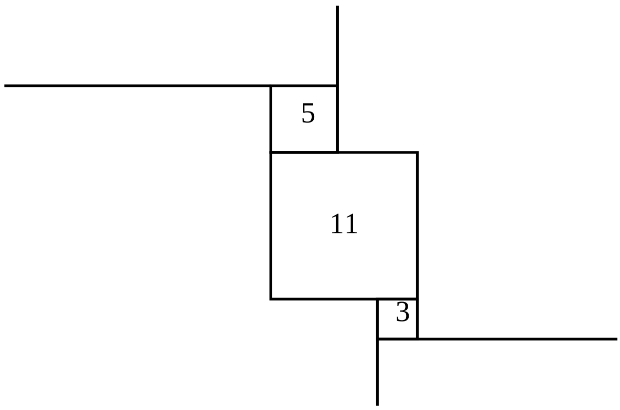
It is easy to check with a spreadsheet that multiples of the sequence **a** by 200, 201, 202, 203, 204, 205, 206, and 207 satisfy the earlier constraints. □

We now have all possibilities for combining tiling and non-tiling sets.

A	B	$A \cup B$	
tiling	tiling	tiling	Prop. 2.1,[9]
tiling	tiling	non-tiling	Cor. 3
tiling	non-tiling	tiling	the evens and the odds, Prop. 1.2,[9]
tiling	non-tiling	non-tiling	some evens, some evens plus two odds
non-tiling	non-tiling	tiling	Cor. 2
non-tiling	non-tiling	non-tiling	two disjoint sets of odds, Prop. 1.2,[9]

Proposition 3. *It is possible to tile the plane using exactly three odd squares.*

Proof. This works by placing three odds as below and creating once again four regions.



Two are simple quadrants and can be tiled with multiples of the generalized Fibonacci sequence in the proof of Proposition 2 by 23 and 24. The upper right region, in the example above, could be tiled by starting with the **14** square, then squares **20**, **34**, **54**, ... The region at the bottom left could be tiled by starting with the **16** square, then squares **24**, **40**, **64**, ... It is easy to check that these four sequences are disjoint for the first levels.

	23	24	
14	46	48	16
20	184	192	24
$\vdots$	$\vdots$	$\vdots$	$\vdots$
602	690	720	712
974	828	864	1152
1576	1472	1536	1864
2550	2990	3120	3016
4126	4462	4656	4880

We also have that in the last two levels, the smallest is greater than the largest in the previous level so the sequences will remain disjoint. $\square$

4. Excessive Growth

We show here that sets growing faster than the Fibonacci numbers cannot tile the plane. The Fibonacci numbers, $\{F_n\}$, grow exponentially with $\frac{F_{n+1}}{F_n} \to \phi = \frac{1+\sqrt{5}}{2}$. We will say that a sequence $\{x_n\}$ "grows faster than the Fibonacci numbers" if from some point on $\frac{x_{n+1}}{x_n} > \phi$.

Proposition 4. *Let X be a subset of $\mathbb{N}$ and let $\{x_i\}_{i\in\mathbb{N}}$ enumerate the elements of X in increasing order. Suppose that i_0 is such that for $i > i_0$, $x_{i+1} > \phi x_i$. Then X doesn't tile the plane.*

Proof. First note that for $i > i_0$, $x_{i+2} > x_{i+1} + x_i$ as follows: We have $x_{n+1} = \phi x_n + \epsilon$ and $x_{n+2} = \phi x_{n+1} + \delta$ with $\epsilon, \delta > 0$. Then

$$\begin{aligned} x_{n+2} &= \phi^2 x_n + \phi\epsilon + \delta \\ &= \phi x_n + x_n + \phi\epsilon + \delta \ (\phi^2 = \phi + 1) \\ &> \phi x_n + x_n + \epsilon + \delta \\ &= x_{n+1} + x_n + \delta \\ &> x_{n+1} + x_n. \end{aligned}$$

Note also that for $i_0 < i < j < k < l$, we will have

$$x_i + x_l > x_j + x_k,$$

since $x_l > x_j + x_k$.

Now suppose there is an X-tiling $\mathcal{X}$ of the plane. We will show that this leads to a contradiction.

Definition 2. A **clump** is any finite set of squares in the tiling such that

(1) the squares form a simply connected set[a] in the plane,
(2) no square in the clump is adjacent to a smaller square outside the clump,

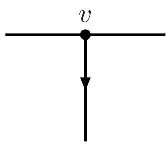
[a] A region of the plane is simply connected if it has no holes.

(3) the squares with sides $\{x_i\}_{i \le i_0}$ are well inside, i.e. not on the edge, of the clump.

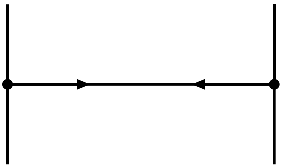
It should be clear that given any set of squares in the tiling, a clump can be formed containing the set.

Next we observe that $\mathcal{X}$ generates a planar graph. The corners of the squares in $\mathcal{X}$ are the vertices of the graph. Vertices are adjacent if they are connected by an edge or a part of an edge of a square.

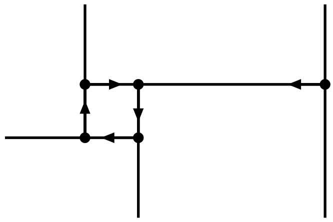
$\mathcal{X}$ also generates directions for some of the edges in the graph. If a vertex v has order 3, then let the one edge from v which is perpendicular to the other two edges be given the direction pointing away from v.



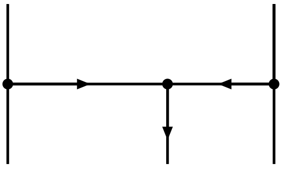
Note that no edge will be given two different directions, since two vertices of order 3 pointing at each other can occur only when two squares of the same side-length are used.



For the same reason, a directed edge cannot point to a vertex of order 4. Thus, there are no sinks, that is, paths consisting of directed edges do not end, they continue forever. They can, of course, loop.

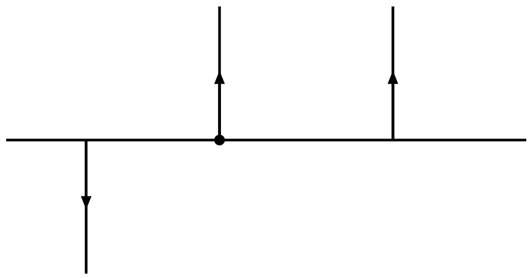


Note that two paths can join (as above), but only in the case where squares of side a and b are lined up with a square of side $a + b$.



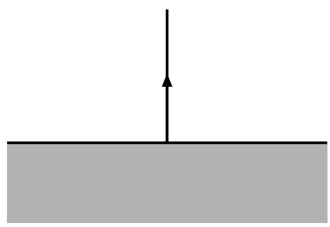
By definition, this can only happen inside a clump.

Definition 3. A directed path is **born** if it has a vertex with no edge leading to it.

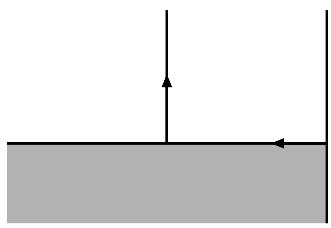


Claim 4.1. *No directed path in the graph which is on the boundary of or contained in the clump can exit the clump.*

A directed path that exits a clump can only do so along an edge. In the figures below, the shaded regions indicate squares in the clump.



The directed edge shown above is not in the clump. If any directed edge is connected to it,



then there would be a square outside the clump adjacent to a larger square in the clump, violating the definition of clump. This proves the claim.

Paths inside a clump must eventually loop in the clump. Thus, any path in a clump must join itself at some point. Note that no path can grow infinitely, since any path can be included in a clump and can't leave it.

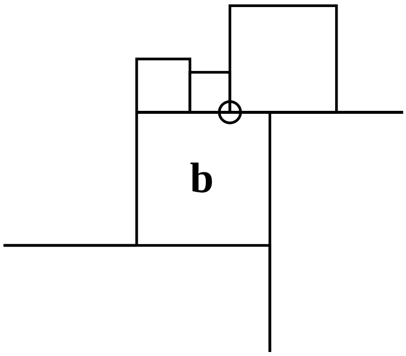
Let n be the number of instances of $a + b = c$, $a, b, c \in X$. It follows that there can be no more than n paths in $\mathcal{X}$, since a path can only join itself when the side of one square is the sum of the sides of two other squares. Thus, there can be only a finite number of births in $\mathcal{X}$.

Definition 4. A **tidy clump** is a clump that contains all births of $\mathcal{X}$ inside the clump (not on the edge).

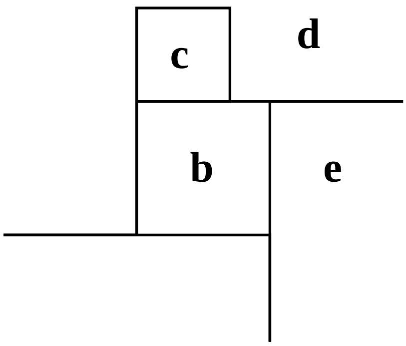
Claim 1 guarantees that tidy clumps exist. For the remainder of this proof, $\mathcal{C}$ will be a tidy clump.

Claim 4.2. *Any square not included in or adjacent to $\mathcal{C}$ must have at least one integral side.*

Suppose that **b** is a square outside and not adjacent to $\mathcal{C}$ and that **b** has no integral side. Then **b** is a pinwheel. We must have that **b** has only one neighbor on each of its sides as follows. It can't have three neighbors or there would be a birth outside $\mathcal{C}$ (circled).

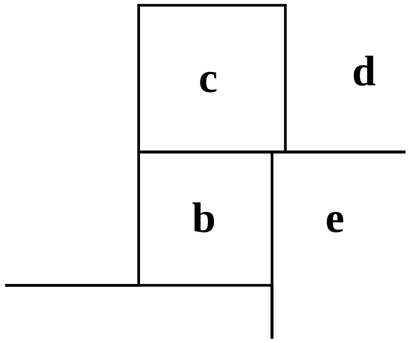


And it can't have two neighbors—if **b** borders on **c**, **d** and **e** as below,



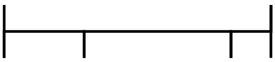
then **d** and **e** must share a corner, otherwise a path will be born at the top right-hand corner of **b**. But that would give us that $c + d = b + e$ which is not possible outside of $\mathcal{C}$.

Now let **c** be the smallest neighbor of **b**.



Again, **d** and **e** must share a corner (or there would be a birth at the bottom right-hand corner of **c**) and again $c + d$ would equal $b + e$. This proves Claim 2.

Claim 4.3. *Any line of edges outside or on the border of $\mathcal{C}$ is finite. It has a corner at each end. It must have exactly two corners along it. And there must be one square on one side, three squares on the other.*



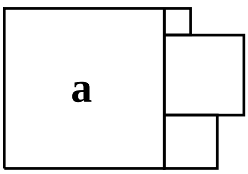
The prohibition of births precludes more than two corners along the line.

If the edge had no corners along it, we would have two squares with the same side-length. If the edge had only one corner along it, we would have numbers $b = c+d$ with **b**, **c** and **d** outside $\mathcal{C}$. Finally, if we had two squares facing two, we would have numbers $b + c = d + e$ with **b**, **c**, **d** and **e** outside $\mathcal{C}$. This establishes the claim.

Let **a** be the smallest square not in $\mathcal{C}$. By Claim 2, **a** is adjacent to $\mathcal{C}$, for otherwise it would have an integral side and hence a neighbor smaller than **a**, contradicting the choice of **a**.

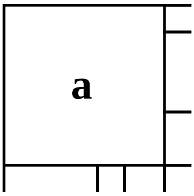
Claim 4.4. *Any edge of* **a** *touching $\mathcal{C}$ is an integral side meeting three squares from $\mathcal{C}$.*

By Claim 3, every edge of **a** is either an integral side, one square facing three or else is one of three squares facing one. The latter is impossible by part (2) of the definition of clump. Thus the edge of **a** is one square facing three. Any square adjacent to **a** along that edge would be smaller than **a**, hence in $\mathcal{C}$ by the choice of **a**.



Claim 4.5. *$\mathcal{C}$ does not abut **a** along adjacent edges of **a**.*

By Claim 4, the two edges would be integral sides which would necessitate a birth along **a**.



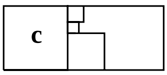
Claims 4 and 5 lead us to a progressive expansion of $\mathcal{C}$. If **a** is adjacent to $\mathcal{C}$ on just one side then $\mathcal{C}_1 = \mathcal{C} \cup \{\mathbf{a}\}$ is still a tidy clump. If **a** is somehow adjacent to $\mathcal{C}$ on more than one side so that there is a hole in $\mathcal{C} \cup \{\mathbf{a}\}$, then a new tidy clump $\mathcal{C}_1$ can be formed by including **a** and all the squares in the hole. In the first case, $\mathcal{C}_1$ will have the same number of corners as $\mathcal{C}$. In the second, $\mathcal{C}_1$ will have fewer corners.

We can now take the smallest square not in $\mathcal{C}_1$ and repeat the procedure, forming $\mathcal{C}_2$. Continuing in this way, we construct a sequence of clumps. The clumps grow larger and the number of corners either decreases or stays the same. Thus, for some n_0, the number of corners in $\mathcal{C}_n$, $n \geq n_0$, is constant.

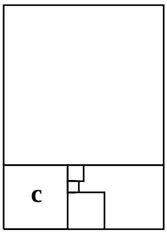
Claim 4.6. *For $n \geq n_0$, $\mathcal{C}_n$ is convex.*

If $\mathcal{C}_n$ is not convex, there would be a square **c** not in $\mathcal{C}_n$ but adjacent to $\mathcal{C}_n$ on two adjoining sides. But for some $m > n$, **c** will be the smallest square outside $\mathcal{C}_m$, contradicting Claim 5.

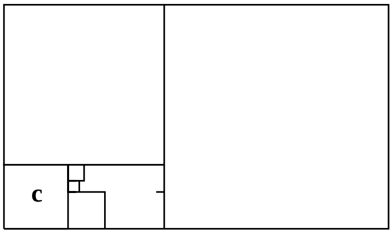
The only convex shape for a clump is a rectangle. Then the smallest square **c** outside $\mathcal{C}_{n_0}$ must fit on one end.



The smallest square outside of $\mathcal{C}_{n_0+1}$ can't fit on the opposite end (it can't be the same size as **c**), so it fits on a neighboring side.



Since the smallest square outside of $\mathcal{C}_{n_0+2}$ must face three squares in $\mathcal{C}_{n_0+2}$ must now fit on the opposite end.



And now there is no place for the smallest square outside $\mathcal{C}_{n_0+3}$; the bottom edge must have at least four squares on it. This contradiction concludes the proof of Proposition 4. $\square$

5. Open Questions

We have shown that there exists a tiling using three particular odd numbers. Can this be done with any set of three odd numbers?

We conjecture that it is possible to tile with n odd numbers if $n \neq 2$. It is possible to tile the plane with 7, 8, or 9 odd squares using squared rectangles. Is it possible to tile the plane using exactly 4 odd squares?

We have shown there exists a tiling using one odd number and then a collection of the even numbers. Can the plane be tiled using the entire set of even numbers and one odd number?

We have shown that there exists a tiling using mostly even numbers and a finite collection of odd numbers. Is there a tiling using mostly odd numbers and a finite collection of even numbers?

We know that the set $\{n : n \equiv 1 \mod 2\}$ does not the tile plane and that $\{n : n \equiv 0 \mod 2\}$ does. Clearly $\{n : n \equiv 0 \mod 3\}$ tiles the plane, but what about $\{n : n \equiv 1 \mod 3\}$, $\{n : n \equiv 2 \mod 3\}$, and $\{n : n \equiv 1, 2 \mod 3\}$? And what about other moduli?

For all of the tilings shown in this paper, in Ref. 8, and in Ref. 9, the ratio of successive members of the tiling set approaches either ϕ or 1. We know that ϕ and 1 are bounds on the ratio of successive terms. Given r, $1 < r < \phi$, can we tile the plane using sets where the ratio of successive terms approaches r?

Acknowledgments

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