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An example of a dissection of the square into pairwise unequal squares.

by

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Herr Stöhr[1] was recently occupied with the question of whether a rectangle of given aspect ratio, in particular a square, can be dissected into finitely many pairwise different squares. Theorem 10 of his dissertation states:

If any rectangle with sides a and b, $a < b$, admits two such dissections, which have no component square in common, and do not contain the square with side a, then the square with side $a + b$ can be dissected in the desired way (namely, into the squares with sides a and b and the two given dissected rectangles.)

In the following we give two dissections of a rectangle with aspect ratio 13 : 16 which satisfy the conditions of this theorem.

The first is shown in Figure 1; it has been based on well-known[2] dissections of the rectangles with aspect ratio 32 : 33 and 47 : 65 into pairwise unequal squares, which coincidentally have no square in common. The numbers denote the side lengths of the component squares.

We name the rectangle of Fig. 1 R_1; it has sides of $5 \cdot 13$ and $5 \cdot 16$ and it does not contain the square with side equal to its smaller side as a component.

In order to find the second decomposition, we first augment R_1 with a square along its longer side, thus producing a rectangle R_2 with sides $5 \cdot 16$ and $5 \cdot 29$. Furthermore, a rectangle R_3 with sides $16 \cdot 16$ and $13 \cdot 29$ is used; its dissection is shown in Figure 2.

The sides of the squares are

in R_2: 1, 3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 17, 18, 19, 22, 23, 24, 25, 33, 80;

in R_3: 7, 10, 28, 54, 61, 68, 75, 113, 115, 123, 133, 141.

[1] A. Stöhr, Über Zerlegungen von Rechtecken in inkongruente Quadrate. (Dissertation.) Schriften des Math. Inst. und des Inst. f. angew. Math. der Universität Berlin, Band 4 (1939), S. 119–140. Vgl. die Literaturangaben auf S. 119 und 120.

[2] Jaremkewycz, Mahrenholz, Sprague, Lösungen der Aufgabe 1242. Zeitsch. f. d. math. u. naturwiss. Unterricht 68 (1937), S. 43.

Linear enlargement of R_2 in the ratio $1:13$ and of R_3 in the ratio $1:5$ yields rectangles R_2' with sides $5 \cdot 13 \cdot 16$ and $5 \cdot 13 \cdot 29$ and R_3' with sides $5 \cdot 16 \cdot 16$ and $5 \cdot 13 \cdot 29$.

Joining R_2' and R_3' at the sides of equal length creates one rectangle R_4 with sides $5 \cdot 13 \cdot 29$ and $5 \cdot 16 \cdot 29$. It does not contain the square with side equal to its smaller side as a component. The rectangle R_4 contains pairwise unequal squares. This is because R_2 and R_3 have this property, and unlike the squares in R_2', no square in R_3, and hence none in R_3', has side a multiple of 13.

No side of a square in R_2 or R_3 is a multiple of 29; therefore, the same is true for R_4. The enlargement of R_1 in the ratio $1:29$ and R_4 thus are two dissections of the rectangle with sides $a = 5 \cdot 13 \cdot 29$ and $b = 5 \cdot 16 \cdot 29$ which fulfill the conditions.

R_1 consists of 20, R_4 of 33 component squares; two more component squares also appear, those with sides a and b. Therefore, each square is dissectable into 55 pairwise unequal squares.