THEORY & PRACTICE OF SQUARED-RECTANGLES & SQUARED-SQUARES

BY BERNARD LESLIE MOSS

INDEX TO SECTIONS & TOPICS COVERED

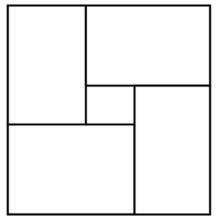
INDEX

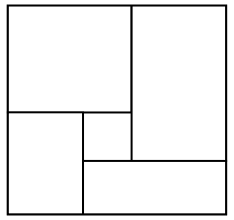
INTRODUCTION TO THIS BOOK

This book deals with all aspects of Squared-Rectangles and Squared-Squares.
Some of the content is in the form of Information and statistics, rather than general principles.

It is an inexhaustible subject.

The book is concerned with how Squared-Rectangles are in Theory and Practice, and it will be seen that whereas both are relevant and true, they often vary. Often constructions can be viewed in different ways.
To show one example, just look at the following!


A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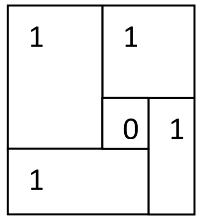
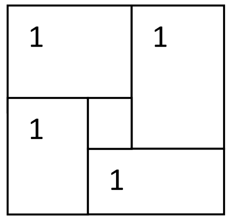


C



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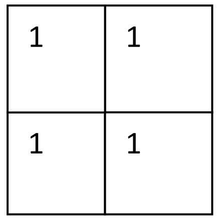


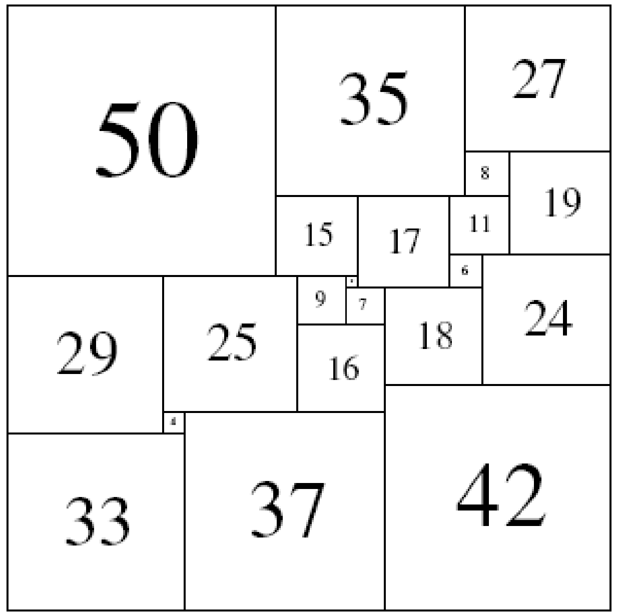
E

In A & B, Patterns have been drawn at random and in C & D, actual Element numbers have been calculated.
Finally, E square has been redrawn to ignore the zero.
Which of the above is Theory, and which Practice?
Which are the 'same' and which 'different'?
Much depends on how we view it! Both can be true according to different viewpoints.

1. **Arguably A B C D & E are different forms of the same Square.**
2. **A & B arguably represent the same Square.**
3. **A & B in Practice have been drawn sensibly ignoring any squares adjacent to each other, but**
4. **C & D arguably represent the same Square, but**
5. **C & D have patterns which are strictly illogical which may be regarded as Theoretic rather than Practice.**
6. **E may arguably be regarded as more correct than C or D so is true to Practice. But are there 4 squares or 5? Well both! There are 5 in theory with a Valid pattern, but 4 in Practice with an Invalid one!**

Theory and Practice often vary in this subject, but often Theory is more useful than the Practice.
The study of Squared-Rectangles & Squares is a very fascinating one as this book will reveal.
Before 1936 little was known about this subject, but a huge amount has now been found thanks to today's powerful Computers.
Today a huge amount and many Solutions can be downloaded from the Internet.

VARIOUS DEFINITIONS BRIEFLY EXPLAINED



1. SCALED DRAWING shown above as the Natural shape but more convenient for display is however:
2. COMPRESSED GRID DRAWING shown left. 3. __ Layer (horizontal) 4. | Layer (vertical) 5. REDUCED DIMENSIONS 104 x 104

```
________________________
|61   |43_____:________| 6.SQUARED-SQUARE
|_____|18_____|25......| 7. Crossover e.g. __|18__ 8. 61 etc. ELEMENTS 9. ORDER 24 (i.e.24 Elements)
|43|18|11 |7__|________| 18|11 |
|  |  |___|8 :  |9  |15| 9. : SLIDE (2 lines in same vertical distance)
|  |  |7|4|___:_|   |  | 10. .. SLIDE (2 lines in same horizontal distance) - not shown
|  |  | |6 |5 |1|___|  | 11. IMPERFECT (Duplication of some Elements e.g. 1,6,8,18,43
|  |__|_|  |  |4 |6 |  |  12. PERFECT (No Duplication of Elements as in drawing above.
|  |25: |  |__|__|  |  | 13. TRIAD i.e. 3 Elements positioned as 61 43 and 18 at left.
|  | :  |__|1|8  |  |  | 14.   SQUARED-RECTANGLE - i.e. Dimensions vary e.g. 33 x 32 unlike those shown.
|  | :  |7   |:  |__|__| 15. FULL DIMENSIONS e.g. Happens to be 37632 x 37632 for solution above (reduces
|  | :  |____|:__|21   | [21] 112 x 112. 16. REDUCTION INDEX e.g. 37632/112 which is 336 in solution above.
|__|__:_|15___:__|_____| 17. SQUARE INDEX e.g. 37632/112/112 which is 3 in solutions above.
```

18. BOUWKAMP CODE e.g. (61,43)(18,25)(43,18,11,7)(8,9,15)(7,4)(6,5,1)(4,6)(25)(1,8)(7)(21)(15) but in this book -
19. CODE +61,43 .18-25, -43.18.11.7 etc... where '+' denotes corner Elements '-' side and '.' Internal ones.
20. DISTORTED ELEMENTS (Showing Elements in varying rectangular shapes as here though all Squares in reality).
21. VALID SOLUTIONS shown here. INVALID either have Zero Elements or Adjacent Elements |1|1| as shown here.
22. COMPOUND Solutions contain at least one rectangle inside the solution.
23. SIDES INDEX 2-2-4-4 denotes number of Elements bordering the 4 sides (or 3-3-3-4) in 1st solution).

A. DEFINITIONS & BASICS ON RECTANGLES

A1.   SQUARED-RECTANGLES

A1.1. INTRODUCTION TO SQUARED-RECTANGLES

A Squared-Rectangle is shown below and comprises a quantity of individual squares, termed Elements - so joined together that they form a rectangle without any gaps occurring. It is like a jigsaw puzzle with square pieces that fit together perfectly.

A Squared-Rectangle may have any positive number of Elements. Below there are nine.

They vary in shape from extremely elongated to exactly square, the latter being termed Squared-squares.

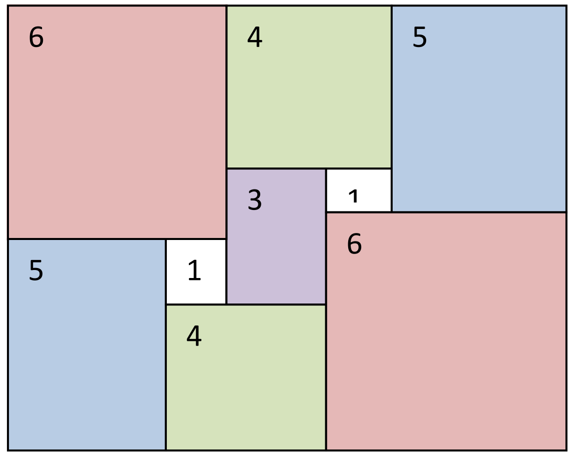
Some rectangles have Elements each with a different size, but others have Elements duplicated to a lesser or greater extent. This Book has Text and Drawings areas. The terms “above” and “below” always refer to Drawings as follows:- “above 2” means the second item in the last drawing. “below 3” means the third item in the next drawing, and so on.

The symbol @ denotes further references on the same subject.

Words used carrying a particular meaning in this book are given an initial capital letter, Squared-Rectangle, Elements and Perfect being examples.

Here is a Squared-Rectangle of 9 separate squares some of which have repeated values, with Dimensions of 15 across and 11 down.

[9] 15 x 11



A1.2. EASY IDEA OF SQUARED-RECTANGLES

What makes this subject attractive is that the idea of Squared-Rectangles is simple,

A COMBINATION OF SQUARES (CALLED ELEMENTS) FORMING RECTANGLES

With no gaps remaining. That is it!

Or if you prefer,

RECTANGLES DISSECTED INTO SQUARES

A1.3. SQUARED-RECTANGLES ARE A COMPLICATED & ABSORBING STUDY!

Despite the easy description above, the study of Squared-Rectangles is often difficult, involved and endless in scope keeping mathematicians occupied forever!

The more the One finds, the more amazing this subject becomes. As soon as something is solved, more features usually arise!

Some theories may be proved by algebra. Other ideas seem obvious but a formal proof hard to find …

A frustrating feature of this subject is the times an interesting relationship is found and upon inspection is true for many Rectangle Solutions. Then the relationship is found to be untrue with other Solutions.

This intention of this book is to cover all aspects of this subject and so in some cases the Reader may think the feature interesting but what use or relevance is it? Possibly none at all!

On the one hand is the general feature of how these Rectangles work - on the other is data given to provide statistics - some interesting and some not at all. All types of Squared-Rectangles & Squared-Squares are considered in this Book with the least interesting quickly dismissed.

A1.4. WHAT HAVE THESE TO DO WITH SQUARED-RECTANGLES?

1. Planar Electrical Circuits?
2. A postman's best route round a circuit of roads?
3. The Fibonacci Series 1 1 2 3 5 8 13 21 34 55 89 ...?
4. Chambered Nautilus and other Spiral Sea shells?
5. The number of petals in a flower?
6. The ideal proportions of the human face?
7. The Golden Ratio 1.618... and its reciprocal 0.618...?
8. God's Order in Creation? Surprisingly much!

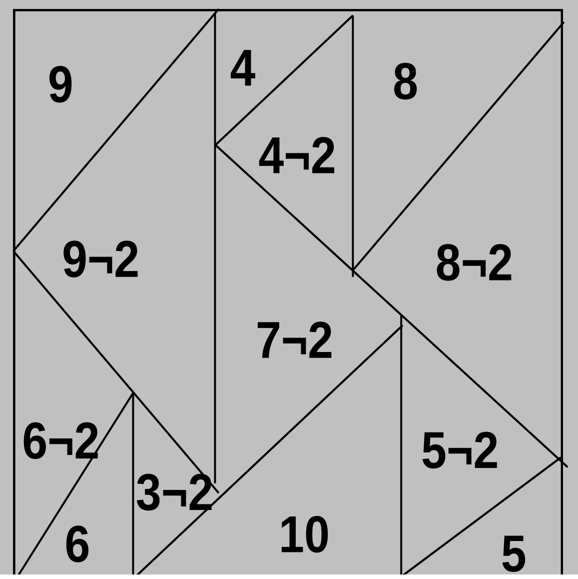
A2. FORMATS DIFFERENT FROM SQUARED-RECTANGLES

There are forms allied to this subject, not actually " Squared-Rectangles" but worth mentioning -

OTHER CONSTRUCTIONS ALLIED TO SQUARED-RECTANGLES BUT OUTSIDE THE SCOPE OF THIS BOOK:-

A2.1. TRIANGLED-RECTANGLES

The example shown below is a Square though Rectangles also exist:-



[13] 26 x 26 SQUARE

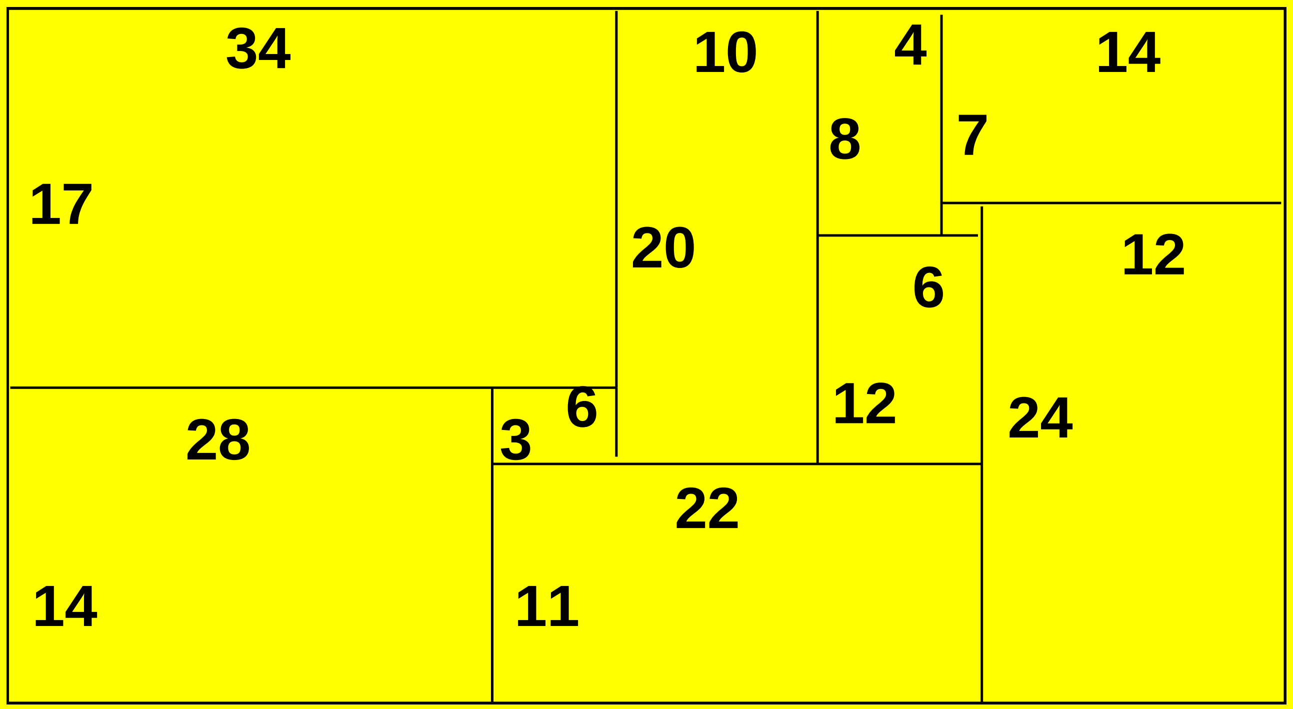
For "¬2" read Square Root of 2

The Triangles are all

Right Angles Isosceles ones.

A2.2. RECTANGLED-RECTANGLES

Although this book is concerned with Dissection into SQUARES the diagram above shows a Dissection into RECTANGLES.
In this case each have the ratio of 2 : 1 or 1 : 2, with some arranged horizontally and the rest vertically.
Numerically there are less of these Solutions than in Squared-Rectangles. There are 2 solutions for Order 10.
The Ratio does not have to be 1 : 2 and could be 1 : 3, 1 : 5, 4 : 7 etc.
Rectangled-Rectangles are not part of this study but may be useful in its general theory.



[10] 62 x 31

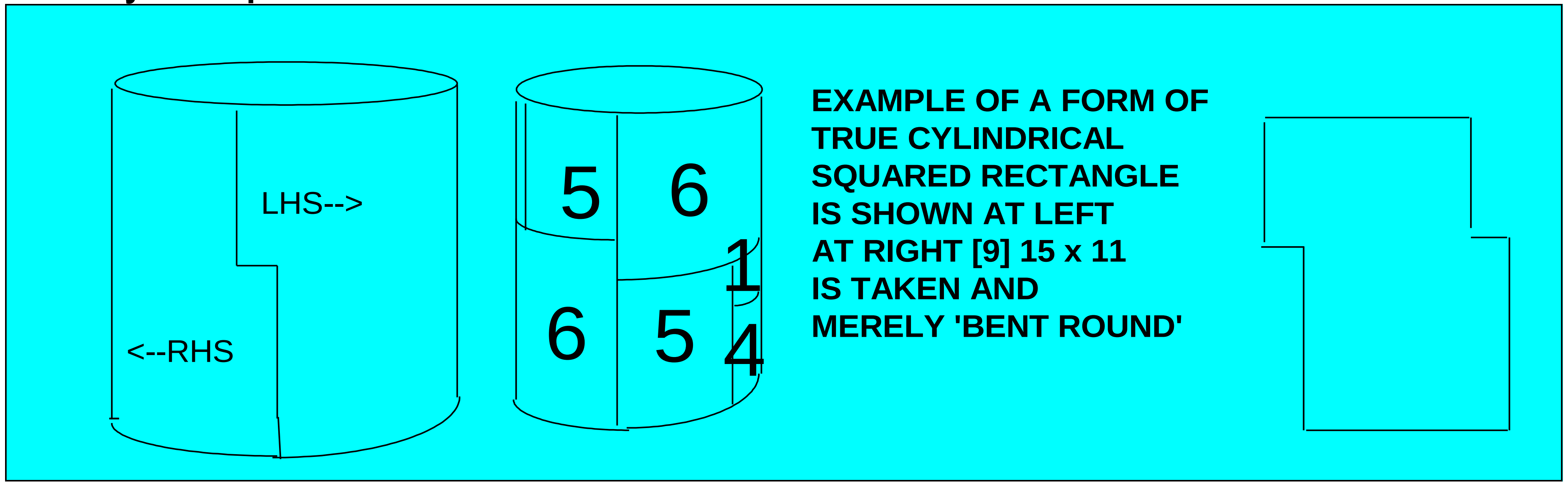
RECTANGLE

DISSECTED

BY RECTANGLES

A2.3. CYLINDRICAL SQUARED-RECTANGLES

If a Squared-Rectangle is constructed in such a way that its right side can be swung round to join its left side and fit then a Cylindrical Squared-Rectangle will be formed. Of course, any given SR can be drawn and then bent round so that the left and right borders meet, and similarly the top with the bottom! See Below 2. But this is of little interest.

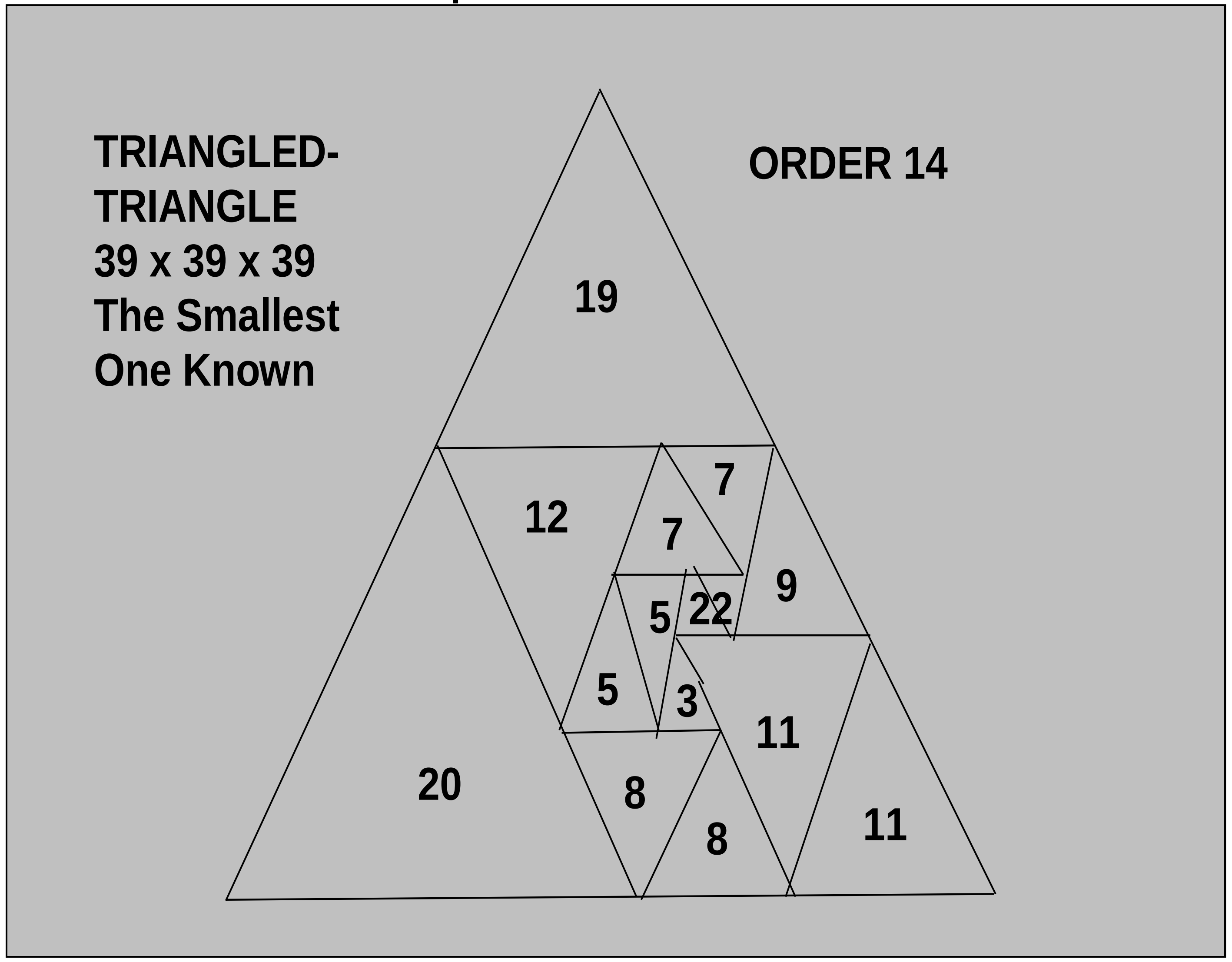


A 'True' CSR is one where the original "unwrapped" rectangle has a kink as in above 3, (or number of kinks) on the lhs repeated exactly on the right hand side.

Constructions are found where cylinders connect vertically and horizontally. But trying to connect both at the same time ends in disaster! For further information on this subject see Section Q.

A2.4. TRIANGLED TRIANGLES

These have been found, and some of these are "Perfect" if this term is construed to include no more than 2 triangles of a particular size. Where two exist one is up-side-down in relation to the other and may be regarded different in this sense.



A3. USES OF SQUARED-RECTANGLES

A3. PRACTICAL APPLICATIONS

Some areas of recreational mathematics have little or no practical application, but surprisingly Squared-Rectangles do as that they are closely related to planar Electrical Circuit theory!

It will be seen later that every Squared-Rectangle has two corresponding electrical circuits. Thus in this study there is a dual theory exists, of Squared-Rectangles on the one hand and electrical circuits on the other. See Section B for details.

There is said to be a practical application for Postmen! With an enclosed network of roads to cover a route may be found which visits each road twice (i.e. for both sides of the road).

To do this we draw a 'box' of roads as a planar circuit and choose a top and bottom as 'poles'. We then have to transform this into a Squared-Rectangle (calculation of the Elements is unnecessary). Then we draw a number of Spider Trails over each Element.

A4. POTTED HISTORY OF SQUARED-RECTANGLES

A4. BACKGROUND TO SQUARED-RECTANGLES SUBJECT

In the famous Canterbury Puzzles, Dudeney made a statement that suggested that to dissect a square into smaller squares all of different sizes was impossible. The mathematician Luzin also claimed the same impossibility.

This challenged four students in 1936 - 1938 to study the subject in depth, named C. A. B. Smith, A. H. Stone, W. F. Tutte and R. L. Brooks.

They ultimately discovered and recorded several Squared-squares, proving it can be done.

In fact, once no limit is put on the amount of Elements used, the quantity of possible Squared-squares is infinite!

Squared-Rectangles are easier to discover than Squared-squares. However inferior types of Squared-squares can be found easily from known solutions.

All existing Squared-Rectangles up to and including Order 15 have been calculated by me for reference.

All solutions to and including Order 18 have been calculated by others. (Now higher than this).

In this subject it is necessary to be precise in defining the various features e.g. what is meant by the "largest" rectangle for a particular Order? Does it mean the Squared-Rectangle with the greatest longer side, or the one with the greatest overall area!

The elementary section that follows introduces the subject by defining various features of Squared-Rectangles.

Some features will seem obvious but lead the way to more productive study. Other features are mentioned in passing, not being that important but part of the entire study. Invalid rectangle is an example.

A5. TERMS USED IN SQUARED-RECTANGLES

Many terms in Squared-Rectangles need to be defined, and most of the terms used by Tutte, Smith, Stone and Brooks form the basis of this book for example, Simple, Imperfect, Perfect and Complexity. Others have been invented by me such as Cornex, "Invalid", "Elongation", Reduction Index, Twin Rectangles and many more.

A5.1. SQUARED-RECTANGLE

A Squared-Rectangle is a rectangle divided up into a number of squares termed Elements in such a way that no gaps result.

Unless qualified, the term Squared-Rectangle covers a wide range of types.

They can be illustrated in a variety of ways, namely

1. Such Rectangles can be drawn showing the individual squares accurately drawn or distorted as oblongs.

2. They may exclude the values (as under A1) or more usual include the values.

3. The values could also be included in appropriate locations with the dissecting squares absent. This has limited value.

4. They may be shown as a Smith Diagram - in fact by two different Smith Diagrams. See Section B for explanation.

5. They may be recorded in a non-geometrical form as a Formula of numbers. There are several useful formats possible which are concise, save space and can be typed out without the need for any geometry.

A5.2. RECTANGLE

Although a Square is strictly a rectangle with the vertical and horizontal sides equal, the term in this book will assume that the vertical and horizontal sides are unequal even if shown to be square after calculation. Otherwise it is a square.

A5.3. DIMENSIONS

The Dimensions of a Squared-Rectangle are measured in units that are always positive integers. Any solutions calculated to be otherwise are easily adjusted to positive integers. In passing, it is observed that any solution can be given negative Elements or even Zero Elements throughout without the shape or structure being altered.

Thus "106 x 105" denotes a rectangle 106 units along and 105 down.

The horizontal dimension is deemed to be the larger one in this book and always shown first. See also full and reduced dimensions further on.

A5.4. ELEMENTS

These are the values of the individual component squares that form the rectangle or square.

The value of each Element is regarded as acting both horizontally and vertically so an Element of 10 is 10 units across and also 10 units down.

A5.5. CORNER, OUTER AND INNER ELEMENTS

Inner Elements are those which do not touch any side of the rectangle. The others are Outer, four of course being Corners.

A5.6. LARGEST & SMALLEST ELEMENTS

The smallest Element (which must be at least zero in size and not regarded as negative), is that one that is smallest, and is always an inner Element. The largest Element is very often in a corner, less frequently on a side, and infrequently internally. In theory negative inner Elements often arise, but in practice can always be made positive by easy slight adjustment of the pattern.

A5.7. ORDER

The Order (capital O) is simply, the total amount of Elements contained in a Squared-Rectangle.

Always a positive integer it is the total quantity of inner and outer Elements.

The Order number is shown inside square brackets [ ], e.g. as [9] 69 x 61.

A5.8. FORMULA

The Formula, later fully explained in A11, is a set of bracketed numbers separated by commas that are written in a convenient Order, from which the Squared-Rectangle can readily be drawn.

It takes much less room to write this than draw the actual solution, and a useful means of recording solutions.

The Formula known as Bouwkamp's Formula can be expressed in different ways and I prefer to replace brackets with ":" and show "+" before Corner Elements and "-" before Side Elements.

A5.9. NUMBER OF SIDES OF RECTANGLE

There are Four of course.

A5.10. TWIN RECTANGLES, OR TWINS

These are two or more different solutions with the same dimensions. Three twins means three solutions of identical size. @G

A5.11. SIMPLE AND COMPOUND RECTANGLES

A rectangle is termed SIMPLE if does not contain any smaller rectangle anywhere inside the rectangle.

COMPOUND solutions contain at least one smaller rectangle.

Any solution containing adjacent Elements is always Compound, but a complicated Squared-Rectangle made into a larger one by the addition of further Elements round two or more sides is also Compound.

A5.12. PERFECT AND IMPERFECT RECTANGLES

A rectangle is termed PERFECT if every Element is a different size.

In an IMPERFECT solution at least one of the Elements is duplicated at least once. Some Imperfect solutions have heavy duplication and are regarded as highly Imperfect whilst many have one duplicated Element only.

A5.13. VALID AND INVALID RECTANGLES

See later for full explanation. An Invalid solution is either one which contains zero Elements or contains at least one Element which has another Element of the same size immediately adjacent to it.

INVALID solutions are inferior to VALID solutions, and always Imperfect. @A4.7

A5.14. SYMMETRIC RECTANGLES

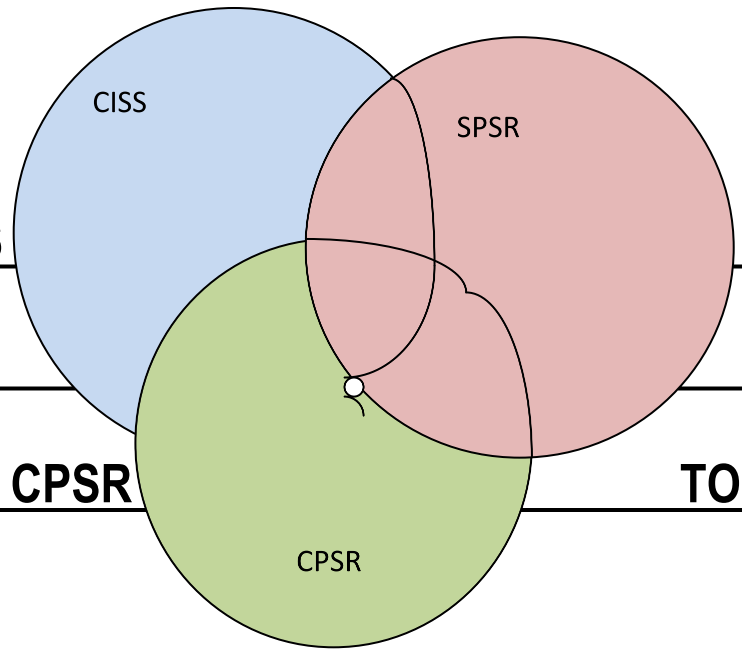
These are solutions which when divided into halves or quarters each half or quarter is identical.

The solution may or may not have a central Element which is common to each half or quarter.

Solutions may have 2-fold or 4-fold symmetry. In the 2-fold cases, the symmetry may be either vertical or diagonal (the rectangle divides II or x through the middle). Such solutions are always Imperfect by definition. @E9.10

A5.15. DIVISION OF MAIN GROUPS

DIFFERENT TYPES OF CONSTRUCTIONS SHOWN GRAPHICALLY



CISS — **SPSR** **C=COMPOUND I = IMPERFECT P = PERFECT**

CPSR & SPSS — **SR = SQUARED-RECTANGLE SS = SQUARED-SQUARE**

CPSR — **TO BE FINISHED**

A5.16. BLIND & WORKED RECTANGLES

BLIND Rectangles are a pattern of Elements drawn at random without being calculated out by Algebra.

WORKED Rectangles are those with Elements calculated .

It is possible to ascertain some information from BLIND Rectangles. See later for explanation of 1 to 5

1. Trails can be drawn and the quantity of them ascertained. See later for meaning of Trails.
2. Elements can be found for possible values of x, y z etc by inspection. Usually they can be found within a single trail.
3. Where any single Trial crosses any Element twice, the value of that Element is always an Even number.
4. In some cases each Element can be shown as an Odd value or an Even value.
5. If there is a Pentad or Octad End then the corresponding Dimension will be divisible by 4 or 15 respectively.

A6. MORE SQUARED-RECTANGLE TERMS

A6.1. CORNEX, SIDEX AND CENTREX RECTANGLES

The largest Element in a Squared-Rectangle is often situated in one of the corners - CORNEX, or it may border a side - SIDEX.

Less frequently the largest Element is internal (not touching any side) - CENTREX. @A3.1 @C3

A6.2. INNER-OUTER ELEMENTS RATIO

This is the ratio obtained by comparing the amount of external Elements with the amount of internal ones, and always shown in this form - 6:7.

A6.3. RATIO

RATIO unless otherwise defined, means the ratio obtained when the largest Element value is divided by the smallest Element value, in the Squared-Rectangle and is shown to two decimal places, e.g. 6.17

A6.4. UNKNOWNS - x y z a b.. ETC.

In using algebra to calculate the relative positive values of the Elements in a Squared-Rectangle, at least two UNKNOWNS need to be employed, and sometimes more.

An xy solution uses two Unknowns, xyz three Unknowns, xyza four Unknowns, xyzab five Unknowns and so on.. This book refers to xy solutions and the like. @C1.1 @C3.4

A6.5. DIVIDING LINES

These are defined later.

A6.6. GULF LINES

These are defined later.

A6.7. REDUCTION INDEX

When calculating a rectangle using algebra, a set of Element values is obtained. Often all the Elements have a common factor (for example all may divide by 3 and remain integers) in which case the rectangle can be shown with reduced Elements. Putting the same idea differently, the values for x, y etc. may have a common factor e.g. x = 12 and y = 24 cancels down to x = 1 and y = 2, giving a Reduction Index of 12. This common factor is termed the REDUCTION INDEX. It is a positive integer often 1 or 2 but can be a huge number.

A6.8. FULL DIMENSIONS

These are the numerical horizontal and vertical values of the rectangle after it has been calculated and before any possible reduction is made. These are shown in the form 368 x 255. @F2.4

A6.9. REDUCED DIMENSIONS

These are the numerical horizontal and vertical values of the rectangle after any possible reduction has been made. In other words, each Element in the full rectangle has been divided by the Reduction Index. Where the Reduction Index is 1 the Full and Reduced dimensions are the same.

A6.10. ELONGATION

This is the ratio of the shorter side divided by the longer side and is expressed as a percentage to two decimal places.

The maximum percentage of 100% occurs when the solution is a square.

Within any given Order a wide range of Elongation exists (e.g. 46.60% to 100.00% for Order 13). As the Order increases so does the range of possible Elongations.

A6.11. TWO-BY-ONE RECTANGLES

These are Rectangles exactly twice as horizontal as vertical (elongation of 50.00%) . Similarly Three-by-One rectangles are possible but very hard to find! Even up to 12 by 1 have been found. @J

A6.12. SEMI-PERIMETER

This is the numerical value of the horizontal side plus vertical side.

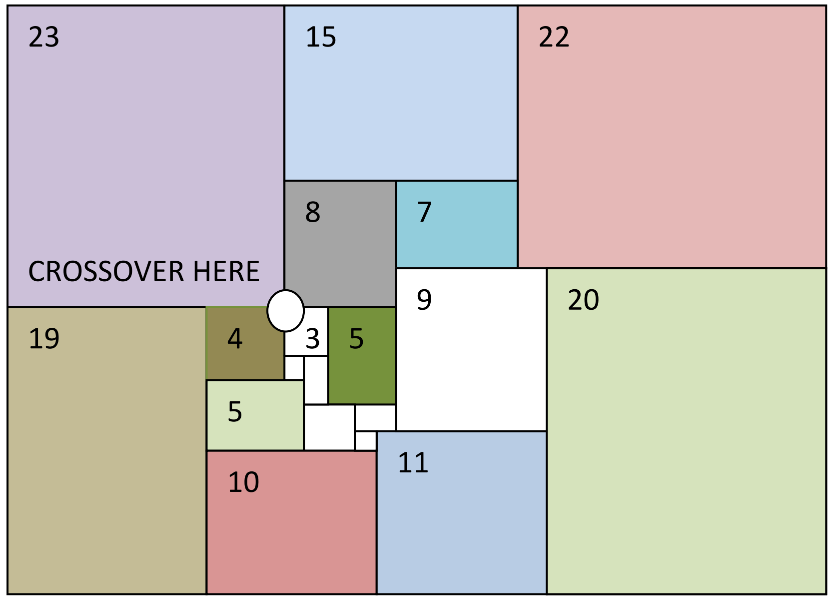
This is important in the theory of this subject.

Note that the Semi-Perimeter relates to the full, not reduced dimensions. @F2

A6.13. CROSSOVER POINTS

A CROSSOVER point occurs when a rectangle has four Elements all meeting at a single point.

On the Internet it is described as a CROSS. This occurs in practice, but not in theory, in all Invalid solutions. However Crossovers are more interesting in Valid and Simple solutions, where they are relatively unusual.



[19] 60 x 42 R270 SHOWING A CROSSOVER (CIRCLED)

A6.14. USE OF DISTORTED DIAGRAMS IN RECTANGLES (DISTORTION)

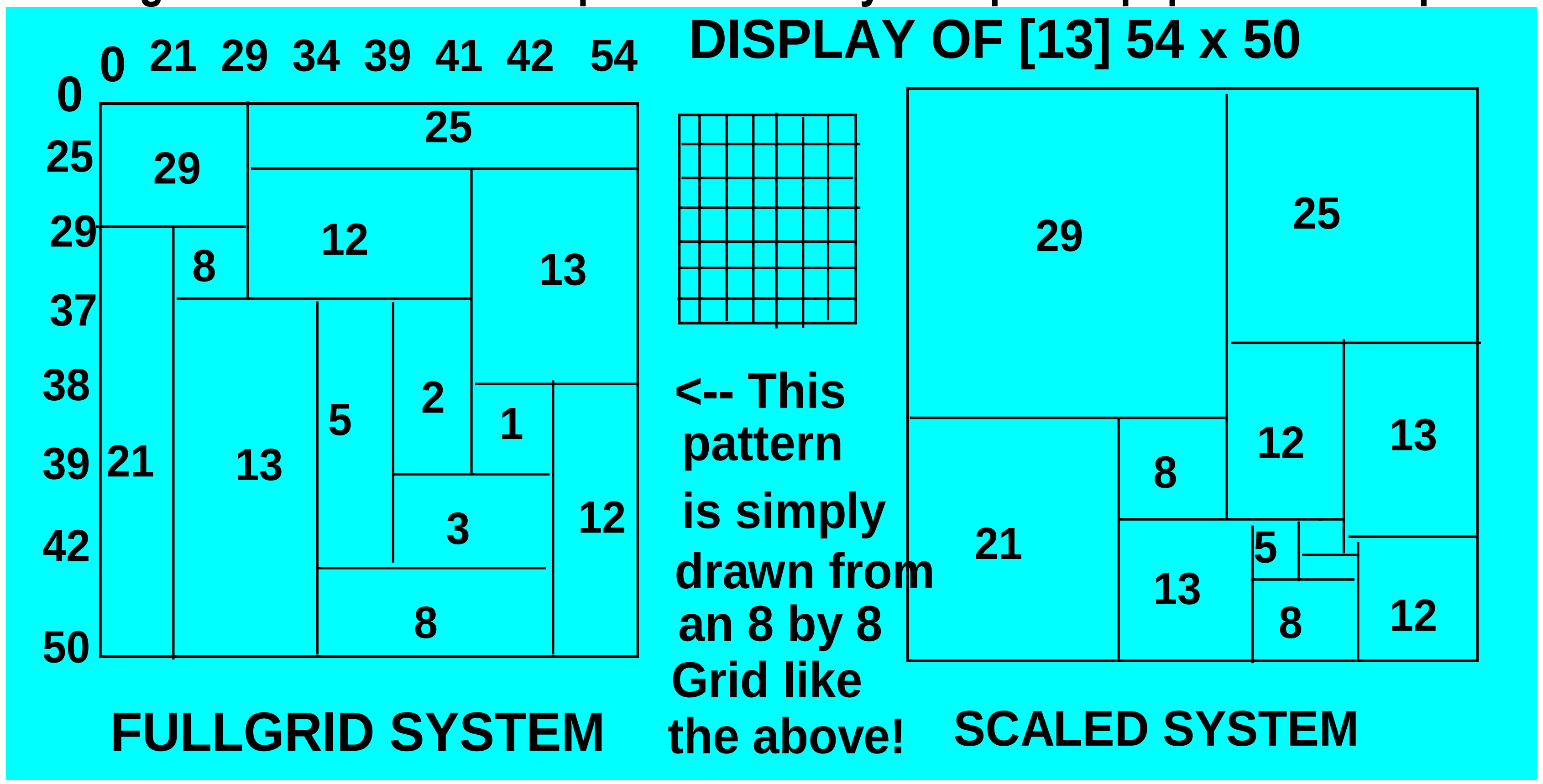
There are several advantages in purposely not drawing Squared-Rectangles to scale, and showing Elements which are really square as long oblongs:-

1. Small Elements such as 1 or 2 can become minute and difficult to insert numerical values. See above.
2. Zero Elements can only be sensibly shown by using distortion.
3. Distorted diagrams serve to show the theoretic patterns for Invalid solutions.
4. To demonstrate various fixed patterns or endings even though negative or zero Elements may arise.
5. Most important of all, many rectangles can be drawn in a suitable and readable way by using Distortion. Often it enables a rectangle to be drawn smaller and still be easier to read.

Drawing squares as very elongated oblongs does not invalidate the rectangle, but can be very useful.

A6.14.1. GRID DRAWINGS FOR SQUARED-RECTANGLES

The following system of Distortion (found by me, but also by others on the Internet) is a most successful way of displaying Rectangles in a fashion that smaller Elements appear bigger and the larger ones smaller than normal. Another advantage is that the pattern for any given Rectangle is fixed and can be produced easily on squared paper. An example is shown:-



+29 +25 TEXTUAL DRAWING. A neat way of producing a visible Squared-Rectangle using text but no
.12 .13 geometry. Here the corners are preceded with +, with - and internal Elements with .
+21.8 Note the fixed columns (7) as well as rows (7).
-13 .5 .2. In any given Rectangle the number of Columns + of Rows = the Order + 1.
.1 +12
.3
_________.8___________

In the GRID DRAWING, a set of horizontal numbers - top to bottom - and vertical - left to right - are shown equidistantly. To enter Element 29 the lines intersect at 29 across and 29 down. To Enter 25 the lines intersect at 54 across (29 + 25) and 25 down, and so on. In some Solutions the matrix will be much more elongated than in this Example.

<u>1. With all numbers omitted,</u> I have no difficulty reducing the size to just half an inch square - but just imagine if the Scale Drawing was reduced to this size! The Element 1 would be .01 inch square!

<u>2. With only the outside numbers</u> - the Coordinates - showing the Rectangle can be drawn in just 1.5 x 1.5 inches in small handwriting. In small print 1” square is easily possible.

<u>3. With Coordinates and Elements shown</u> - this solution can still be printed just 1.5 x 1.5 inches.

Why all this concern about small size? It is to solve the problem of how to store many thousands of Squared-Rectangle results graphically rather than by a listing of numbers, without taking up a huge amount of space.

I have thousands of Solutions drawn on paper 6x 5½ inches - showing only one at a time, and the data is still cramped. And the paper is over 6 feet thick!

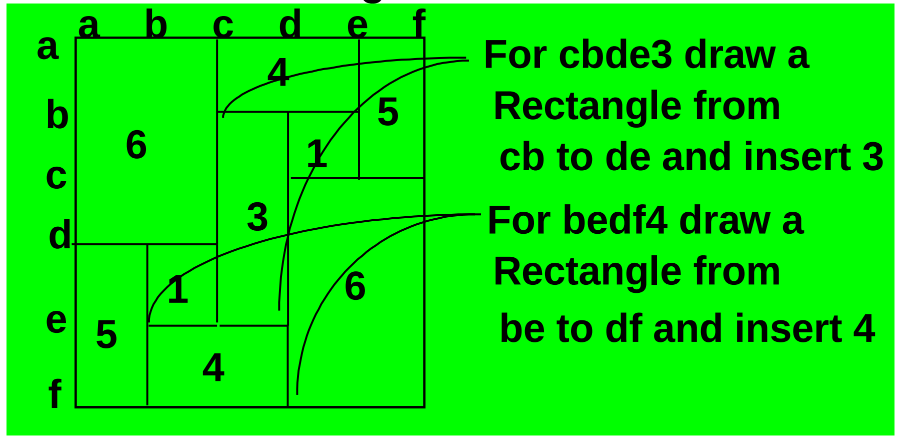
It is useful to note under 2. Coordinates and Pattern only, that the Elements can be easily calculated from this data.

The Vertical Intervals may be one amount and the Horizontal another. Thus another benefit of this system is that Rectangles can be presented in a predetermined size on a Computer screen - regardless of the Order or Dimensions involved.

A6.14.2. RECORDING GRID DRAWINGS TO ENABLE RECTANGLES TO BE DRAWN

I eventually found a good and easy way of writing down SR's in such a way that the SR can be drawn easily, although clear that many squares would need to be distorted into rectangles.

But by using the Grid and adding some code letters before each Element it is possible. a=1 b=2 c=3... with across before down. Thus aacd6 is a rectangle drawn from aa to cd in the Grid as Top Left to Bottom Right, with 6 inserted. See Below.



<u>NOTE: -</u> The above Solution could be recorded as briefly as <u>"[9[15 x 11 a6c4e5:c3d1:d6:a5b1:b4"</u> a USEFUL FORMULA. The ACROSS coordinates do not need to be included since the first group (6 4 & 5) will always be coordinate. A & : will then denote b the next : c and so on! Using FOUR letters enables any Element to be drawn in easily - and independent of all other Elements. Thus the Elements can be listed in any pattern we please, e.g. they could be ascending as 1 1 3 4 4 5 5 6 6 or as calculated

1 3 4 5 6 4 1 6 5. Or Top Down Formula 6 4 5 3 1 6 5 1 4 etc.

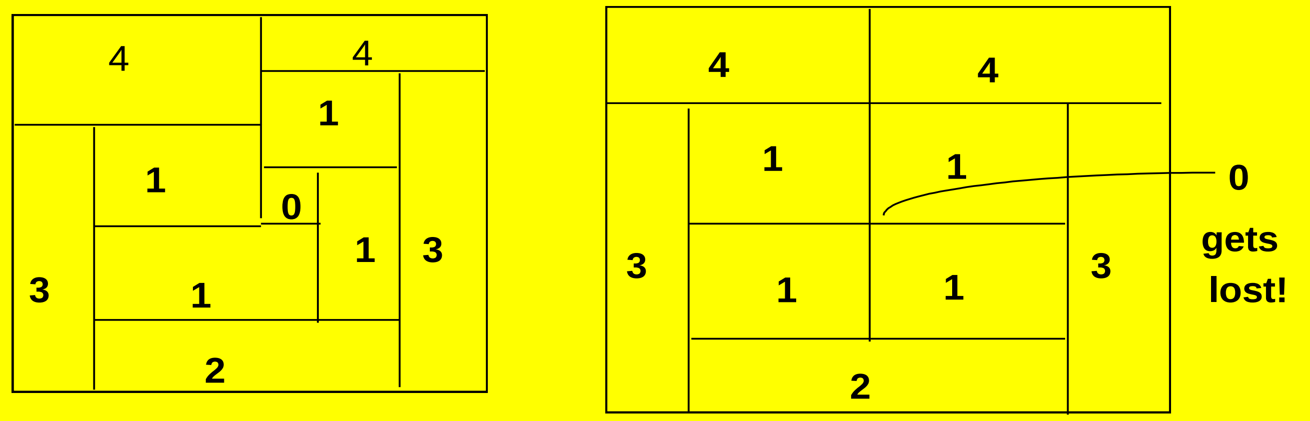
A6.14.3. TWO TYPES OF GRID DRAWINGS HAVE TO BE CONSIDERED

The use of Grid Drawings is an excellent way of displaying SR's conveniently and compactly. However once they are looked at in detail certain types of Rectangles present a slight challenge, as the Grid Drawings can be drawn in two distinct ways.

1. Although Squares could be displayed two ways - horizontal or vertical, this problem is overcome if we use using the higher series of numbers along the top. E.g. 100 59 42.. is greater than 100 45 61.. or 100 59 36 .. If this distinction is not made we have two Grid Patterns for the same Square which is unhelpful.

2. Zero Solutions. Though it is better to show the Zero Elements rather than omit them, it means a particular type of Grid Drawing is necessary.
3. Invalid Solutions.
4. Solutions containing Slides
5. Solutions containing a Crossover.
6. |-----| Type Symmetric Solutions which by their construction contain Slides.
Consider the random drawing of a Squared-Rectangle pattern. Often the actual calculated Rectangle
1. Will be the same shape (not requiring any modification)
2. Not contain any zeros
3. Contain any negative quantities
4. No Slides
5. No Crossovers.
If so the Grid Pattern is fixed, easily determined with only one unmistakable format possible.
But in other cases we then have a choice of using the FULLGRID or the CLOSEDGRID Pattern.

e.g. [10] 8 x 6 INVALID solution -



FULLGRID SYSTEM CLOSEDGRID SYSTEM

The first shows 7 horizontal & 6 vertical lines, and has the advantage of showing the original pattern, but the lines appear illogical e.g. the two 4's finish on different levels as if different - but are the same!

The second is misleading as it shows just 5 horizontal & 5 vertical lines, and the Element 0 disappears. But it does show what we may argue, the correct positioning of the Lines. This has 2 Crossovers (no Slides).

Which is better? The FULLGRID SYSTEM is better one to use for INVALID SOLUTIONS by which we mean
1. Those where Zero Elements occur.
and/or 2. Those where Adjacent Elements occur.
This is the ***Theoretical*** one, whereas the CLOSEDGRID SYSTEM which is the ***Practical*** one!

In the case of VALID SOLUTIONS the CLOSEDGRID SYSTEM is probably better as it is simpler, smaller and easier on the eye.
In my catalogue SRSMALL I have used the CLOSEDGRID SYSTEM but shown dots to indicate where

1. Crossovers occur
2. Where Slides occur.

A6.14.4. DISPLAYING GRID SYSTEMS - 2 TYPES OF HORIZONTAL SPACING

We saw Above a Solution produced with the Vertical Columns evenly spaced. e.g. |21______|
|10 |6 |5 | etc...

1. EVEN SPACING
If you drew Solutions in a squared exercise book you would have to use this system..
But a problem with this is that either all Element Numbers will need to be shrunk (if they are to be the same size), or vary in size with larger numbers being cramped sometimes in small spaces!
Probably better is to have e.g.
|21___|
|10|6|5| etc. ... with the Elements dictating the width of the Columns and keeping the same size.
Below shows a more compacted system spaced according to width of Elements.
2. NUMBER SPACING CLOSEDGRID NUMBER SYSTEM

```
________________________
|106  |96______________|
|     |20|12|7___|9 |48|
|     |  |  |5 |2|__|  |
|_____|  |  |  |11  |  |
|96|10|  |__|__|    |  |
|..|__|__|8|9..|____|..|
|  |38     |___|3_|8|  |
|  |       |7|2|__| |  |
|  |       |_|5___|_|  |
|  |       |20____ _|__|
|  |_______|10__|58    |
|__|48__________|______|
```

This is probably the Best Possible way to show Solutions. (I say ‘probably’ since it depends how you wish to use and display the solutions).

A6.15. QUANTITY OF HORIZONTAL & VERTICAL LINES FOR EACH ORDER

In the Above Example the Distorted Pattern showed 7 horizontal & 6 vertical Lines totaling 13, three more than the Order. This is also true for any Order, so all Order 22 Solutions have a total 25 Lines, providing the following adjustments are made: <u>Lines = Order + 3</u>
1. The right and bottom border Lines are included. <u>Otherwise Lines = Order + 1</u>

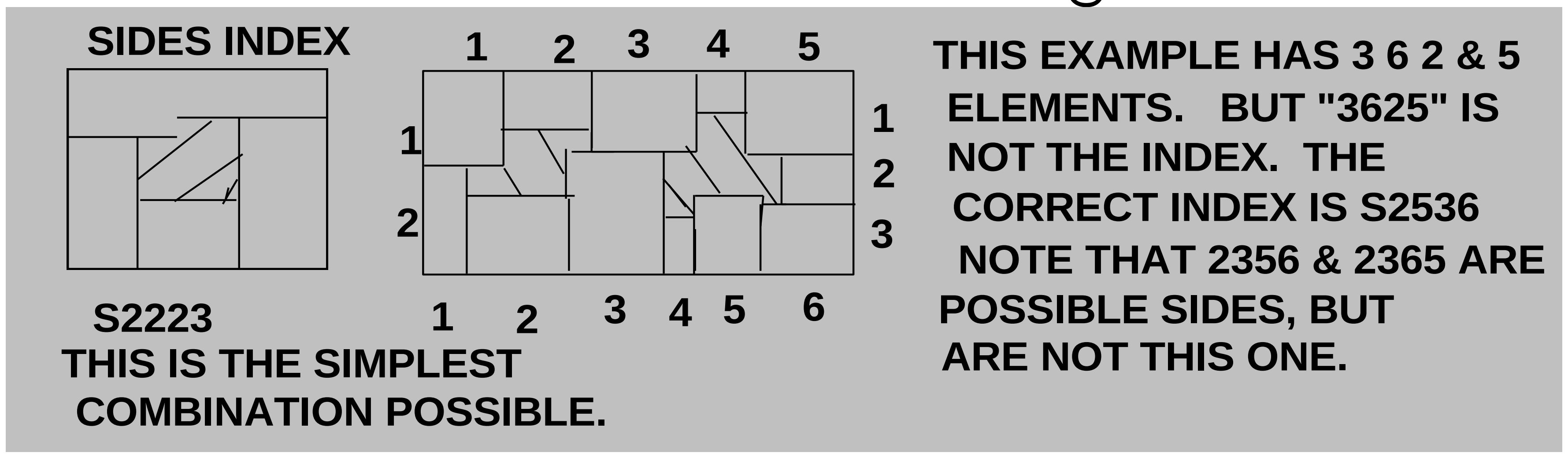
2. Add one Line for any and every Slide.
3. Add two Lines for any and every Crossover. NB. A Crossover will always occur whenever an Element 0 occurs.

A6.16. SIDES FORMULA

Rectangles may be usefully classed according to the numbers of Elements bordering each side as shown below.

The formula involved is preceded by S thus S2223 is a rectangle with 2,2,2 and 3 Elements.

Whether the numbers are taken clockwise or not depends on the rectangle concerned, but that combination which produces the lowest possible number is the one used- e.g. S3222, S2322 and S2232 are shown as S2223. S2253 as S2235. S3232 as S2323. 2433 as S2334 - note this is not the same as S2343. S3453 as S3345 and so on. See @E1.



A6.17. MINIMUM ORDERS FOR EACH SIDES FORMULA

Obviously the larger the Order, the larger the range of Side Indices possible. But low Orders such as 7 to 10 have a restricted range. If we look at [9] 15 x 11 (above) which has sides S2323 we can produce solutions of S2424, S2525, S2626 simply by adding Triads repeatedly to one (or either) end. This raises the Order by three each time -

1. S2323 Order 9, S2424 Order 12, S2525 Order 15, S2626 Order 18, S2727 Order 21, S2828 for Order 24... and these represent the *smallest* Orders possible for these Sides, all of them VALID.

2. For S2223 there is an Invalid Solution [7] 8 x 7 but the smallest Valid Solution is [9] 69 x 61. For S2224 there is an Invalid Solution [9] 6 x 5 but the smallest Valid Solution is [10] and adding Triads to this we get INVALID SERIES S2223 Order 7, S2224 Order 9, S2225 Order 12, S2226 Order 15, S2227 Order 18 ...
but VALID SERIES S2223 Order 9, S2224 Order 12, S2225 Order 15, S2226 Order 18, S2227 Order 21 ...
Exploiting the idea of adding Triad repeatedly we can easily work out the series -

2A. S2223 Order 7 (INVALID), S2324 Order 10 (VALID), S2425 Order 13, S2526 Order 16, S2627 and so on (after 7 all are Valid)...

3. S2233 Order 9 (VALID), S2334 Order 12. S2435 Order 15, S2536 Order 18, S2637 Order 21 and so on...

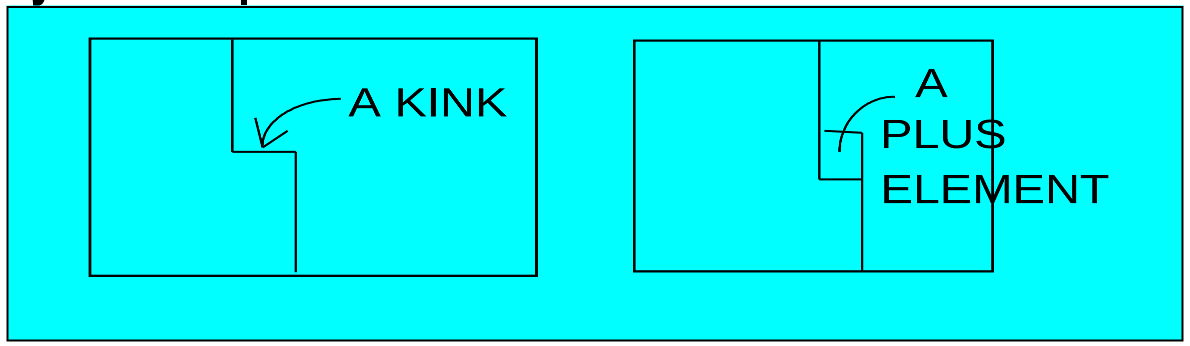
4. INVALID SERIES - S2333 Order 9, S2334 Order 12, S2435 Order 15, S2536 Order 18, S2637 Order 21, S2738 Order 24 ... Following table needs serious checking! y means verified. Where Two numbers, first is for Invalid

Side	3	4	5	6	7	8	Side	3	4	5	6	7	8
222	7y 9y	9y 10y	11?	14i	15	18	253	14	15	17	18y	20	22
223	9y	11y	13y	15y	17y	19y	254	16	17	19?	20?	22	24
224	11	13y	15y	17y	19y	21	255	18	19	21	23	25	27
225	13	15	17	19	21	23	256	20	21?	23	25	27	29
226	15	17	19	21	23	25	257	22	23	25	27	29	31
227	17	19	21	23	25	27	262	14	15	17	18	20	22
232	8y 9y	10	12	14	16	18	263	16	17	19	20?	21i 22?	24
233	11	12	14	16	18	20	264	18	19	21			
234	13	14	16	18	20	22	265	20	21?	23	25	27	29
235	15	16	18	20	22	24	266	22	23?	25	27	29	31
236	17	18	20	22	24	26	267	24	25?	27	29	31	33
237	19	20	22	24	26	28	272	16					
242	10	12	13	15	17	19	273	18					24i
243	12	14	15	17	19	21	274	20					
244	14	16	17	19	21	23	275	22					
245	16	18	19	21	23	25	276	24					
246	18	20	21?	23	25	27	277	26					
247	20	22	23?	25	27	29							
252	12	13	15	17	18 OK	20							
Side	3	4	5	6	7	8							
333	12 OK	14	16	18	20	22							
334	14	16	18	20	22	24							
335	16	18	20	22	24	26							
336	18	20	22	24	26	28							
343	14	16	17?	20?	22?	24							
344	16	18	19										
345	18	20	21										
346	20	22?	23?										
353	16	17	19	21?									

354	18	19	21?											
355	20	21	23											
363	18	20	22?	24	26	28								
364	20	22	24?	26	28	30								
444	18	20	22											

A6.18. KINKS AND PLUS ELEMENTS

In many solutions it is possible to divide them into two sectors with just three lines. The internal line which may face left or right (or top or down) give rise to a Kink. Kinks may occur several times in a solution, just once or not at all.
Any Kink will either contain an Element as in below 2 or it won't. Where it exists it is termed a PLUS Element - particularly as part of a symmetric pattern.



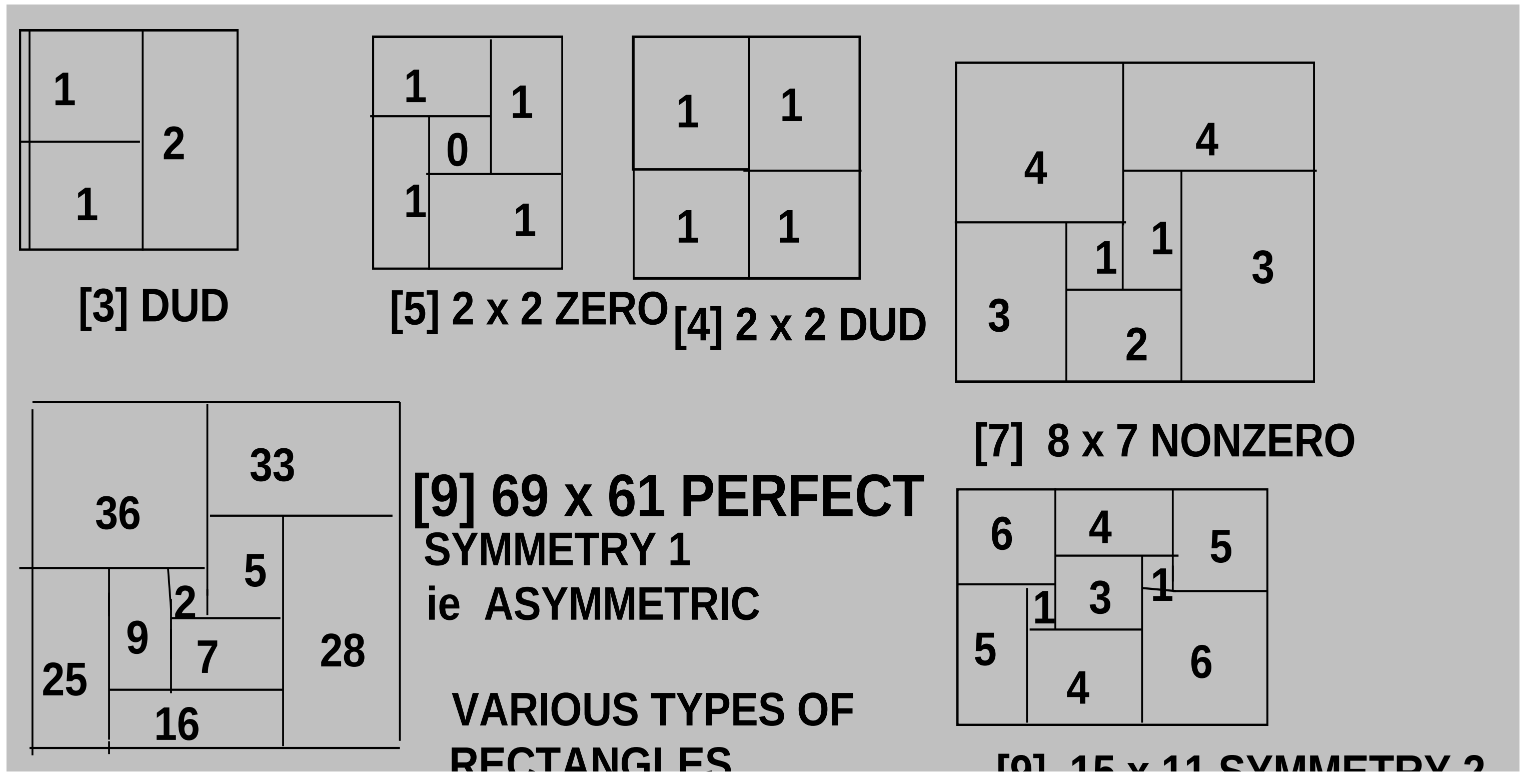
A7. GROUPING OF SQUARED-RECTANGLES

To show some of the definitions this solution below left is a Rectangle Order 9 with 9 separate values.
It is VALID that is, without zero Elements or adjacent Elements.
Its REDUCED DIMENSIONS are 69 x 61 the same as the Full Dimensions.
It is PERFECT with no duplication of any Elements.
Full size happens to be 69x 61 and the REDUCTION INDEX 1.
This example is SIMPLE as it contains no smaller rectangles.
It is an xy solution - can be calculated with a minimum of two UNKNOWNS.
It is CORNEX with largest Element 36 in corner.

It is not SYMMETRIC .

It has no CROSSOVER points.

The SEMI-PERIMETER is 130 or 69 + 61.



The following definitions appear complicated but are readily explained by actual examples.

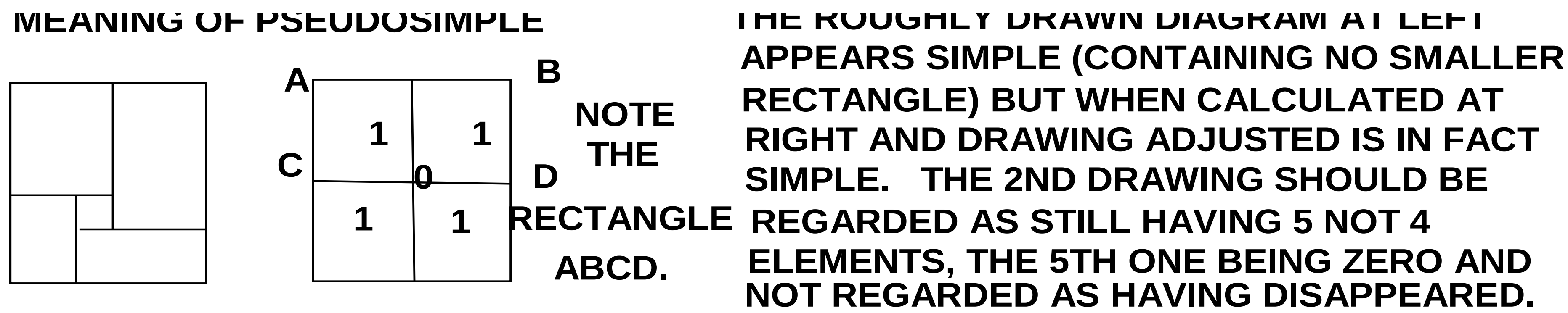
Most of the items also apply to Squared-squares, but see that section for a full description of Squared-square types.

A7.1. SIMPLE OR COMPOUND

All Squared-Rectangles which contain one or more smaller rectangles within it, however large or small they are, are termed Compound. So any rectangle with a single Element on one side is always Compound, and this means any Simple rectangle must have at least two Elements bordering each side.

If an Element 132 can be added to the solution above the result is Compound.

(A7.1)



A7.2. PSEUDO-SIMPLE

These are solutions which appear Simple when drawn from a rough diagram but in practice turn out to be Compound. The resultant rectangles are either Zero or Non-zero. See Above.

A7.3. DUDS

These are the least desirable rectangles of all and except for theoretical purposes are really beyond the scope of this book. An example is a box of 4 ones. They are all Compound.

A7.4. REPEATERS

These are otherwise normal rectangles but at least one corner has 2 or more repeated adjacent Elements which makes them Compound.

A7.5. SINGLE ENDED SOLUTIONS TERMED SINGLENDS

These look normal rectangles but have one or two Elements completely bordering the end(s) which makes them Compound. Unacceptable as proper solutions, they are useful in studying theory.

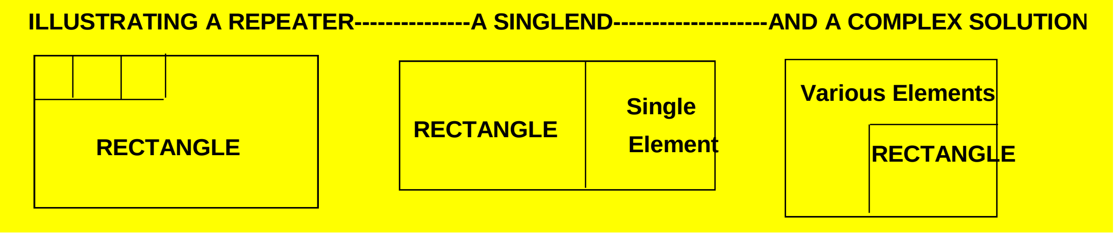
A7.6. COMPLEX

Where a rectangle contains at least one normal Squared-Rectangle within it, it is termed Complex.
They are Compound (but not Pseudo-simple).

A7.7. VALID OR INVALID

All solutions are one or the other. Any rectangles containing a zero Element are termed Invalid. Also all solutions with Elements of the same size which are exactly adjacent are also regarded as Invalid.

Although some solutions seem of little interest, they need to be considered in the general theory. See A3.4.



A7.8. PERFECT OR IMPERFECT

All solutions are one or the other. A solution is called PERFECT if every Element differs in size.

Otherwise the rectangle is called IMPERFECT. In the case of Imperfect the duplication may vary from one to many Elements. It follows that all Invalid solutions are also Imperfect. Compound solutions may be Perfect or Imperfect.

Most of this book is concerned with this best group.

A7.9. ZERO AND NON-ZERO

All solutions containing at least one zero Element are termed ZERO. Other Invalid solutions which would have adjacent Elements if the solution was drawn to scale but not containing zero Elements are termed NON-ZERO.

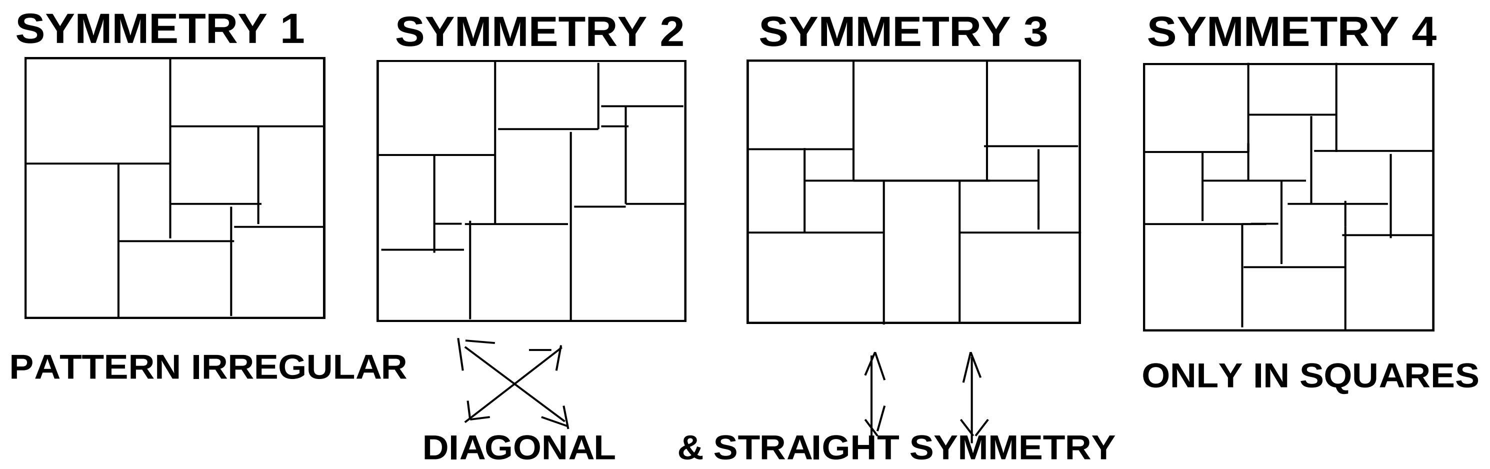
These solutions are both Invalid and Pseudo-simple.

A7.10. SYMMETRY 1, 2, 3 & 4

All solutions are one of these. A rectangle has SYMMETRY when half the rectangle has the same pattern as the other half. The two halves fit together, but if the Order is an odd number like 13 an unduplicated extra central Element is present.

Where the Order is an even number all Elements are duplicated. Note the difference in SYMMETRY 2 and SYMMETRY 3 below. SYMMETRY 3 is more common and has diagonal symmetry. Symmetry 4 only applies to squared-squares.

All solutions not Symmetry 2 3 or 4 are Asymmetrical and are in effect Symmetry 1. All symmetries include rectangles which are Invalid and Compound, but especially Symmetry 3. Of course Symmetry 2-4 solutions are always



Imperfect.

A8. RECTANGLES LEAGUE STATUS

A8.1. LEAGUE

Some Squared-Rectangles are clearly more interesting and satisfactory than others.
The types may be graded in status as follows within three separated groups:-
(A8.1)

Lower the number the higher the status of the rectangle	Status diagram	
1. Perfect	1. Complex Perfect	1. Square symmetry 1
2. Imperfect	2. Complex Imperfect	2. Square symmetry 2

3. Non-zero	3. Complex Non-zero	3 Square symmetry 4
4. Zero	4. Complex Zero	4. Symmetry 1
5. Singlend		5. Symmetry 2
6. Repeater		
7. Duds		

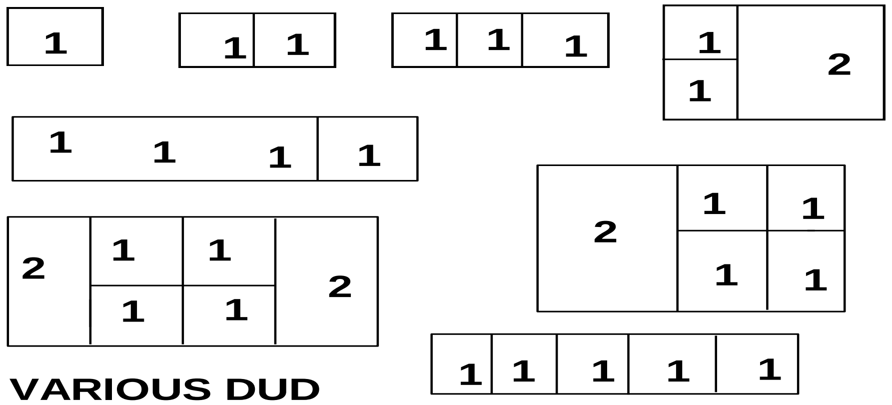
A8.2. ALGEBRAIC NOTATION

Not yet settled -
Centex j Sidex k Cornex l
Full dimensions f**x** Reduced dimensions
Semi-perimeter SP
Higher dimension m Lower dimension n (m x n)
Reduction Index r**
Elongation e1 (square) e2 (Two by One Rectangle) e3 (etc.)
Single a Duplicated b c d etc. (following reduced dimensions
p Mid-plus m Mid a b Reciprocal Pairs t “Twins”
Crossover c
Unknowns (x, y, z, a, b …) u2 u3 u4 … or xy xyz etc.
Sides S2223 S2223-3 followed by quantity of internal Elements
Smith Diagram- Ordinary, Box, y type
Poles, Terminals, Potential, Complexity Rotor Stators

A9. NUMBERS OF SOLUTIONS EXISTING TO ORDER 18 AND BEYOND

A9.1. ORDERS 1 2 3 4 & 5

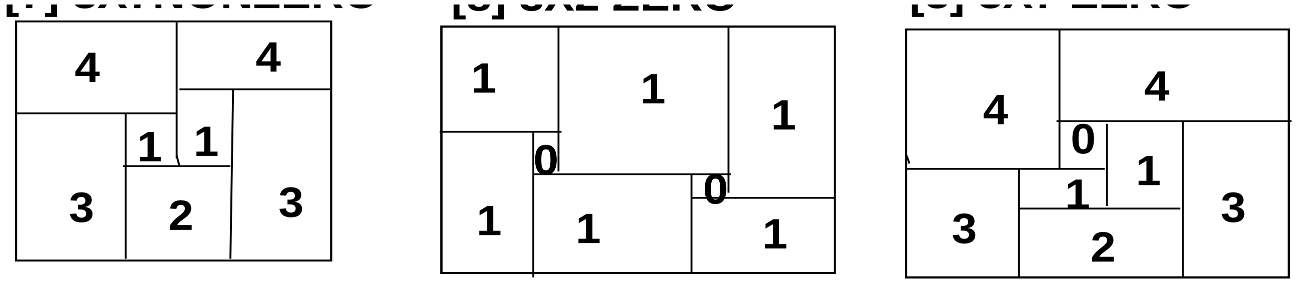
The only solutions are dud or Invalid types of little interest. These are shown in passing.
All solutions are dud except for one Invalid [5] 2x2 with sides 2222.

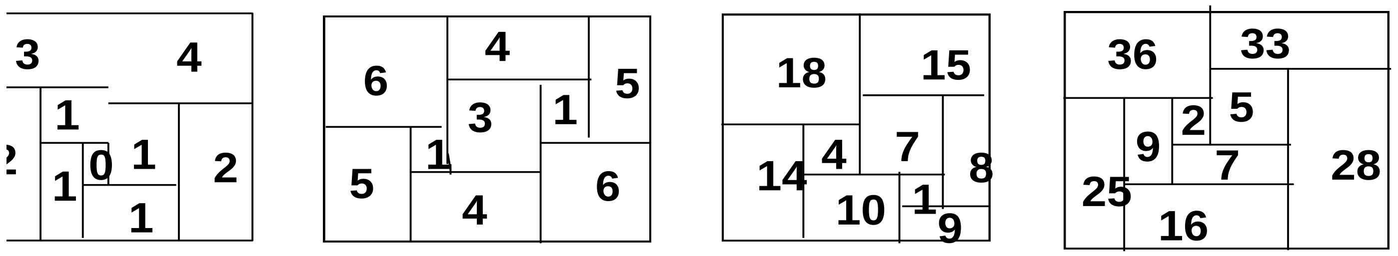


VARIOUS DUD SOLUTIONS ORDERS 1 TO 5.

A9.2. ORDERS 6 7 & 8

Apart from Duds not shown there are only 3 which are [7] 8 x 7 non-zero [8] 3 x 2 zero and [8] 8 x 7 zero. None are Valid





| 8X6 ZERO [9] 15X11 SYM 1 [9] 33X32 PERFECT [9] 69X61

T IS NECESSARY TO MAINTAIN THE ORIGINAL PATTERNS! THIS IS WHY ADJACENT NES & FOURS ARE SHOWN AS IF THEY ARE DIFFERENT IN SIZE.)

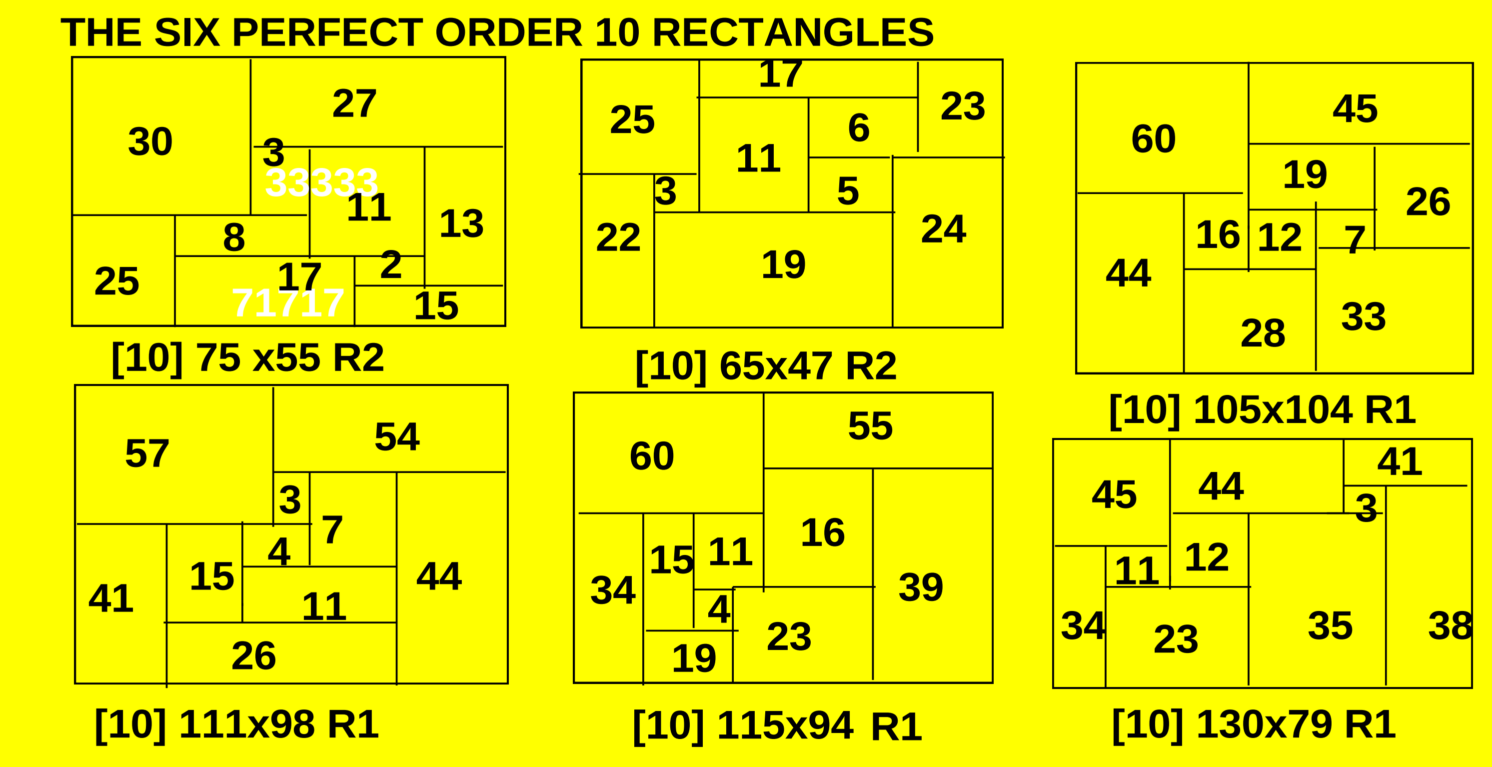
A9.3. ORDER 9

This is the smallest Order where Perfect and Imperfect rectangles exist. Ignoring Duds there are 4 solutions: [9] 6 x 5 Zero [9] 15 x 11 Imperfect symmetry 2 [9] 33 x 32 Perfect and [9] 69 x 61 Perfect. See Above 4-7. 3 Valid

A9.4. ORDER 10

There are 5 zero or non-zero solutions which are [10] 6 x 5, 8 x 6, 10 x 9, 11 x 8 and 30 x 26 all are Invalid.

There are no Imperfect solutions.
There are 6 Perfect solutions. [10] 57x 55, 65 x 47, 105 x 104, 111 x 98, 115 x 94 and 130 x 79 all Shown Below.
6 Valid.



A9.5. ORDER 11 UPWARDS

There are 11 Zero or Non-zero solutions, but oddly no Imperfect solutions.
There are 22 Perfect solutions, all catalogued by me. Total Valid, 22

ORDER 12

The amount of solutions accelerates and the proportion of low grade rectangles reduces.
There are 18 Invalid solutions (Non-zero and Zero). 9 Imperfect solutions and 67 Perfect solutions. I have worked out all.
Total Valid, 76

ORDER 13

51 Invalid solutions, 34 Imperfect and 213 Perfect solutions. Total Valid, 247
I have catalogued all these.

ORDER 14

Found on an IBM 650 Computer were 104 Imperfect and 744 Perfect solutions. Total Valid, 848

ORDER 15

Found on an IBM 650 Computer were 283 Imperfect and 2,609 Perfect solutions. Total Valid, 2,892

ORDER 16 TO 25 PERFECT SOLUTIONS

Found and shown on the Internet to year 2011.

Order 16,	957 Imperfect	9,016	Perfect solutions	Total Valid, 9,973
Order 17,	3,033 Imperfect	31,426	Perfect solutions	Total Valid, 35,259
Order 18,	9,494 Imperfect	110,381	Perfect solutions	Total Valid, 119,875
Order 19,	30,301 Imperfect	390,223	Perfect solutions	
Order 20,	98,889 Imperfect	1,383,905	Perfect solutions	
Order 21,		4,931,307	Perfect Solutions	
Order 22,		17,633,469	Perfect Solutions	
Order 23,		63,301,415	Perfect Solutions	
Order 24,		228,130,900	Perfect Solutions	
Order 25,		825,000,000	Perfect (Approx.)	

These figures suggest there may be roughly three times more solutions for each increase in Order.

In studying the methods of creating rectangles it is obvious that each higher Order has far more solutions than the one before. (Refer to Section on Adding and Diminishing)

Amazingly for any Order over 6, there are far more solutions for that Order than the total for all lower Orders!

A9.6 ORDERS 20 TO 47 - QUANTITY ACTUALS & ESTIMATES

Using advanced computers, the total quantities of Perfect & Imperfect Solutions has been calculated to Order 35!
A Formula has been devised connecting these values has been found (which includes the square root of pi!) and estimated quantities have been calculated to Order 47 (and more) as shown.
NB This formula is not absolute - it is an approximation.

Order	Quantity of all Valid Solutions (Perfect & Imperfect)	
20	1,427,065	
21	5,052,780	about 3.54 x Order 20
22	17,992,102	
23	64,398,982	
24	231,595,693	about 3.60 x Order 23
25	836,505,020	
26	3,033,508.350	
27	11,041,527,171	
28	40,327,701,410	
29	147,762,266,421	
30	543,019,459,457	about 3.68 x Order 29
31	2,001,125,317,966	

32	7,393,729,651,680
33	27,385,057,313,051
34	101,662,597,313,051
35	378,223,383,714,871
36	1,410,010,107,776,020
37	5,266,646,375,738,230 about 3.73 x Order 36

38 19,707,860,022,106,100
39 73,874,904,808,387,400
40 277,375,734,235,367,000
41 1,043,082,820,594,150,000
42 3,928,395,734,235,367,400
43 14,815,874,077,269,400,000
44 55,953,430,375,235,800,000
45 211,586,091,106,156,000,000
46 801,094,614,153,181,000,000
47 3,036,643,000,848,710,000,000 about 3.79 x Order 46 Approx. 3000 MMM!

A9.7. SQUARED-SQUARES QUANTITY COMPARED TO SQUARED-SQUARES

As we might guess, there are much fewer Squared-Squares than Squared-Rectangles, and ignoring Order 1 and Invalid types there are no occurrences to and including Order 12, or of Order 14.

But Squares exist for Order 13 and for all Orders from 15 upwards, in rapidly increasing number.

There is ignoring Invalid solutions, one for Order 13, 3 for Order 15, 5 for Order 16, 15 for Order 17, 19 for Order 19 and 58 for Order 20.

Not one of these 101 solutions however are Perfect as all have at least one Element with size duplication.

For some unclear reason the quantities are relatively more for Odd Orders than Even Orders.

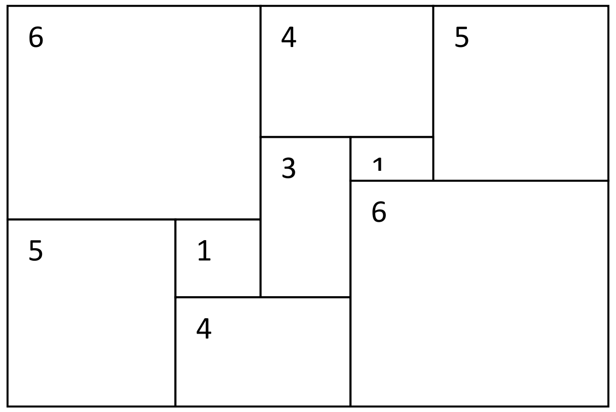
A10. NEGATIVE OR ZERO ELEMENTS IN SOLUTIONS

A10.1. RANGE OF INTEGRAL VALUES IN SOLUTIONS

This Book is concerned with Elements which are positive integers and when solutions arise which include fractions, they are multiplied up to be integers.

Odd internal negative Elements are easily converted into positive by adjustments of position.

The full size of the Solution below is (30,20,25)(15,5)(30)(25,5)(20) which reduces to (6,4,5),(3,1),(6),(5,1),(4). But note that (-6,-4,-5)(-3,-1)(-6)(-5,-1)(4) or even (0,0,0)(0,0),(0)(0,0),(0) would theoretically satisfy this pattern.



A10.2. A RECTANGLE WITH INTEGRAL SIDES MAY HAVE NON-INTEGRAL ELEMENTS

Another minor point in passing. Take [11] 98 x 86. If Elements are halved the odd ones would end in halves, but the dimensions would be 49 x 43.

A11. MEANS OF RECORDING SOLUTIONS

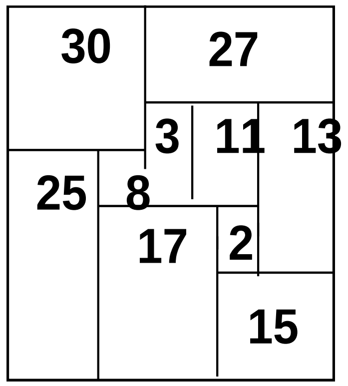
A11.1. RECORDING SOLUTIONS

The solutions when shown in normal geometric form take up much space and time to construct where many of them are required for study.

There are more compact ways of recording them without the need for patterns. In fact there are several ways each with advantages and otherwise, and finding a system which is always useful needs much thought.

A11.2. FORMULA

A formula is used which is easy to apply, and from which the diagram can be readily drawn. Sets of horizontally adjacent Elements are written from the top to bottom as Below:- There are a number of ways of showing this, and some are shown.



[30,27]
[3,11,13]
[25,8]
[17,2]
[15]

CONTRUCTING THE FORMULA
FROM THE HORIZONTAL ROWS
FROM TOP TO BOTTOM.

[30,27][3.11.13][25,8][17,2][15]

[30 27+3 11 13+25 8+17 2+15]

30 27,3 11 13,25 8,17 2,15

The layout of the formula is affected when the solution is rotated or reflected thus [25,17,15] [2,13] [8,11] [3,13] [27] is the Above solution upside-down. Arranged horizontally, the formula again bears little similarity to the others.

This is a possible drawback to the system, but by convention the horizontal formula with the largest corner at left top will be used.

This system was the means of recording Solutions proposed in the 1930's when Brook and his friends worked on the subject of Squared-Rectangles.

A11.3. ALTERNATE WAYS OF RECORDING ELEMENTS

There are several possibilities with merits and demerits. The Above solution can be coded in the following ways-

1. BY FORMULA

*30 / 27*3 / 11 / 13*25 / 8*17 / 2*15* or other variant as previously mentioned, is a good method but marred by not being unique unless rules are applied.

2. SIZE ORDER FORMULA

2 3 8 11 13 15 17 25 27 30 is useful for some purposes but useless for the construction of the solution.

3. BY ORDER THEN INNER ELEMENTS CLOCKWISE

30 27 13 15 17 25 and 3 11 2 8 where the inner Elements (after and) start at top left. Good for constructing rectangle. Corner Elements might be indicated by dots following thus 30.27.13 15.17 25. and...

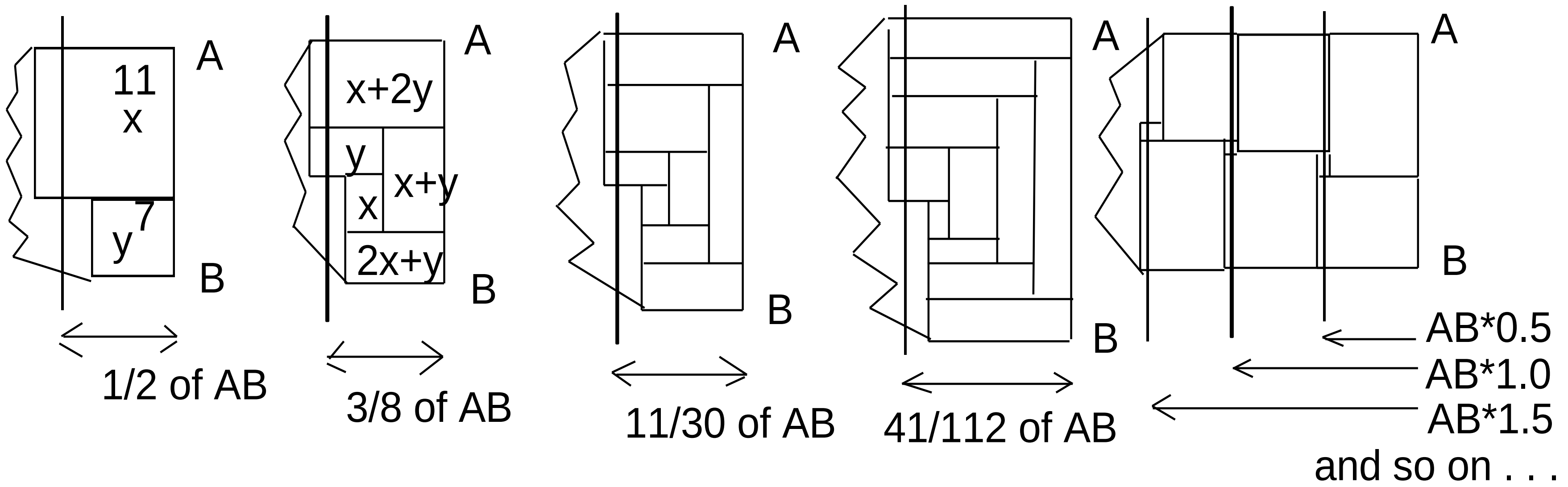
4. BY x y CALCULATION ORDER

2 11 13 15 17 8 3 25 27 30 Solutions can be constructed though not always satisfactorily in one go. If the solution has very small Elements it may be very difficult to draw - or occasionally even impossible. The series is never fixed which is a pity, there being other choices of x and y available.

A12. HORIZONTAL-VERTICAL RATIOS FOR END PATTERNS

A12.1. HORIZONTAL-VERTICAL RATIO FOR SOME END PATTERNS

Look at the End patterns below which are termed Diad, Pentad, Octad, Undecad in this book.
A vertical line has been drawn to bisect the 'kink' between x and y. Is there any relationship between the depth of AB and the distance to AB? Yes! The horizontal is always one-half of the vertical in the case of below 1 (e.g. with 11 and 7 the line is 9 which is half of 11 + 7 or 18).
In the Pentad the line is found to be three-eighths of the Vertical AB whatever values are given to x & y. The Reader can check this from the algebra if required.
In the Octad, the ratio is 11/30 and 11-Add 41/112 of the Vertical AB.

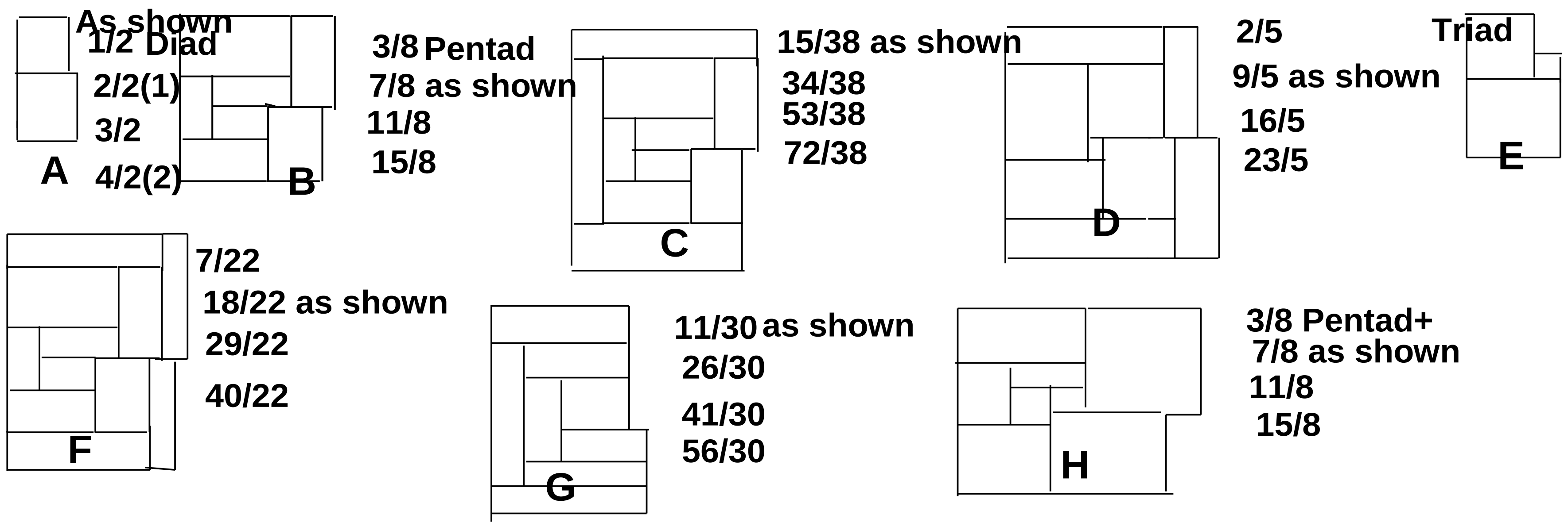


This series can be continued for 14 17 20 Elements... Note 1 + 2 = 3 and 3 + 8 = 11 and 11 + 30 = 41 so the next value number must be 41 + 112 = 153! The second values are double 1, 4, 15, 56 each being 4 times the last value less the previous one. Now 56 x 4 - 15 is 209 so our next Ratio is 153/209!

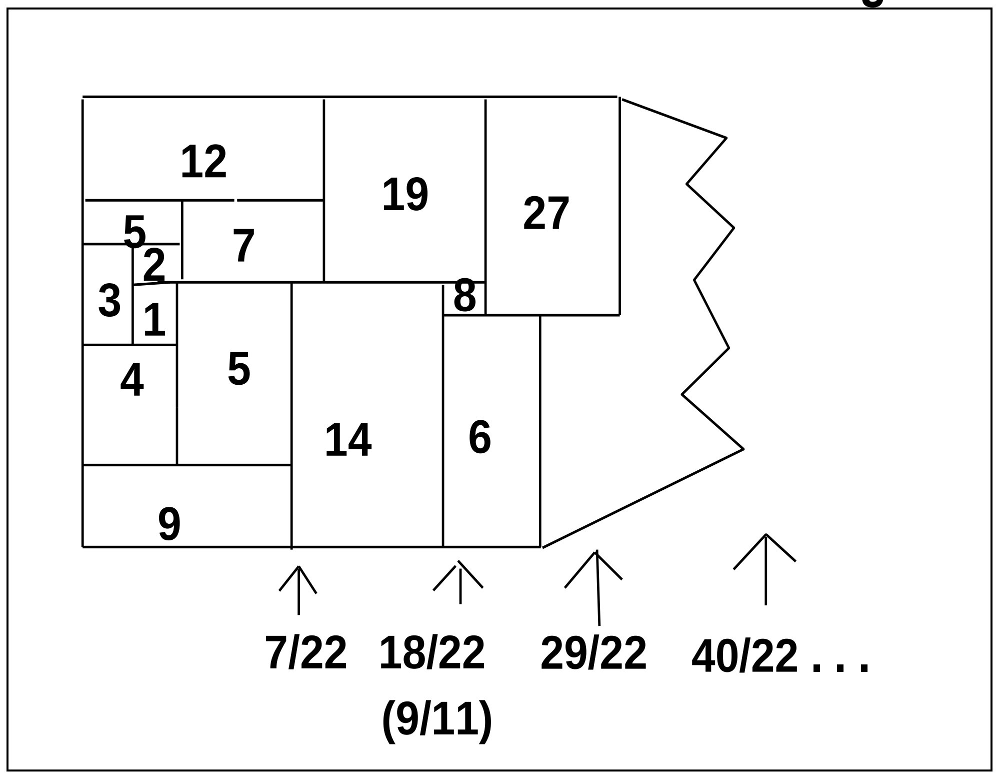
Above 5 shows patterns called Double and Triple. Algebra will show that the Ratios rise by a half each time! So a vertical line driven through the middle of the small Element E is the same distance from A as the length of AB! The above relationship is particularly useful in Section D later.

A12.2. HORIZ-VERT RATIOS APPLY TO ANY SYMMETRIC END

A quick check will reveal that in any End which is Asymmetric that this ratio is not fixed and so does not apply. However it does for absolutely any Symmetric Pattern.



The above shows some series of Ratios - the pattern shown relating to the Ratio "as shown". The other Ratios apply when Triads are added. For the Triad pattern E the Ratios are the same as in Diad pattern A. The same is true with patterns B and H where the series of patterns *is* different but the Ratios remain unchanged.



Another pattern with Triads at right is shown above. Here the Ratios rise by a half or 11/22 each time a Triad is added. These are verified by the dummy Elements shown. It will observed that some Ratios cancel down to 9/11 20/11 (31/11 42/11 ths etc.). It is evident that whatever is added to the pattern at right hand side, the lesser dimension must always be divisible by 11 (in the Above case 33 is divisible by 5). We shall term this number 11 as the Pattern Factor or PPF and the next section will deal with these.

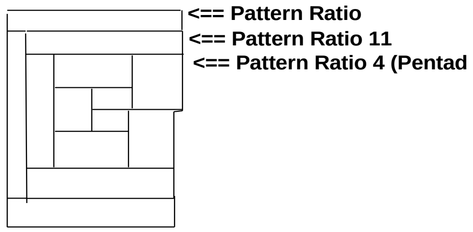
A12.3. PATTERN FACTORS

A number of symmetric patterns have already been shown the simplest being the Dad and Triad which both have a Pattern Factor of 1. In a Pentad however the PPF is 4. In other words the distance straddling the Pentad is always a multiple of 4 (where all Integers are integers of course).

A Pentad is a Dad with one Claw added. An Octal is a Diad with two Claws added. The Pattern factor now becomes 15. For 11-Add the PF is 56 and for 14-Add the PF is 209.

This series 1 4 15 56 209 is easily constructed by multiplying each number in term by 4 and deducting the one before. Thus 209 x 4 - 56 = 780 and 56 x 4 -15 = 209 and so on. The series remains true no matter how many Triads are subsequently added. In fact adding Triads make no difference to the Pattern Factor.

1 4 15 56 -- is not the only series. We saw above a Pattern Factor of 11. Needs tidying up! ***

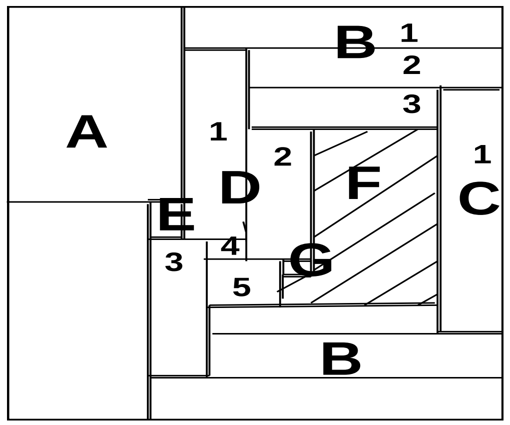


A13. CODING SYSTEM FOR SYMMETRIC PATTERNS

A13.1. SYSTEM OF CODING SYMMETRIC LINKS

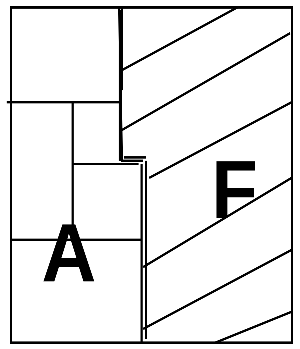
I have spent ages attempting to evolve a perfect Coding System that will embrace all Symmetric patterns, which is easy to apply without being too cumbersome and lengthy. Finally a good System has been found.

Below a Sample pattern is shown, which can be varied in infinite ways. Note that other patterns will look extremely different from this. See 2nd diagram.



A LEFT Forms a Regular Pattern
B TOP (BOTTOM is same reversed)
C RIGHT
D INTERNAL Is a pattern, with or without --
E PLUS A single Element, often omitted
F BLOCK - A variable Irregular pattern
G PLUS A single Element, often omitted

CODE - A;531



CODE - C 000

1. Each pattern must contain at least Sectors A - some sort of REGULAR Pattern, and F - some IRREGULAR Pattern.
2. The Pattern may or may not contain any or all of the other Sectors B C D E and G.

3. Ignoring BLOCK F there are four Sectors - LEFT TOP RIGHT and INTERNAL, but 4. The BOTTOM is simply regarded as a reversed repeat of the TOP.
5. Where PLUS E and PLUS G exist, they occur each side of D, INTERNAL, but are coded separately.
6. There are often several variations with the same Code which not unique, unless separated by further explanation and coding.
7. The Pattern for A is denoted by two code letters like -Y or BH - see later for list.
8. B is easily recorded by the Number of Elements at the Top, 3 in above. However other patterns may share the same code.
9. C is easily recorded by the Number of Elements at the Right, 1 in above. However other patterns can share the same code. 10. B C and D are each coded as 0 if these sectors do not exist.
11 The INTERNAL area D may not exist at all, or exist with Elements at E, G, E and G, or without E and G. Where D does not exist but a single PLUS does then this PLUS must be regarded as E only with G absent.

A13.2. COMPILING THE CODE

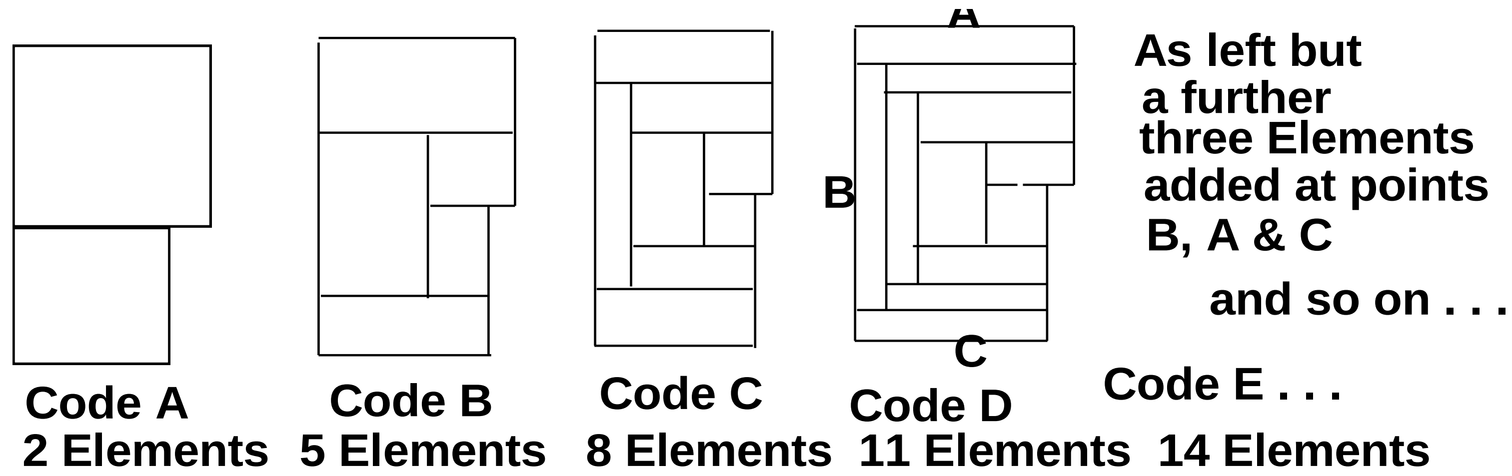
The procedure is as follows:-
1. The Code letters for the Regular pattern at A is shown first. The Diad has been labelled A.
2. This is followed by a space, comma, hyphen or semicolon according to the presence of E and / or G -
(SPACE) No Element at either E or G. , (COMMA) Element at G only. ' (HYPHEN) Element at E only. ; (SEMICOLON) Elements at both E and G as Above 1. Where there is no Internal area D but there is a single PLUS then ' (HYPHEN) is used. 3. This is followed by three numbers which are simply the quantity of Elements in D B and C respectively.
Above there are 5 in D 3 in B at top and 1 at C. Thus the codes for Above is A-;531 and C-000.

A13.3. ADVANTAGE OF CODED PATTERNS

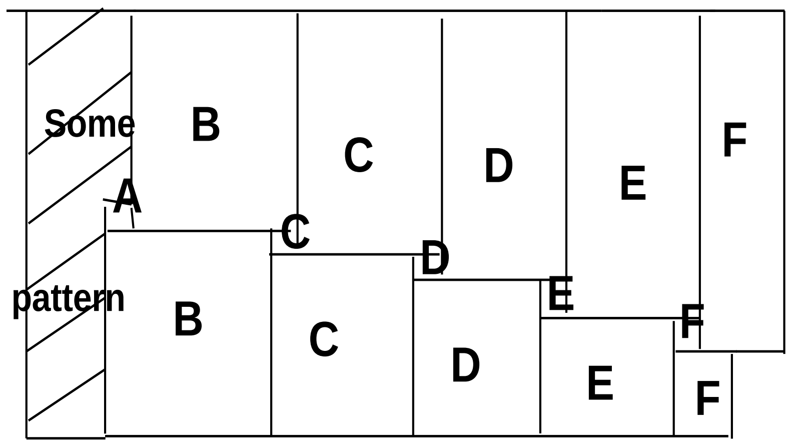
Codes not only form a useful short reference to complicated patterns, but they can also be processed in various ways.
In particular, new links can be readily found by knowing which amounts to add or subtract to numbers in the Code.
They are useful in determining which links are still to be found, and a check as to when the list may be finally complete!

A13.4. EXPANDING THE CODING ON THE LEFT HAND SIDE

The pattern at A - the left hand side - mentioned earlier, can be an infinite number of patterns and so a Coding system for this first part has also been devised. This is actually more straightforward than it at first looks!
First -



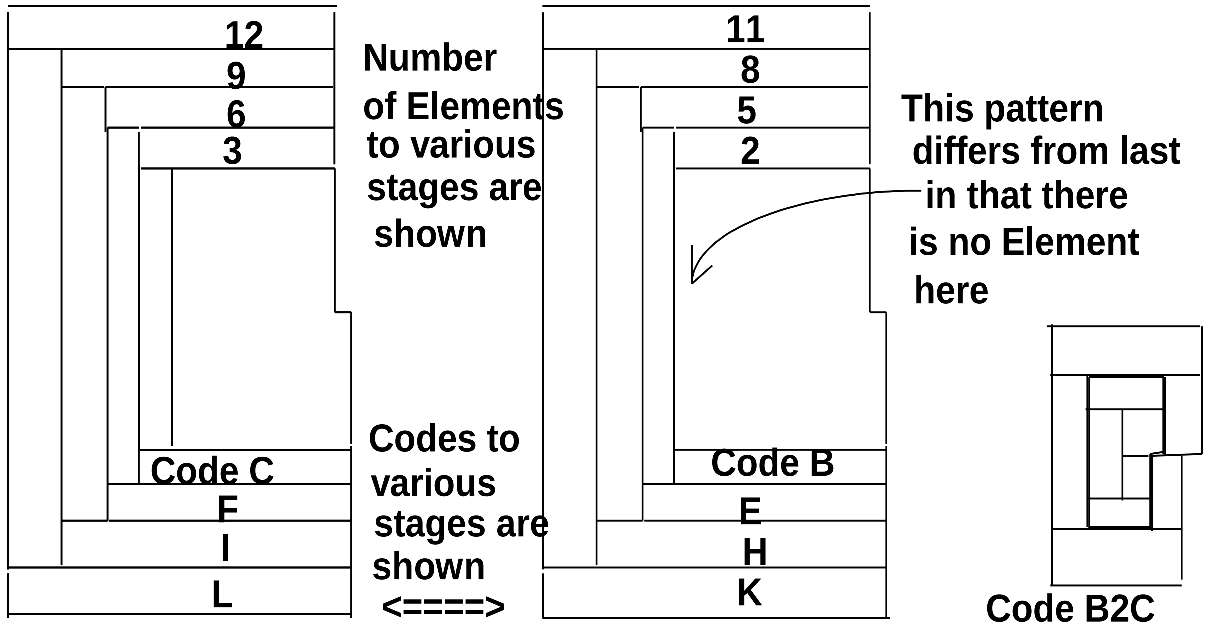
Absolutely any Symmetric Pattern will comprise a bit as Code A, but some will have 5 Elements as Code B or one of the other patterns. This A, B & so on, forms the first part of the Code.
Next, it may be that there is also a pattern to the right of this containing 2, 3, 5, 6, 8, 9. . or more Elements. See below.



For example there may be three more Elements shown as A & B Above. If C also applies then there are 6 more Elements. If D also applies there are 9 more Elements and so on. In this part we simply show the number of additional Elements. So if "Some Pattern" is Code A and Elements A & B in the last appear (but not C) then our Code becomes A-3. If A to F all appear with Code B at left then the Code is B-15. And so on.

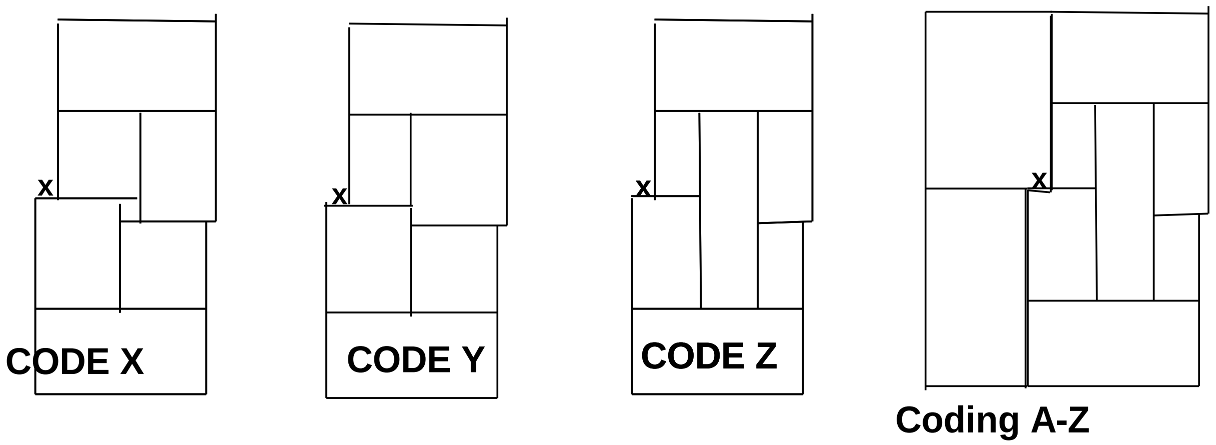
Now with some other patterns at the Left the single Element A may not be present, and so additional Elements may be 2, 5, 8, 11, 14 . . . But if A does exist as multiples of 3. (3, 6, 9, 12 ...) may do.

Next to the pattern already considered there may or may not be, more Elements added at top and bottom and possibly on the left hand side. These will be coded B for 2 Elements, C for 3, E for 5, F for 6, H for 8, I for 9, K for 11, L for 12 etc. according to the position in the alphabet.



In above 3, note that our original pattern - a Pentad & code B becomes internal after 2 Elements at right are added, followed by three & top at left, the third letter in the alphabet being C.

Lastly some Symmetric patterns are more complicated as they contain one of the following on the right hand side -



In above 4 note that the start pattern, a Diad, is this time external. Note too that where x has been shown it is possible to have the same patterns with a further Element added here. If so add “1” after the previous letter, i.e. “A - Z” becomes “A - 1Z” and “A - Y” would become “A - 1Y” and so on!

A14. 1. VERTICAL & HORIZONTAL BARS

Within any Rectangle or Square are a collection of Lines (horizontal or vertical) which border a number of Elements on both sides. The Lines will be referred to as Bars when the bordering Elements are considered.

The number of Elements is always three or more.

1. Only Adjacent solutions can have two Elements.
2. Where there are three Elements 1 on one side, two on the other. Clearly the size of all must be different.
3. Four Elements - Either all Elements will be different, or there will be two pairs or duplicates. E.G.

|23_ |15| |15|36__|
|18|20_ | |36__|15|

4. **Five Elements - a. 1 Element bordering 4. Usually all Elements will be different. Occasionally there is a duplicate pair.**
   b. Two Elements bordering three (or vice versa). There may or may not be a duplicated pair. More about this is dealt with in *Section S.*
5. **Six Elements – a. 1 Element bordering 5 or b. 2 Elements bordering 4 or c 3 Elements bordering 3.**
6. **Seven Elements & Larger.**

A14.2. OCCURRENCE OF DUPLICATE ELEMENTS WITHIN BARS.

Many Bars of 5 or more Elements are found with a pair of Duplicate Elements. Why so many? Others have all different Elements. Are there any rules to determine which constructions have Duplicates? Well, the Trailing System dealt with at length in *Section S* may help.
Section S12 shows that two Pentads put back-to-back result in a duplicate pair of Elements. Ditto with two Octads or two Undecads etc.

B. SMITH DIAGRAM BASICS

B. INTRODUCTION - VARIOUS REPRESENTATIONS OF SQUARED-RECTANGLES

Apart from the normal illustrations of Squared-Rectangles (whether shown as exact squares or distorted rectangles) and the listing of Elements according to various formulas, there are other ways of representing solutions - some more useful and general than others -

1. NETWORKS (Like Smith Diagrams without positions for poles) *B7*

In all types of Smith Diagrams noted below there are always distinct patterns for the Vertical and Horizontal formats B5...

2. SMITH DIAGRAMS (Unfixed format). The basic type used *B1.1 etc…*

----

3. SMITH DIAGRAMS (Fixed formats) - Various:-

a. SMITH DIAGRAMS TYPE Y *B13.3* and TYPE X *B13* etc.

b. SMITH DIAGRAMS BOX TYPE *B13.5*

c. SMITH DIAGRAMS O TYPE *B13.6*

d. SMITH DIAGRAMS G TYPE (GRID TYPE) *B13.8*

e. SMITH DIAGRAMS TRIANGLE, SQUARE, PENTAGON & HEXAGON TYPE with or without additional Stator Wires.

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4. C-NETS *B16* Essentially a restricted form of Smith Diagram arranged as a Triangle.

5. P-NETS *B16* As above but a wire (element) removed.

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6. MATRIX (plural MATRICES) *B16.1*. A series of lines with numbers.

7.* MOSS DIAGRAMS *B14.* Elements regarded as points *(not lines as in SD's) joined by lines in a square format.

B1. SMITH DIAGRAMS - TERMS USED

B1.1. SMITH DIAGRAM (‘DIAGRAM’ IN SHORT)

This is a means of representing geometrically a Squared-Rectangle by a diagram which closely resembles a planar electrical network with two poles and connecting wires. Proper explanation is given later.

B1.2. POLES

The two poles in a Smith Diagram are the two points from which the current flows from, being the positive pole, and current flows to, the negative pole. The number of Poles is always TWO.

B.1.3. COMPLEXITY

This is fully explained later, but is the full current flowing from the positive pole, and flowing to the negative pole. The COMPLEXITY is equivalent to the full horizontal value of the Squared-Rectangle.

There happens to be a fixed Complexity for every electrical circuit irrespective of which poles are chosen.

B1.4. NETWORK

This relates to the entire pattern of wires in the Smith Diagram.

B1.5. CIRCUIT OR INDIVIDUAL CIRCUIT

This relates to a single piece of the entire network in the Diagram. In this book “Circuit” means an individual circuit.

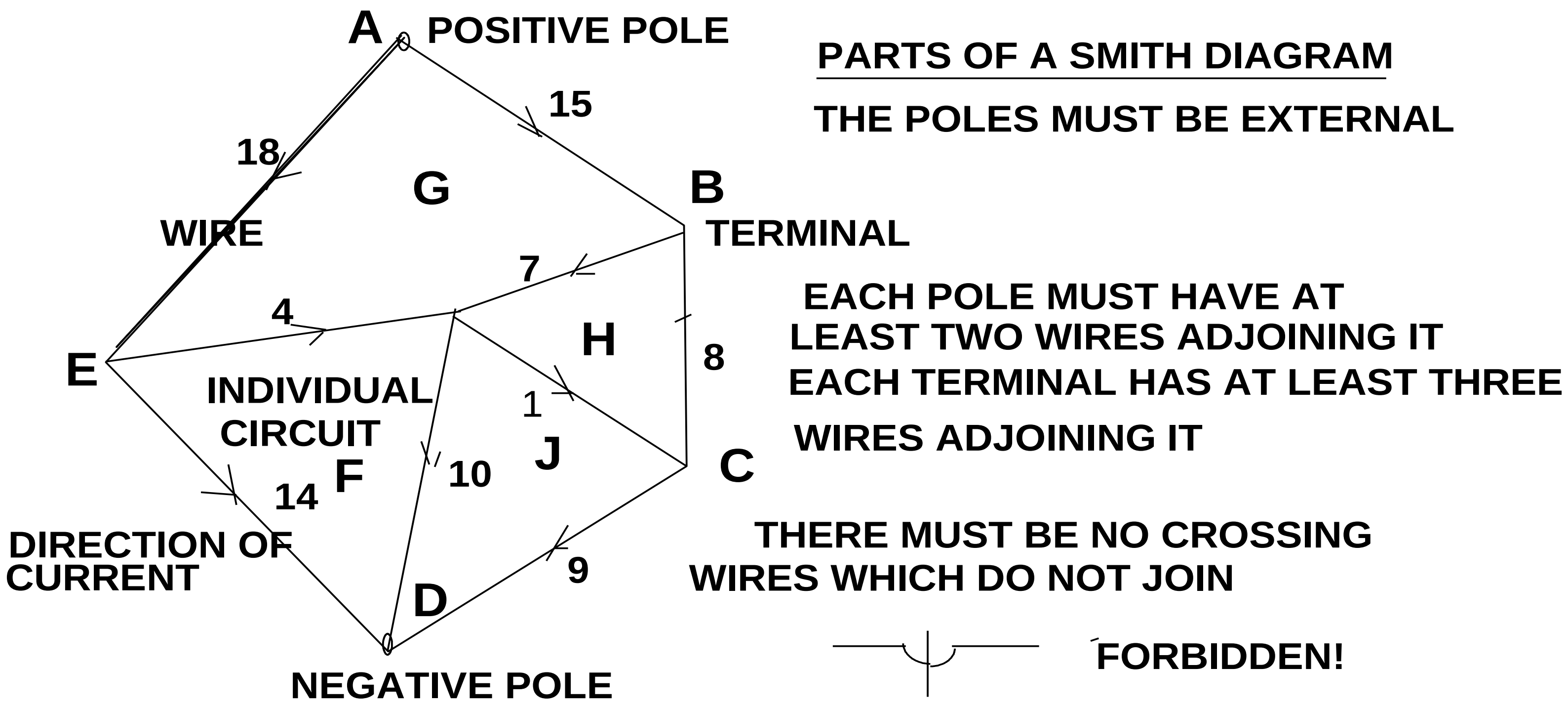
B1.6. ROTORS and STATORS

This is fully explained later. Sometimes part of a Smith Diagram can be reversed. This is called the ROTOR. The remaining part which cannot be reversed is called the STATOR. In many rectangles the whole rectangle can be regarded as a Stator without any Rotor.

B1.7. “Y” SMITH DIAGRAMS

This is fully explained later. Many rectangles calculated with two unknowns (x, y in algebra) can be represented by a special type of Smith Diagram with a centre point and three arms in the shape of a “Y” with lines connecting them.

All Squared-Rectangles may be converted into what are termed “SMITH DIAGRAMS”. In fact a Smith diagram is a representation in a different form of a Squared-Rectangle.



B1.8. PARTS OF A SMITH DIAGRAM

The features are best illustrated. Above [9] 33x32 horizontal is shown and each Element replaced by a wire of same value.

It resembles a river starting at one place, the POSITIVE POLE, and flowing via devious routes to another, the NEGATIVE POLE. Thus the POTENTIAL at both Poles is the same, namely 33. Note that the length of each wire is irrelevant and has no effect on the structure which means that the diagram can be pressed and pushed into many shapes in infinitely different ways. Hence this diagram is termed UNFIXED. See later for various attempts to obtain FIXED Diagrams.

B2. SIMILARITY TO PLANAR ELECTRIC NETWORKS

Basically a Squared- rectangle is essentially the same as an Electrical Network consisting of wires joining together and arranged in such a manner as to be planar, i.e. no wires crossing. In real electrical network the network is not bound to be planar (flat) nor the currents bound to be positive integers or zero.

Two features that do not apply to Squared-Rectangles:-

In a correctly drawn Rectangle all Elements are naturally always positive, but in Diagrams the flow of current along each wire (which effectively replaces each square), the flow of current must be shown by means of arrows to distinguish positive and negative flows. A flow of 6 in one direction is the same as a flow of -6 in the other.

B3. HISTORY OF SMITH DIAGRAMS

Smith Diagrams were originally devised by Mr. C. A. B. Smith, a student of Trinity College, Cambridge in 1936 when he and three friends made an extensive study of Squared-Rectangles. Smith objected to the name, but this term was adopted. Squared-Rectangles can come in various forms and adaptations, and there are other possible diagrams.

B4. KIRCHOFF'S LAWS

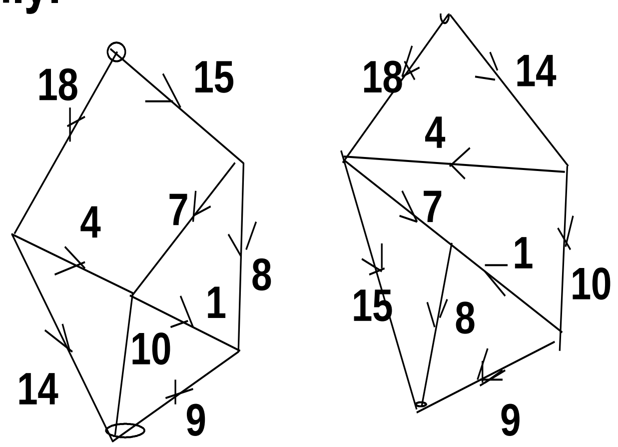
Kirchoff in electrical theory stated that for a flow of current in an electrical circuit,
1. Except at a pole, the algebraic sum of the currents flowing to any terminal is zero, and
2. The algebraic sum of the currents in any individual circuit is zero .
To check these points look at terminals abcde and circuits FGH and J in the diagram where A. pole B. 15 - 7 - 8 = 0 C. 8 + 1 - 9 = 0 D. pole E. 18 - 4 - 14 = 0 and circuits F. 4 + 10 = 14 G. 15 + 7 = 18 + 4 H. 7 + 1=8 J. 1 + 9 = 10.

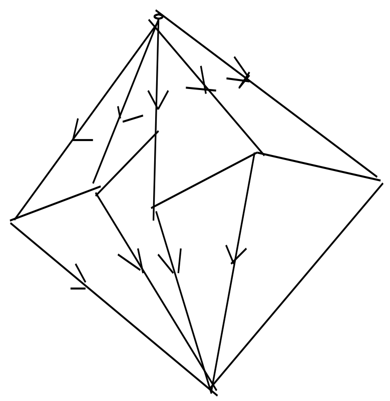
B5. VERTICAL AND HORIZONTAL DIAGRAMS

B5.1. VERTICAL AND HORIZONTAL DIAGRAMS

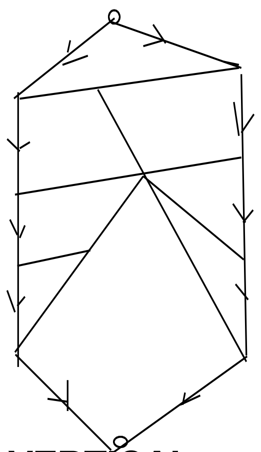
Any Squared-Rectangle can be turned upside-down and keep the same pattern. Obviously mirror images of both up and down versions do not alter the pattern. There are 4 ways of displaying any Diagram, but by convention the largest corner is put at top left to prevent confusion.

In this book Squared- rectangles are drawn with the positive pole at the top and most arrows pointing downwards. It is possible to display Squared-Rectangles so that all arrows point downwards rather than upwards or on the horizontal. Clearly all external wires point downwards only.





A HORIZONTAL SMITH DIAGRAM



& A VERTICAL

SMITH DIAGRAMS FOR [9] 33X32
(THE SECOND MAY BE REGARDED AS FOR {9} 32x33)

B5.2. THE TWO DISTINCT SMITH DIAGRAMS FOR ANY RECTANGLE

Any Rectangle can be drawn horizontally or vertically and a Squared-Rectangle drawn for each. There is no practical difference in the rectangles but the two resulting Diagrams are very different, and seem to be unrelated to each other!

See Above for an example for [9] 33 x 32. For each Rectangle there are two diagrams and thus the total possible Smith Diagrams for any given Order is an even number. A horizontal Smith Diagram may well have more wires adjoining the poles than between, and the reverse true for vertical Diagram. From given Smith Diagrams it is often impossible to guess whether it represents the horizontal or vertical SR but the 3rd diagram above is clearly horizontal since the Rectangle is 5 Elements wide and 2 deep. The 4th diagram is clearly vertical since the Rectangle is 2 Elements wide and 5 deep.

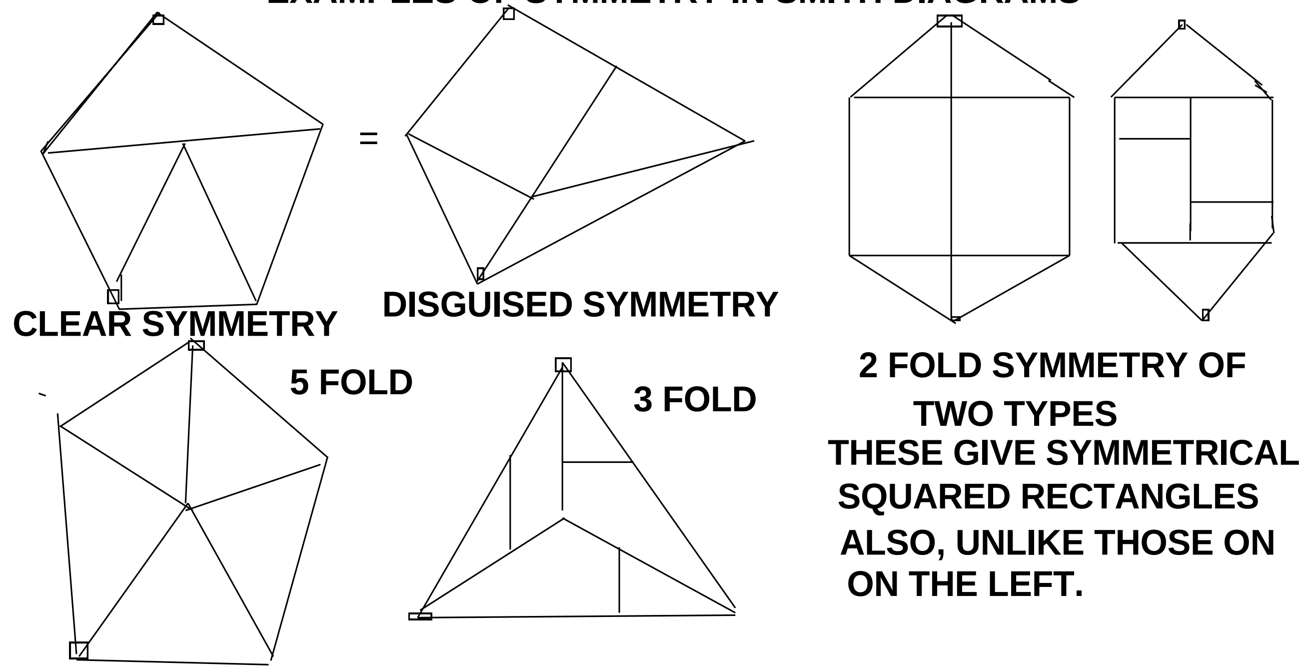
Also in these diagrams arrows show the direction of current where it is obvious, but it is not possible on quick inspection to insert arrows for the remaining wires, and it is incorrect to assume that arrows must always point downwards.

B6. SMITH DIAGRAM SYMMETRY

B6.1. INTERNAL ELEMENTS VERSUS INTERNAL WIRES

Surprisingly the number of Internal Elements in a SR and internal wires in a Diagram are not the same! In a Diagram only the Elements at extreme left and extreme right of a rectangle show externally. In Below 3 all wires with arrows are actually external in the solution. These are either external in the diagram or radiating from either pole.

EXAMPLES OF SYMMETRY IN SMITH DIAGRAMS



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B6.2. SYMMETRIC SMITH DIAGRAMS VERSUS SYMMETRIC RECTANGLES

In rectangles symmetry is easily seen. In Squared-Rectangles this is complicated by certain features.

Look at the Rectangle for [9] 33 x 32 (horizontal) above - it is symmetric! But the rectangle is not!

It is found that the Diagram for a symmetric rectangle is always symmetric too, but the diagram may sometimes look otherwise until suitably arranged. As usual, drawings show this clearer than text.

The reason why a Squared-Rectangle can sometimes be asymmetric yet have a symmetric Smith Diagram is down to the choice of poles. The diagrams to the left have no symmetry between the poles chosen. On the right symmetry between the poles exists. This is why the rectangle is symmetric also.

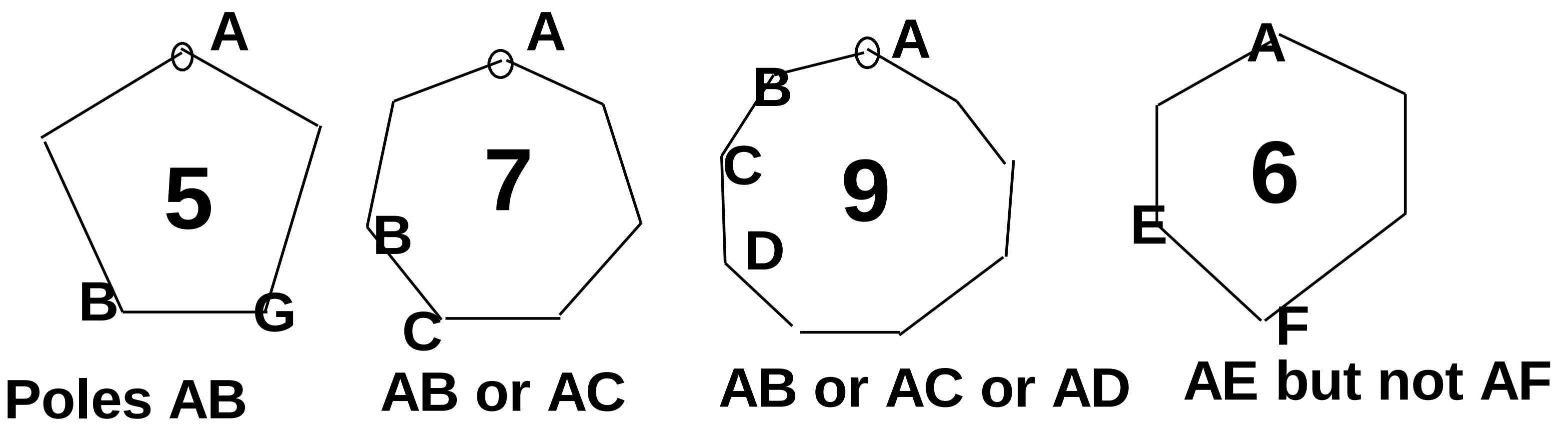
A further complication is that Diagrams can have 2 fold, 3 fold, 4 fold, 5 fold… to any number!

Hence symmetry in Diagrams has a different meaning. It is not evident from inspection of any rectangle whether or not symmetry exists in the Diagrams even if the Diagram is drawn, it can easily be missed in cases where it exists.

B6.3. MORE ON SYMMETRIC DIAGRAMS GIVING ASYMMETRIC RECTANGLES

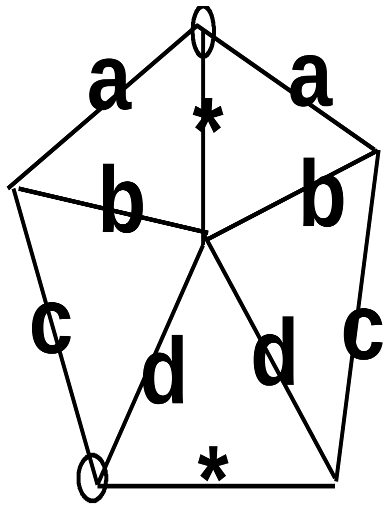
As already said, it is the choice of Poles which dictates this.
In the case of any Symmetric Diagram having an ODD number of outer Element wires it is clear that the bottom Pole cannot fall exactly half way along. Look at AF in below 4 which has a hexagonal pattern with six outer wires. Here any Rectangle must always be Symmetric since the left half is mirrored by the right half. But notice AE in below 4 may well give an Asymmetric Solution despite the number of outer wires being EVEN.

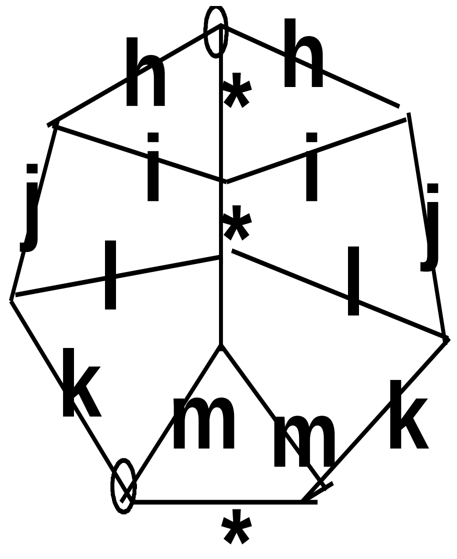
Although some Solutions may be dud, Invalid or Symmetric occasionally, the majority of pole choices for 5 7 9 11... Wires will give Asymmetric Solutions. Note that in Below 1 AG will give a Solution but it is a repeat of AB so may be ignored. But with 5 wires one pattern from Pole AB is possible, two (AB AC) for 7 wires, three (AB AC AD) for 9 wires, and so on. .



6.4. RELATING PAIRS AND SINGLE WIRES IN SYMMETRIC DIAGRAMS

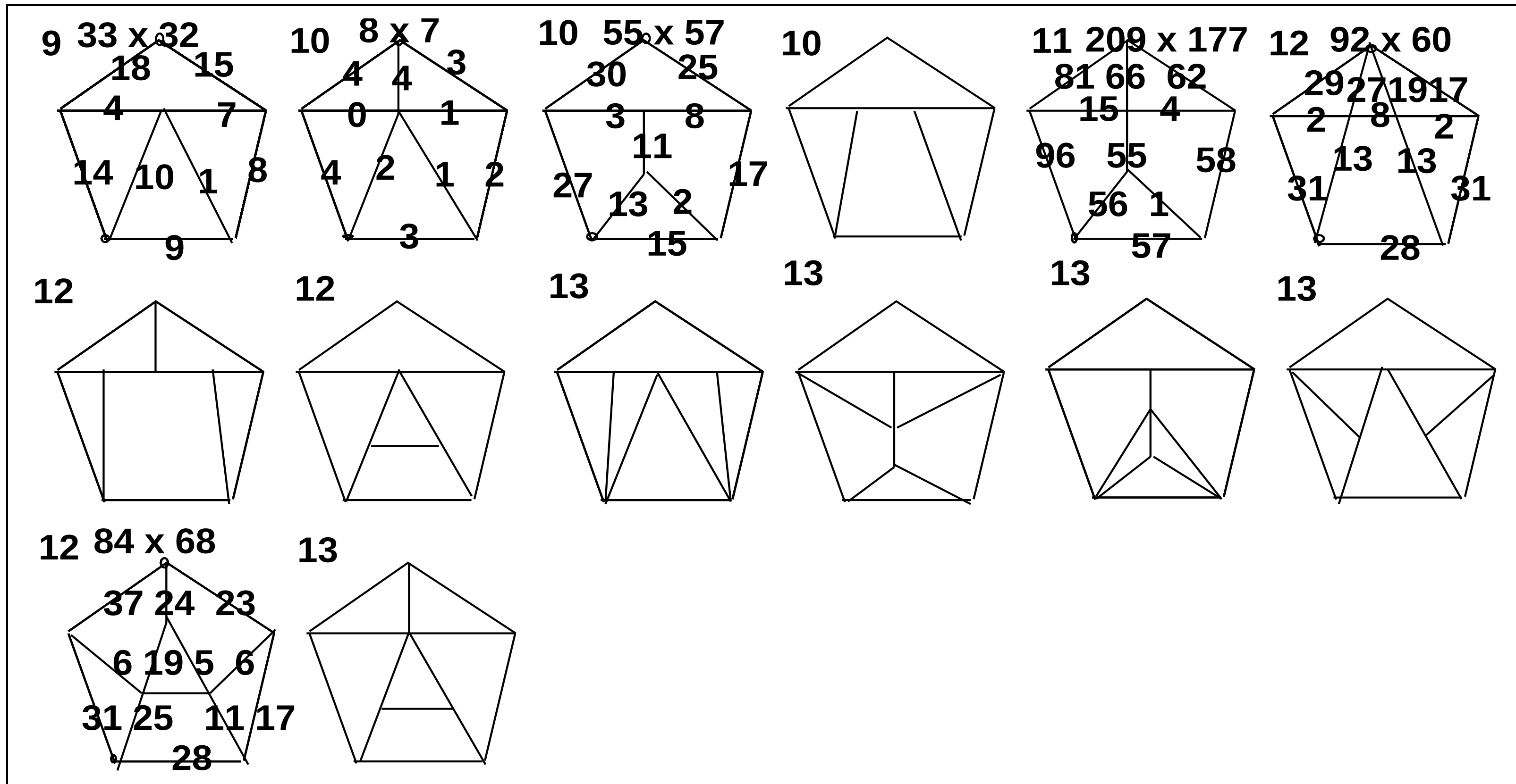
In the first pattern below note it divides into corresponding pairs a a b c c d d. Likewise the second has pairs hh ii jj ll and mm. But the remaining starred wires are Single ones.





B6.5. SOLUTIONS FOR SYMMETRIC DIAGRAMS

Shown Below are some patterns which are Symmetric as far as the Smith Diagram is concerned, yet give Asymmetric Rectangles. The number of external Elements in the Diagram (not the Rectangle) is 5 in each case. The Poles chosen are at the top and SW corner. Not all are Valid solutions. NOT YET FINISHED.



B7. NETWORK PATTERNS

B7.1. NETWORK PATTERNS

This is simply a Smith Diagram pattern with no poles positions shown, from which a number of choices of poles combinations can be made. It is possible for several rectangles to share a Network pattern, but each having differing pair of poles. Each group of rectangles has the same Complexity.

B7.2. COMPLEXITIES

With every electrical circuit there is associated a number, called the Complexity of the circuit. The COMPLEXITY is the full current flowing from the positive pole and also the full current flowing to the negative pole.

The word “full” refers to that current before any reduction has been made, i.e. before each currents is divided by the Reduction Index. Thus in [9] 15 x 11 of full size 75 x 55 the Complexities are 75 horizontal and 55 vertical.

Often a particular amount may have several distinct Complexities, 593 being a case in Order 13, so they are lettered 593A, 593B, 593C and so on.

B7.3. POLES ONE WIRE APART IN A NETWORK

In any circuit or Complexity any two outer terminals may be chosen as Poles.
The fixed Complexity rule is not broken when poles one wire apart are chosen.
However a Singlend solution results and so effectively a rectangle of the Order Below if the single end is axed.

B7.4. CHOOSING POLES 2 OR MORE WIRES APART IN A NETWORK

By doing this Singlend solutions are avoided. Where only 2 wires connect a pole, no other choices of this pole are available and for 222# solutions only a single solution for the complexity exists. Other networks may provide many pole choices, e.g. the horizontal Diagram for [13] 593 x 480 where changes of poles also provide 593 x 472, 593 x 510, 593 x 473 and 593 x 465 if Singlends are ignored.

B8. AMOUNT OF SOLUTIONS FROM ONE COMPLEXITY

B8.1. AMOUNT OF SOLUTIONS FROM EACH COMPLEXITY

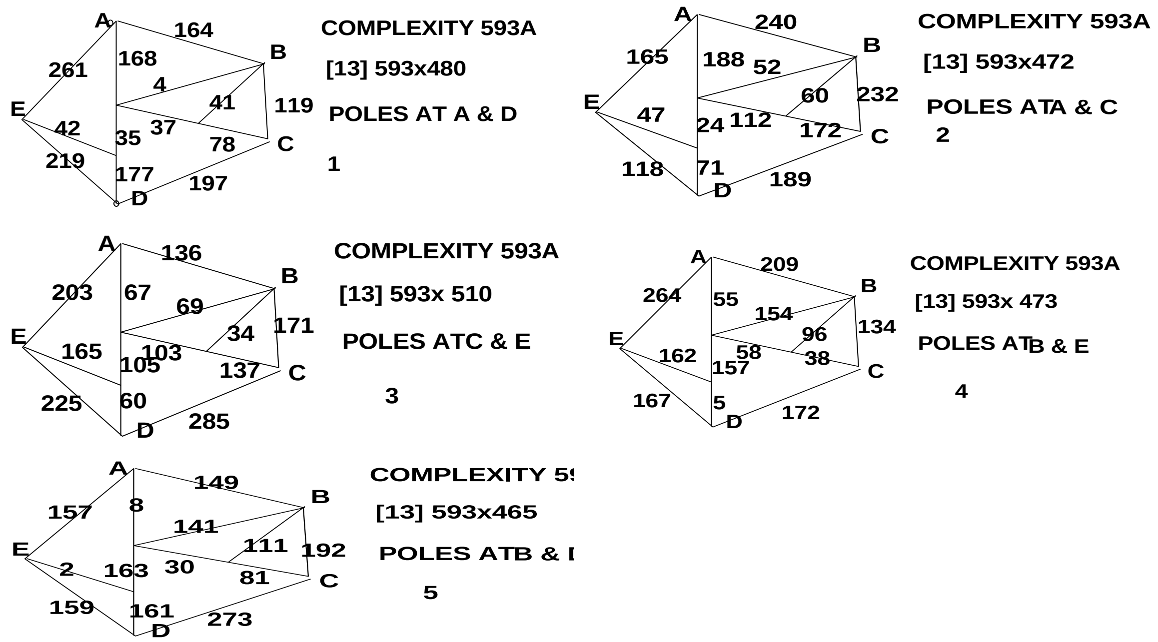
(External Pole choices only - see B8.2 also)

There are Simple rules which apply to this -

1. Cases where 3 or more wires adjoining both poles, which means or 3 or more horizontal Elements at top and at bottom of rectangle: Add the amount of Elements bordering the vertical sides of the rectangle - minimum 4, which equates to the amount of external wires in the Diagram.

Call the terminals abcd ... etc. how many choices of 2 or more apart can be made? For ABCD only AC and BD. For ABCDE AC AD (NB not AE) BD BE and CE (five).
For ABCDEF AC AD AE BD BE BF CE CF and DF (nine)
For ABCDEFG AC AD AE AF BD BE BF BG CE CF CG DF DG e.g. (fourteen).

The series runs 2, 5, 9, 14 ... with differences 3,4,5 …



2. Cases with 3 or more wires at each pole. ABCD AC BD (two) ABCDE AC AD BD BE CE (five) ABCDEF AC AD AE BD BE BF CE CF DF (nine) ABCDEFG AC AD AE AF BD DE DF BG CE CF CG DF DG EG (fourteen).
An example of a series of 5 is shown Above for Complexity 593A.
3. Two cases with 2 wires at each pole: 1 solution only.
4. Cases with 2 wires at one Pole only: one Pole fixed, other can vary. ABCD AC only (one) ABCDE AC AD (two) ABCDEF AC AD AE (three), the series runs 1,2,3,4,5 ...
A Table can be constructed as follows:- but see B8.2 below.
Amount of vertical Elements (left + right) 0 fixed 1 fixed 2 fixed

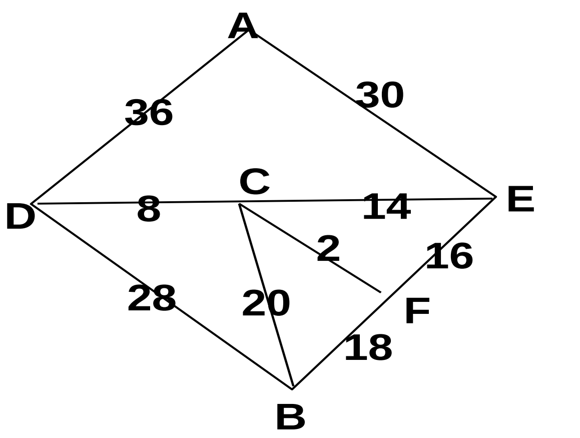
4 ABCD
2
1
1
5 ABCDE
5
2
1
6 ABCDEF
9
3
1
7 ABCDEFG

14
4
1
8 ABCDEFGH
20
5
1
9 ABCDEFGHI
27
6
1
10 ABCDEFGHIJ
35
7
1
11 ABCDEFGHIJK
35
7
1

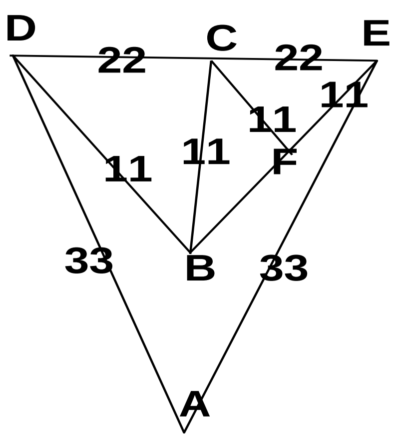
B8.2 ADDITIONAL SOLUTIONS POSSIBLE FOR A GIVEN COMPLEXITY USING SWIVELLING

The following appears tricky but is actually a simple idea!

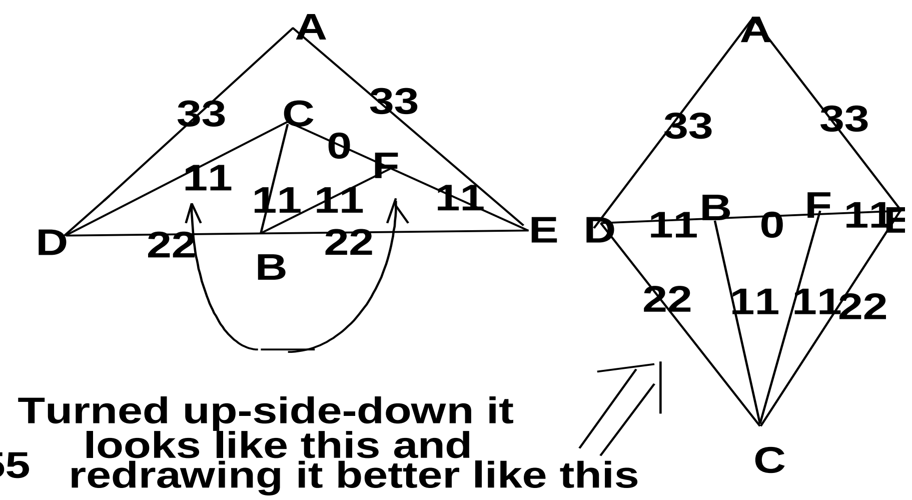
In the case of Complexity 593A above the choices of pairs of Poles were all <u>external</u> and in this case to have pairs which include an <u>internal</u> terminal is impossible. But this is not always the case:



[9] 33 x 32 as [9] 66 x 64



[9] 6 x 5 as [9] 66 x 55



Look at the horizontal Smith Diagram for [9] 33 x 32 above. The full Dimensions for this are 66 x 64 and the full Elements are shown so the Complexity is 66. Now notice that

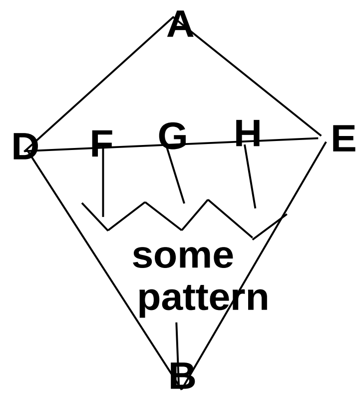
1. There are only two Elements at the top of the Rectangle (36 & 33 shown as AD and AE) and
2. There are more than one Elements joining D with E. (Two here)

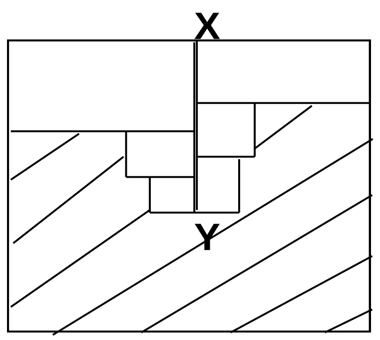
In Above 2 the pattern for the area bounded by B D E has been repeated unchanged except that lines AD and AE have been swiveled from top to the bottom like a skipping rope! This gives a different Rectangle [9] 66 x 55 which reduces to
[9] 6 x 5. But pattern above 2 is not ideal since the largest Elements (33 & 33) appear at the bottom, whereas they would be better at the top in accordance with the usual convention.

So instead of Swiveling AD & AE we Swivel the rest of the area instead leaving AD & AE in the same position as before we get Above 3 which is simply Above 2 up-side-down. Still not ideal this pattern can always be improved by 'flattening' the upper wires between DE and pushing the rest of the pattern downwards as shown in above 4.

The important outcome of all this is that the Complexity in above 1 and above 4 is the same, 66 and 66!
So the choice of ANY two Poles in a network external or internal, *providing that choice is actually possible*, gives rise to a Rectangle with the same Complexity.

So in above 1 we can select Poles A & B or Poles A & F (actually still [9] 33 x 32) or Poles A & C where C is initially internal!



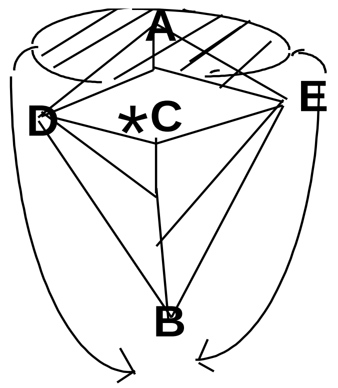


Corresponding Squared Rectangle for the Diagram at left.

DE must comprise more than one Element but may be 2 or higher. Here FOUR are shown and Solutions with Pole Choices A-B A-F A-G and A-H are all possible with each having the SAME COMPLEXITY

Solutions with a Triad top, using an Internal Terminal as one of the Poles are impossible. But if XY in above 2 is bordered by 4 or more Elements, 6 are shown, additional Solutions may be found involving a single Internal Pole choice. In above AF AG & AH are all possible choices, each with the same Complexity.

B8.3. OTHER NETWORKS WITH INTERNAL POLE CHOICES



*
Note that there is a space between D & E, i.e. no Element lines appearing inbetween.

Whenever a Network is so arranged such that there is an unbroken area existing between D and E (see above), at least one Solution using an "Internal" Pole is possible. For instance in above the pattern bounded by BDCE can be Swiveled upside down and a Solution using Poles A and C can be found. Swiveling ADFE instead (as shown above) has the same effect.

B9. TRANSLATING SMITH DIAGRAMS TO SQUARED-RECTANGLES AND VICE VERSA

B9.1. TRANSLATING AN UN-CALCULATED SMITH DIAGRAM TO A RECTANGLE

If a rough Smith Diagram at drawn at random without calculating any Elements it is possible with some trial and error at least to draw the rectangle. This can be done by looking at the notes in B9.2 and applying them in reverse.

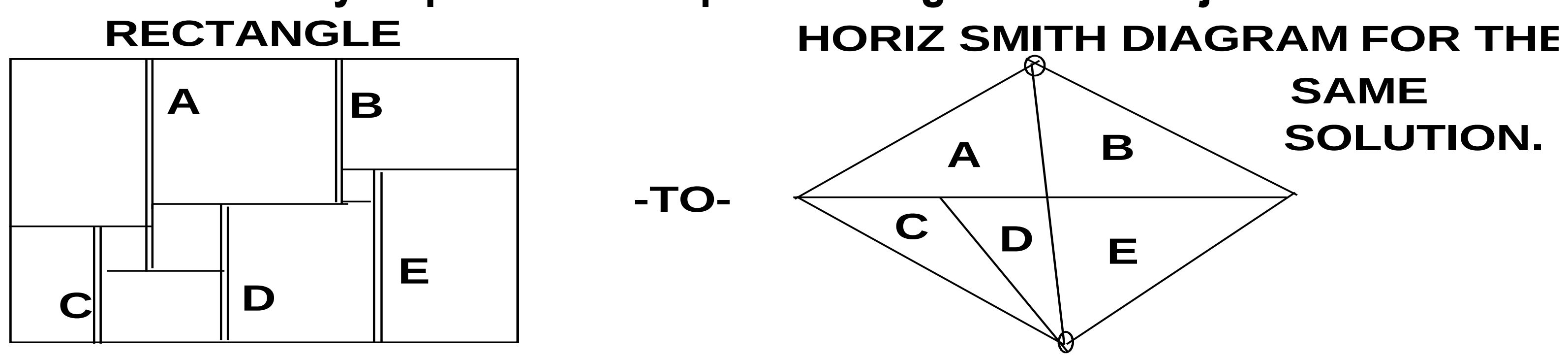
The procedure will involve some trial and error and probably adjustment to the diagram. It is not easy and in fact more tricky than doing it in reverse.

B9.2. TRANSLATING AN UN-CALCULATED RECTANGLE TO A SMITH DIAGRAM

If we draw a rough rectangle at random without calculating out the Elements it is also possible with some trial and error at least to draw the Smith Diagram. An example below gives a clue to how this is done.

Firstly draw the outside of the Smith Diagram observing that 3 Elements radiate from the top and 4 Elements will meet at the bottom, and that 2 Elements will connect each pole at left and two at right.

Then, in below 1 consider each internal vertical line - shown as abcde and observe the Elements to the left and right of each. Each vertical line will need to be replaced with a circuit containing the same amount of Elements e.g. line a is bordered by 4 Elements and area a in below 2 by 4 Elements. It may require some experimenting and later adjustment to do it correctly.



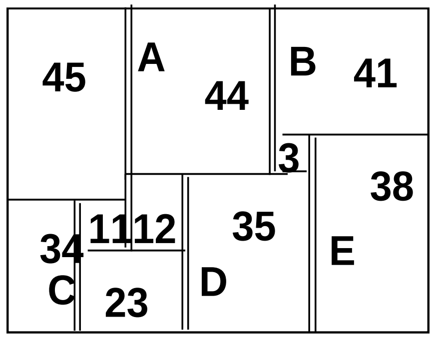
COMPLARE VERTICAL LINES HERE.. WITH CIRCUITS HERE.

B9.3. TRANSLATING A CALCULATED SMITH DIAGRAM TO A RECTANGLE

The information in B9.1 and B9.2 applies and the procedure although simpler with Element numbers shown, still needs explanation which appears more complicated than it actually is.

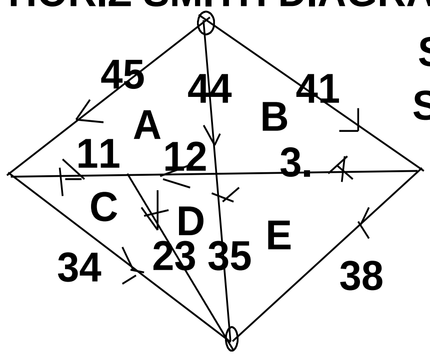
1. Draw radiating lines from top pole showing numbers across the top horizontal line 45, 44, 41 with downward arrows.
2. Look at vertical left Elements 45 and 34, 34 must adjoin bottom pole.
3. Look at vertical right Elements 41 and 38. 38 must also adjoin bottom pole.
4. Look at bottom horizontal line and insert any Elements not included so far 23 and 35, as 44 and 35 join the lines for 44 and 35.
5. Insert inner Elements. Remember to replace each vertical line with a circuit. Make sure arrow direction is correct e.g. in the vertical adjoining 45 11 44 and 12, the arrows for 44 and 12 are in an opposing direction since 45+11 = 44+12.

RECTANGLE



-TO-

HORIZ SMITH DIAGRAM FOR THE SAME SOLUTION.



COMPARE VERTICAL LINES HERE.. WITH CIRCUITS HERE.

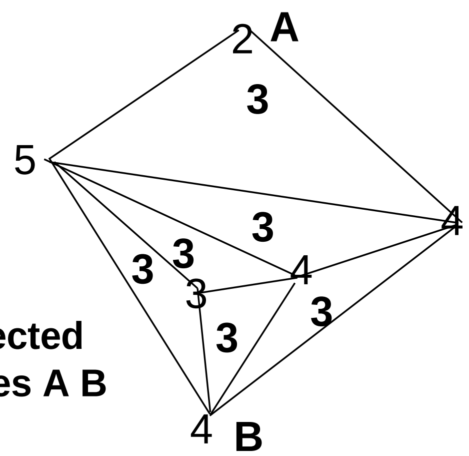
B9.4. TRANSLATING A CALCULATED RECTANGLE TO A SMITH DIAGRAM

The information in B9.1. and B9.2. applies . As it is easier to carefully scrutinize the above diagram to do this than follow long confusing instructions, no further comment!

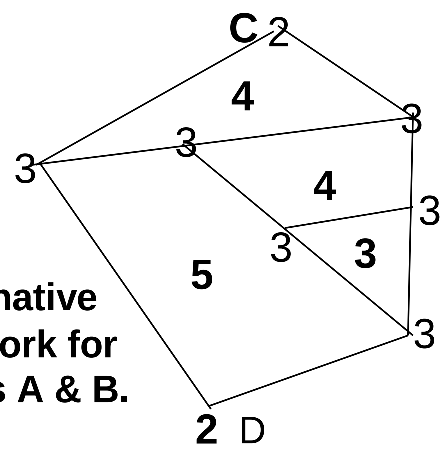
B9.5. TRANSLATING BETWEEN HORIZONTAL AND VERTICAL NETWORKS

Networks differ from Smith Diagrams as

1. No currents values are shown - nor are direction currents and
2. No Poles are shown - but choices of Poles are available to look at. In the circuit chosen below the light numbers shown are the amount of wires at each TERMINAL and the bold numbers the amount of wires in that individual CIRCUIT -



Selected Poles A B



Alternative Network for Poles A & B.

Above 1 and 2 represent the same Squared-Rectangle (it does not matter which one here), and it will now be shown how the second can be found from the first .

1. Firstly observe that the circuits in above one are 3-3-3-3-3 and the terminals in Above two are also 3-3-3-3-3 if the poles at C and D are ignored.
2. Reversing this circuits in above two are 3-4-4-5 and terminals in Above 1 are also 3-3-4-5 if the poles at A and B are ignored.
3. Now notice that there are 2 wires at A and 4 wires at B - total 6. There are 6 outer wires in above two.
4. Reversing no 3 there are 4 wires touching c and d. there are 4 outer wires in above 1.
5. The circuits surrounding b are 3-3-3 and at A, 3. Likewise the terminals between C and D are 3 and 3-3-3.
6. Reversing no. 4 the circuits surrounding C and D are 4 & 5. Likewise the terminals between A and B are 4 and 5.

From the above information, Above 2 can be found by degrees - the important thing to remember is that the terminals numbers in one must translate to the circuits numbers in the other, and vice versa, throughout.

Though not easy it can be done with patience.

B9.5.1. OBTAINING ALL THE OTHER CIRCUITS RELATED TO A NETWORK

Above 1 happened to contain only one choice of poles at A and B. But for many networks many choices are possible and using the above guides it is possible to construct all the connecting networks by the same method.

These networks are unrelated to each other, but all relate to the original.

B9.5.2. CHECKING TWO GIVEN NETWORKS WILL FORM THE SAME OR DIFFERENT RECTANGLE

Suppose above 1 and 2 are given to us, it is possible from the guides given in B9.5 to establish that they are the same Squared-Rectangle. Conversely, had they related to different rectangles, the guides would prove incorrect sooner or later. This has important consequences in the study of Squared-squares where two suitably different networks are found with the same Complexity value, so that a check for a Squared-square can be made.

By investigating various combinations of poles available. See Section L for more on this.

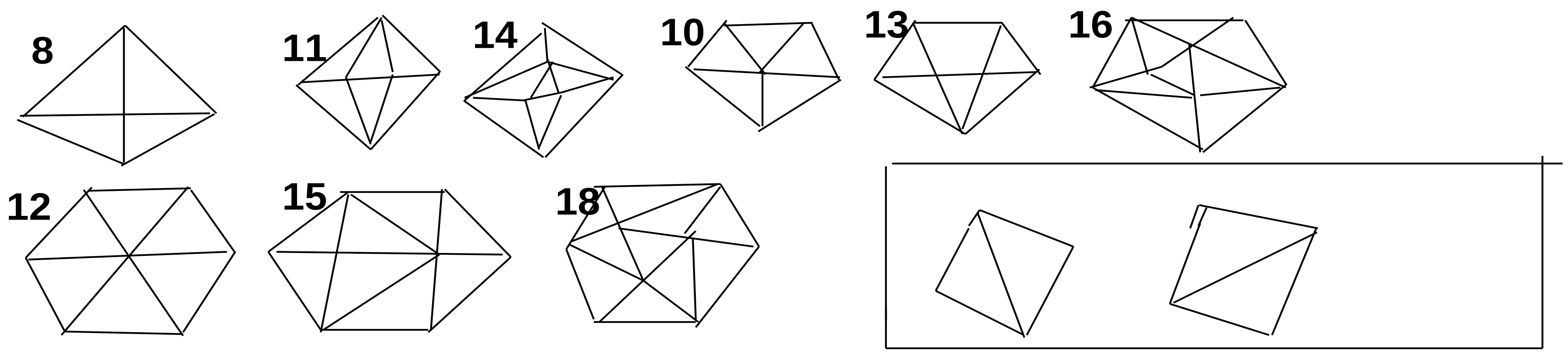
B9.6. MORE ABOUT NETWORKS

In the following, external Smith Diagram wires are shown as “e”, and external wires as "f". Note that internal wires sometimes represent external Elements in the actual rectangle.

Clearly e can be any integer of at least 4, but t can be any integer from 3. Inside any network are points each of which must have a minimum of three wires radiating from it. These are shown as “p” and are any integer from 1 up.

B9.6.1. MAXIMUM ORDERS RELATING TO POINTS AND EXTERNAL WIRES

It is easily seen that the highest Order to be found must be one which consists entirely of triangles. Below are networks for e4-p1 e4-p2 e4-p3 … to e6-p3. When these are extended a formula for the maximum Order becomes evident.



NETWORKS WITH 1,2 & 3 POINTS SHOWING MAXIMUM ORDERS

For each group the Order conveniently increases by three each time, with external wires from 4 upward and points from 1 upward. Notice the several possible patterns for some, but notice also that this will not alter the Order, e.g. where 4 wires appear in a circuit a line can be drawn left-right or right to left (above 10 and 11) to form the necessary triangles.

p = 1 or one point orders 8 10 12 14 16 18… for t = 4,5,6,7,8,9…

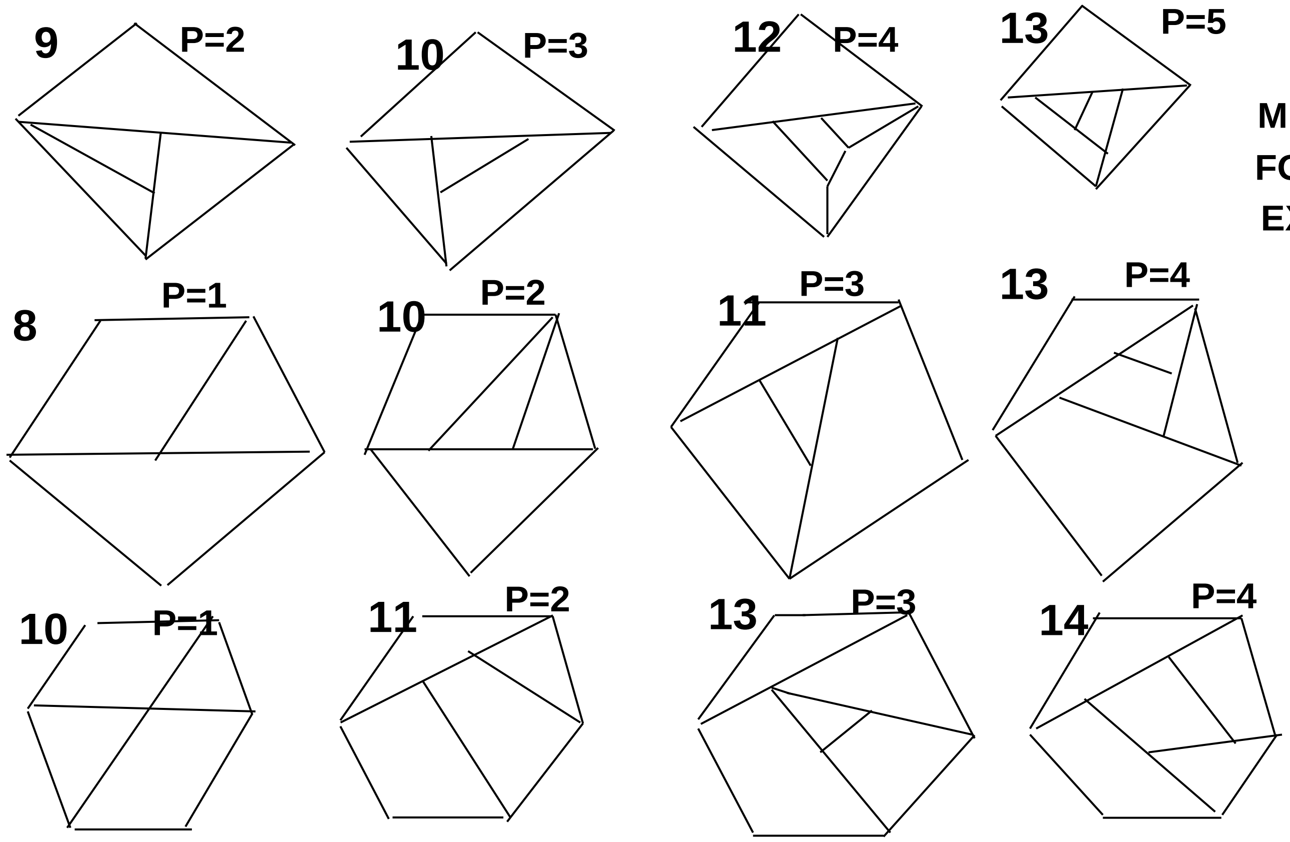
p = 2 orders 11 13 15 17 1 9 21... p = 3 orders 14 16 18 20 22 24… p = 4 orders 17 19 21 23 25 27…

From this the formula of <u>MAXIMUM ORDER = 2e +3t - 3</u> can be found.

B9.6.2. MINIMUM ORDERS RELATING TO POINTS AND EXTERNAL WIRES

This is more complicated! Where e=4 it is best to show a S2223 solution, that is, show one pole with 2 wires only.

Where e > 4 it is best to show both poles with 2 wires only. Each point must have at least three wires joining it, so the trick is to make it only 3 wires. The individual circuits will often have 4, 5 + wires, but some are forced to be only triangular.



E=4
MINIMUM ORDERS
FOR POINTS &
EXTERNAL WIRES

E=5

E=6

Table of values of minimum orders

Minimum	e = 4	e = 5	e = 6	e = 7	e = 8	e = 9
p=1	7 *	8	10	12	14	16 then by intervals of two
p=2	9 *	10 *	11	13	15	17 then by intervals of two
p=3	10 *	11 *	13 *	14	16	18 then by intervals of two
p=4	12 ***	13 **	14 *	16 *	17	19 then by intervals of two
p=5	13 ***	15 ***	16 *	17 *	19 *	20 then by intervals of two
p=6	14 ***	16 ***	17 *	19 *	20 *	22 * then 23 and intervals of two

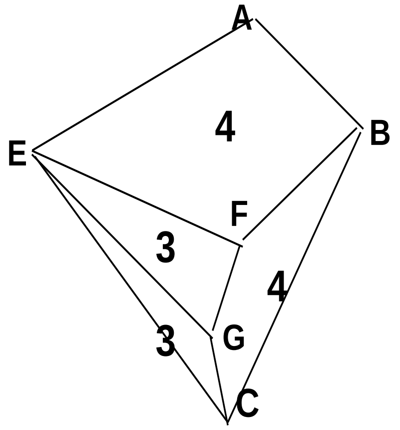
In any network the network can be considered in three parts, namely
1. Wires on the edge, or external wires = e and this amount is fixed regardless of minimum or maximum orders.
2. Wires from edge to a point = d.
If e = 4 d can be 3 (allowing for a single pole of 2 wires), or higher.
If e is 5 or more d can be e - 2 (allowing for two poles with 2 wires), or higher.
3. Wires connecting points = c. (d + c = f the amount of internal wires).
As 3 points can be joined by only 2 wires, 4 by 3, 5 by 4 points can be joined by p-1 wires, or higher. Putting these together if conditions are suitable, the minimum Order is e (from 1.) plus e - 2 (from 2.) + p - 1 (from 3.) that is 2 x e + p - 3. However, sometimes an extra wire or more is necessary to prevent some terminals (other than poles) being joined by only 2 wires. Hence the lack of exact pattern in the above table, where 2 x e + p - 3 applies to some values only. The amount of asterisks is the amount of extra wires found necessary.

If e is at least 4 more than p there is no problem and the minimum Order is then 2x e + p - 3.

B9.7. QUANTITY OF SOLUTIONS FOR EACH ORDER

If a network for particular e and p values is selected showing a minimum Order, it is possible with some problems to find the amount of solutions possible for it and higher orders.

At present, the possibility of duplications is ignored. Look at the diagram below e4 and p2 Order 9:-



A LINE AT EB OR AT AF CAN BE DRAWN.

ALSO A LINE BG OR AT FC CAN BE DRAWN.

THERE ARE 4 WAYS OF INCREASING THE ORDER BY 1.

THERE ARE ALSO 4 WAYS OF INCREASING THE ORDER BY 2, EB & BG, EB & FC, AF & BG, AF & FC, EACH OF WHICH CONVERTS THE NETWORK INTO TRIANGLES ONLY - WHICH MEANS NO HIGHER ORDER RECTANGLES CAN EXIST.

Here is 1 solution for Order 9 since poles at EB cannot be chosen. 4 patterns for Order 10 and a further 4 for Order 11. Nothing is possible for orders 12 and beyond.

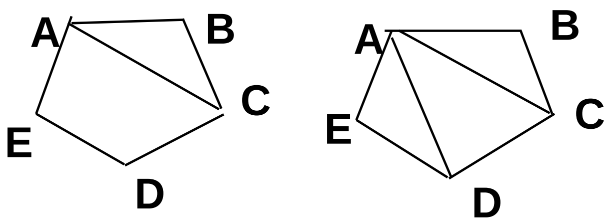
If AF is drawn, note that poles ED become valid. Therefore, there appear to 1 x 2 + 3 x 1 or 5 solutions of Order 10 possible without altering a, b, c, d, e, f, g in any way. Also there are 2 x 2 + 2 x 1 or 6 solutions for Order 6.

12 solutions are possible if none are duplications.

Further wires can be drawn for circuits of 4, 5, 6, 7… but not of course 3, and there are a fixed amount of combinations possible. For 4 we have seen two combinations are possible, i.e. AF and EB in above.

B9.8. COMBINATIONS POSSIBLE FOR CIRCUITS OF 4, 5 AND 6

It is easily seen that 2 lines can be drawn within a circuit of 5, 3 within 6, 4 within 7 and so on. Finding how many combinations of 1,2 and 3 lines etc. can be drawn for circuits of 5,6,7 etc. can be confusing as it is easy to include unwanted duplications if not done with care.



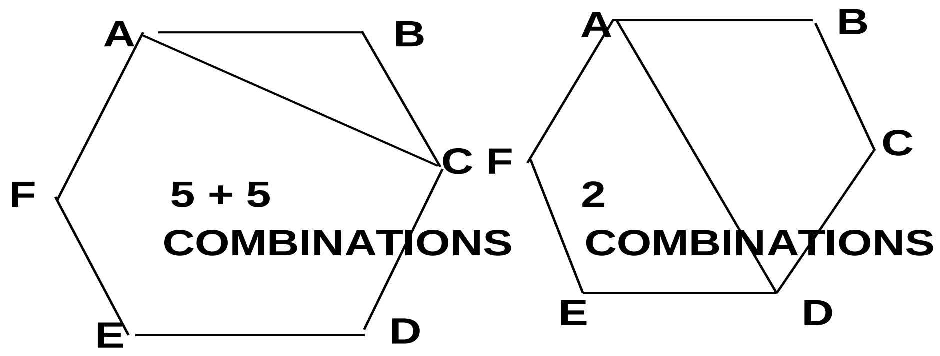
INDIVIDUAL CIRCUITS OF 5 (NOT COMPLETE NETWORKS)

DRAWING SINGLE WIRES IS POSSIBLE AT

AC AD BD BE & CE 5 POSSIBILITIES

DRAWING TWO WIRES - AC+AD, BD+BE, CE+CA, DA+DB & EB+EC ARE 5 POSSIBILITIES

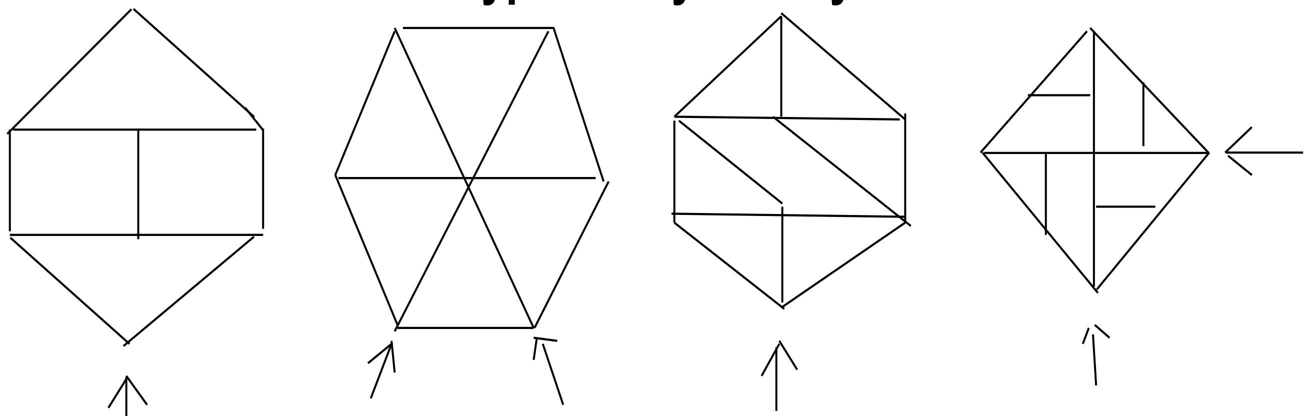
Hexagonal circuits are more complicated! Below there are 9, 24 and 12 combinations for 1, 2 and 3 lines - a total of 45.



1 LINE AC....FB (6) & AD,BE & CF (3) = 9 COM
2 LINES AC+AD,AE,CF,CE 4x6 = 24 COMBS
3 LINES - AC+AD+AE & SIMILAR (6)
ALSO "N" SHAPED (6) = 12
45 COMBINATIONS!

B9.9. DETECTING SYMMETRIC NETWORK PATTERNS

Below are various types of symmetry where e = even numbers



2-FOLD 6-FOLD 2- FOLD ROTOR 4- FOLD ROTOR

B10. CALCULATING FROM SMITH DIAGRAMS

B10.1. CALCULATING RELATIVE VALUES OF CURRENT IN SD'S - INTRODUCTION

In certain respects there are some similarities in calculating out smith diagrams as in Squared-Rectangles but it is a little more tricky with Smith diagrams for at least two reasons -

1. Difficult to decide which wires to choose as x and y
2. The direction arrows are often confusing. Only calculate from Diagrams if absolutely necessary. Always use rectangles if available. You may consider translating to a rectangle is desirable before calculating!

B10.2. CALCULATING VALUES - UNKNOWNS xy - CHOOSING WIRES FOR x AND y

1. To calculate out the diagram below 1, first show the direction of current for each external wire and those adjoining the poles. Make arrows face downwards. The # wires are left. Guess directions for these. Possibly downwards is best but avoid a > < > flow. e.g. Make CE ED DF > > > and AB BD DF > > >

2. Choose two unknowns x and y. some practice is necessary but as a guide A. choose 2 internal wires B. meeting at a point where only c. 3 wires join. BE and ED will do. It is now stressed that there are many differing ways of calculation possible and the method shown is just one of many.

B10.3. CALCULATING BIT BY BIT A SMITH DIAGRAM

1. Call BD = x and CD =y. consider terminal D. now DF is the addition of both (imagine two rivers flowing into one) so is x + y.
2. Now look at the circuit BDF the current BD + DF = x + x + y so BF is 2x + y. Note that a current of 2x + y flows in one direction and 2x + y in the other. Do not be fooled by the fact all three arrows point downwards!
3. Now EF at terminal E = DE + DF which is x + 2y. Again observe the flows of x + 2y in two directions.
4. Now CE = DE + EF = y + x + 2y or x + 3y, at terminal E.
5. Now consider circuit ACED. In one direction there is x + 3y and y (= x + 4y). In the other x and BC. x + 4y - x gives 4y for BC. at terminal C AC = BC + CE = 4y + x + 3y= x + 7y. Lastly at circuit ABC BC + CB = AB
so AB = x + 7y + 4y = x + 11y. The values are now complete.

B10.4. FINDING THE EQUATION AND RELATIVE VALUES OF x AND y

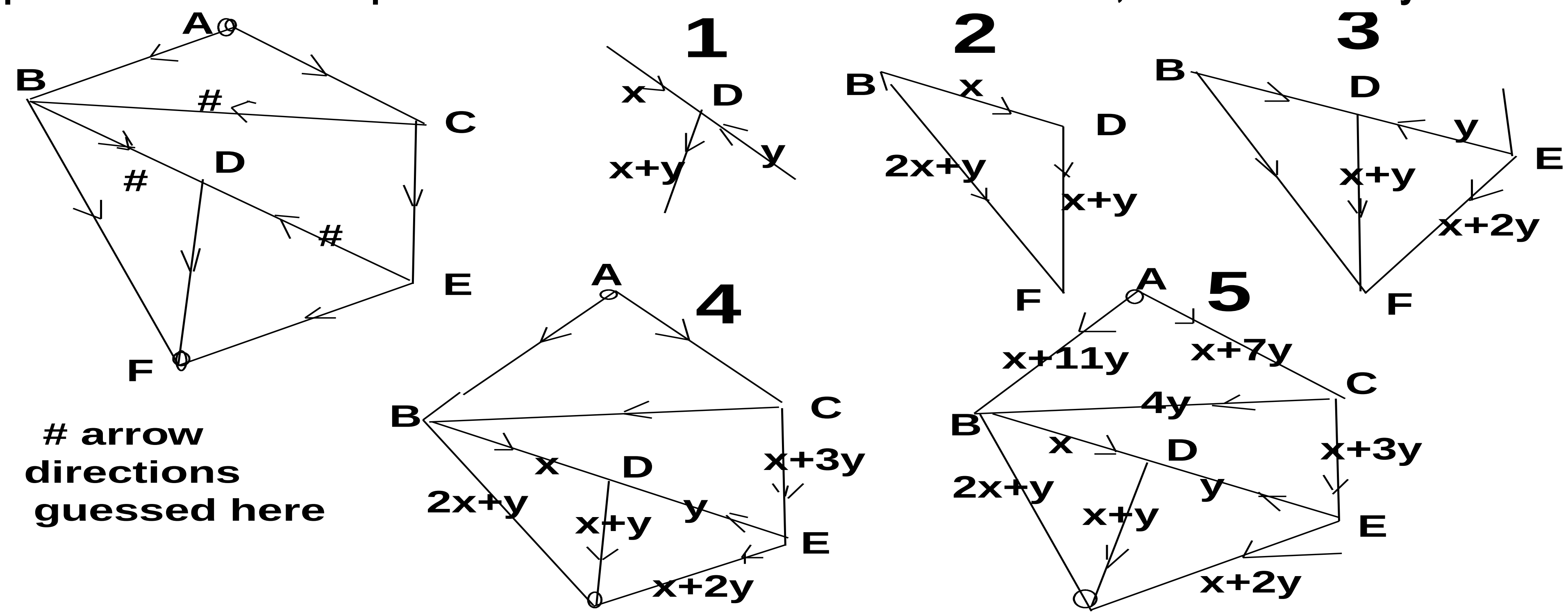
6. Consider terminal B where arrows towards B total x + 11y + 4y = x + 15y. This equates with arrows from b total x + 2x + y = 3x + y. by Kirchoff's law x + 15y =3x + y. 14y = 2x which is satisfied with x = 14 and y = 2. Both have a common factor of 2, so cancel down to x = 7 and y = 1.

Note 1. If minuses occur reverse the direction of the arrow(s).

Note 2. If an expression such as 5x + 7y = 5x +7y occurs this is an Invalid equation and another terminal or circuit will need to be chosen.

B10.5. CALCULATING ACTUAL VALUES OF ELEMENTS

7. Finally calculate each value in turn and translate diagram into a rectangle. [9] 33x32 is obtained. Note 3. You can equate the wires at the poles or between the poles instead - one is fine the other useless, so look carefully.



After some practice the instructions will be clearer.

B11. xyz CALCULATION WITH xyz

B11.1. CALCULATING WITH MORE THAN TWO UNKNOWNS x y z

Often with a more complicated Smith Diagram, only a small proportion of the diagram can be expressed in terms of x and y. When this happens, a suitably adjoining wire is to be called z and the remaining wires calculated in terms of x y and z.

It may be that a 4th 5th… Unknown may prove necessary.

The amount of equations required is always one less than the amount of unknowns employed. Some equations may not include every unknown. With xyz simultaneous equations like $3x + 2y - 5z = 0$ and $4x + 5y = 7z$ are typical.

x y or z may turn out positive or negative or even zero, and if fractions arise will need suitable multiplying to make them all integers. As already mentioned calculating out Rectangles rather than Smith Diagrams is always preferable.

The procedure is pure algebra so no more comment is necessary in this book.

B12. ONE AND TWOFOLD SYMMETRY

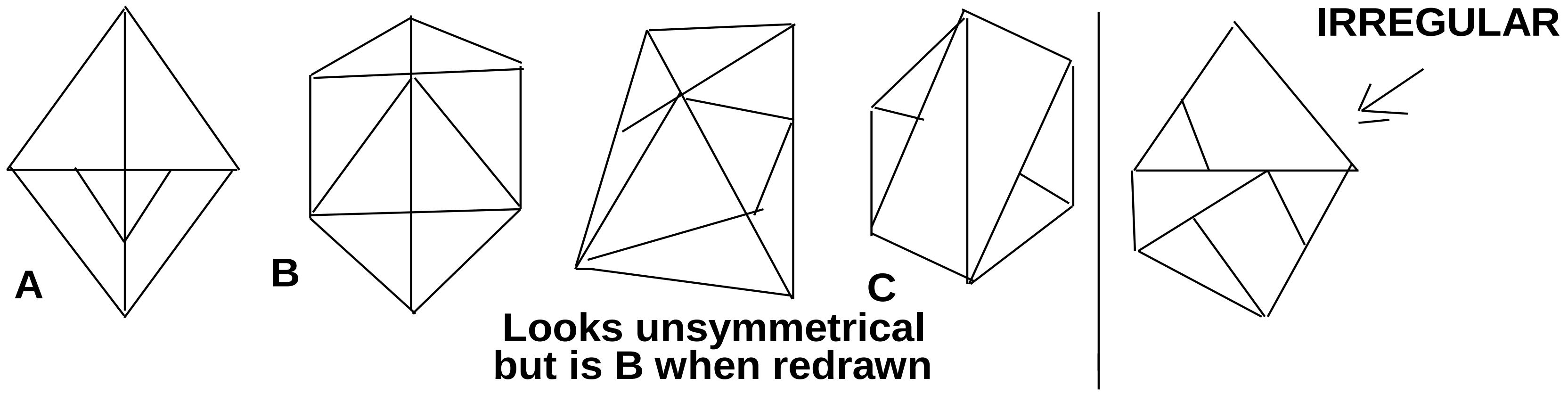
B12.1. SMITH DIAGRAM SYMMETRY

1 Fold symmetry or Asymmetric is just another way to describe irregular patterns in Smith Diagrams, and to distinguish them from 2 and higher fold symmetry. Whereas all symmetric rectangles give symmetric diagrams, the opposite is not true. If poles which are not symmetrically aligned are chosen in a 2-fold diagram an asymmetrical rectangle always results.

The same is true for 3, 4, 5, 6… fold.

Note that symmetric Smith Diagrams are not always immediately recognizable as such, as in below 3.

EXAMPLES OF 2 FOLD SYMMETRY IN SMITH DIAGRAMS & 1 FOLD i.e.

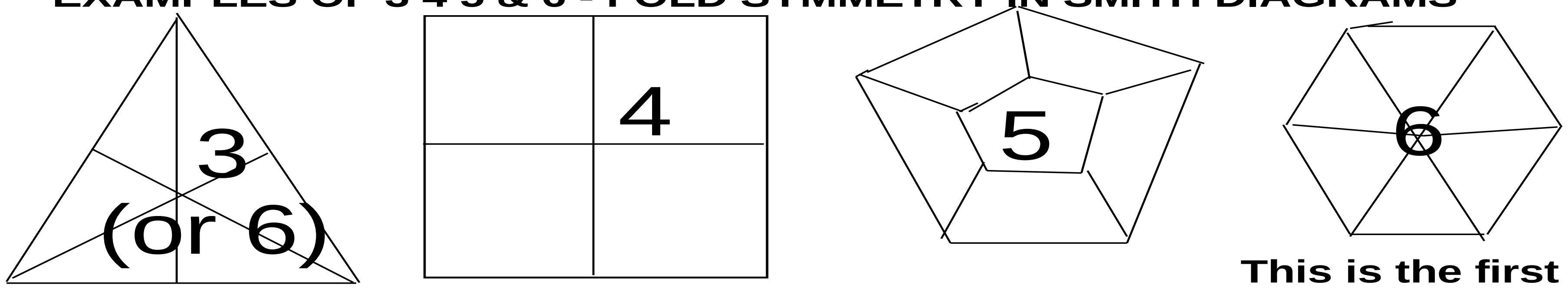


NOTE THAT SYMMETRICAL PATTERNS CAN GIVE ASYMMETRIC RECTANGLES AS WELL AS SYMMETRIC ONES, ACCORDING TO POLES.

B12.2. THREEFOLD AND HIGH SMITH DIAGRAM SYMMETRIES

Smith Diagrams can be conveniently shown in triangular form for 3-fold, square form for 4 fold, pentagonal form for 5 fold and hexagonal form for 6-fold patterns

EXAMPLES OF 3 4 5 & 6 - FOLD SYMMETRY IN SMITH DIAGRAMS



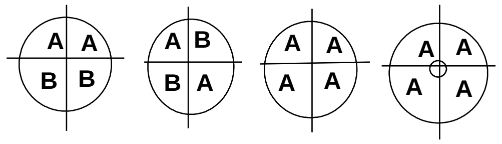
Could also be regarded as 6-fold Symmetry.

This is the first diagram redrawn.

B12.3. SYMMETRIC NETWORKS

Symmetry in networks can be confusing and it is necessary to define the different types. To do this the networks is shown in circles. (See later for O smith diagrams).

Imagine the network divided into four quadrants some or all of which are the same as indicated below:-



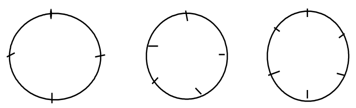
A & B DENOTE DIFFERENT PATTERNS WHICH ARE REPEATED WITHIN THE QUADRANTS SHOWN.

ADJACENT SYMMETRY CROSSING SYMMETRY FOURFOLD SYMMETRY CLOCKWISE SYMMETRY

To show the symmetry, draw a circle and fix a point at top middle.

(Note there are some patterns which appear to require points elsewhere, but these can be suitably redrawn to introduce the fixed point, so this is not a problem).

Next the amount of external Elements is spaced out evenly as below-

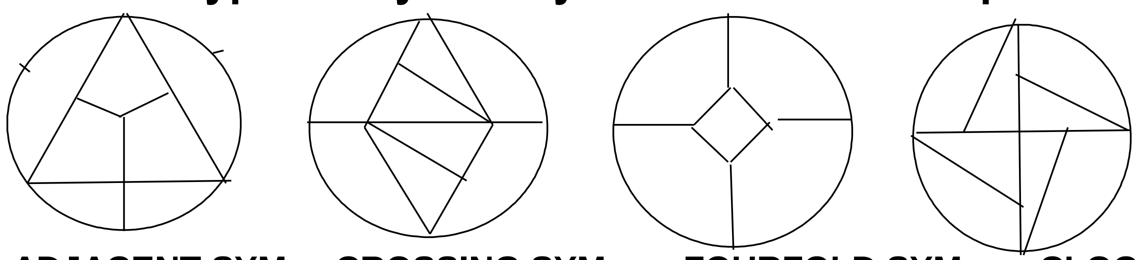


DIVIDING NETWORK FOR 4, 5 OR 6 EXTERNAL ELEMENTS

External Elements refers to the amount of Elements down the left side of a rectangle plus the number at the right.

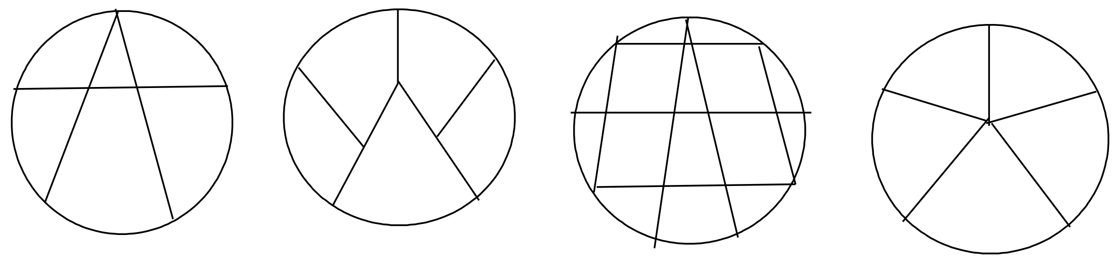
This may be even or odd. When odd, the only symmetry possible is found to be <u>adjacent symmetry</u>, whereas in even quantities of Elements all four types are possible.

Where the external Elements are an odd number, surprisingly symmetric Squared-Rectangles do result as can be seen Below. Examples of the four types of symmetry are shown for comparison -



ADJACENT SYM. CROSSING SYM. FOURFOLD SYM. CLOCKWISE SYMMETRY

AA AB AA AA
BB BA AA AA



ADJACENT SYMMETRY FOR ODDS (5 & 9 ELEMENTS EXTERNAL)

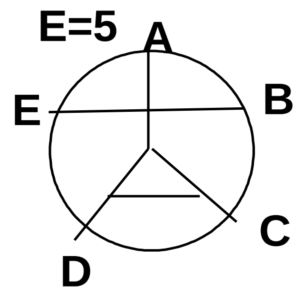
B12.4. CHOICES OF POLES WITH SYMMETRIC NETWORKS - ODD SERIES

The patterns below deliberately have at least one wires touching any point of the circles. (If no wire touches then obviously that point has to be a pole and the choice severely limited, and is less than that shown below).

The points are labelled ABCDE .. Clock-wise. # means actual rectangle asymmetric and * symmetric.

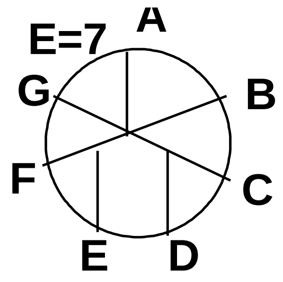
Many other patterns could have been shown, but this does not affect the pole choices.

3 out of 5 are possible in e = 5 cases, 8 out of 14 for e = 7 cases and 15 out of 27 for e = 9 cases.

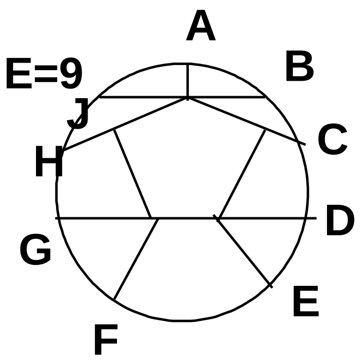


POLE CHOICES

#AC (=AD)
#BD (=CE)
*BE   3 OUT OF 5



#AC (=AF)
#AD (=AE)
#BD (=EG)
#BE (=DG)
#BF (=CG)
*BG
#CE (=DF)
*CF   8 OUT OF 14

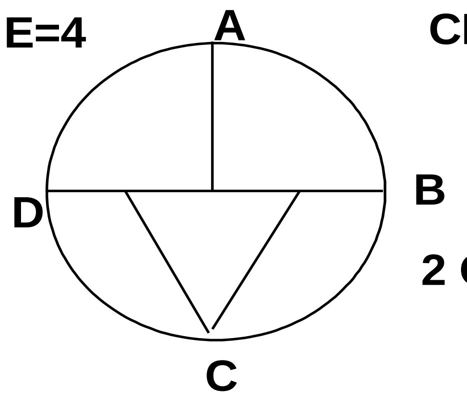


#AC (=AH)
#AD (=AG)
#AE (=AF)
#BD (=GJ)
#BE (=FJ)
#BF (=EJ
#BH (=CJ)
*BJ
#CE (=FH)
#CF (=EH)
#CG (=DH)
*CH
#DF (=FG)
*DG

15 OUT OF 27

B12.5. CHOICES OF POLES FOR SYMMETRIC NETWORKS - EVEN SERIES

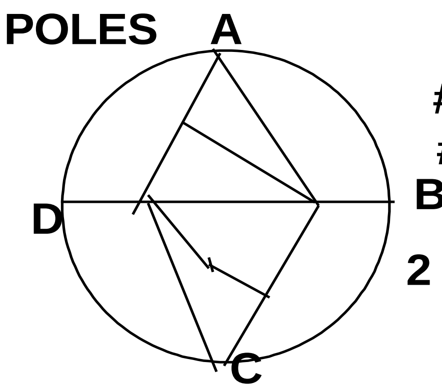
As previously explained there are four types, not one as in the odd series, but as two give the same results, only three are shown. Firstly, e = 4 cases.



CHOICES OF POLES

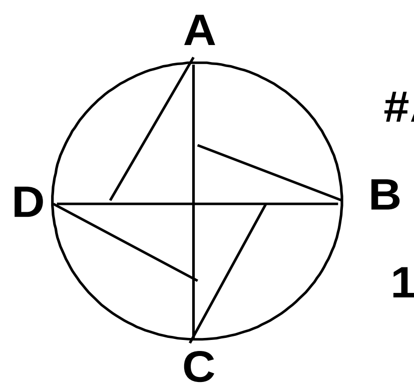
#AC
#BD

2 OUT OF 2



#AC
#BD

2 OUT OF 2

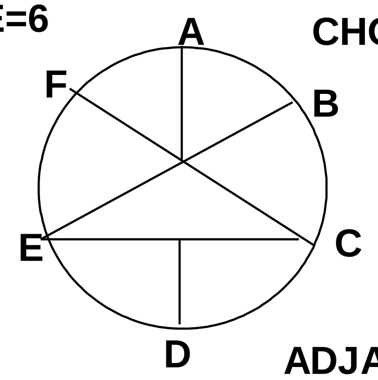


#AC (=BD)

1 OUT OF 2

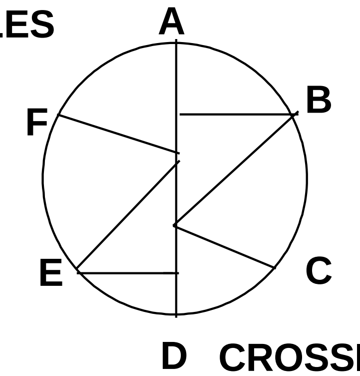
Now for e = 6 cases - the chart Below shows variations in the amount of choices of poles - 2,4 or 6 according to the type of symmetry. Once again some rectangles are symmetric and the others not.

E=6 CHOICES OF POLES



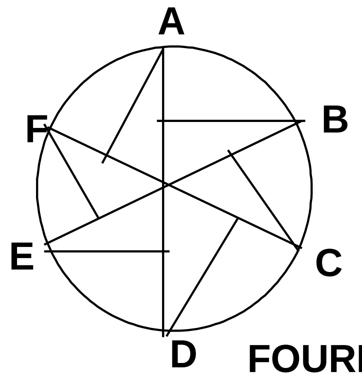
ADJACENT

#AC=AE #BD=CF
*AD *BF
#BE=CF *CE
6 OUT OF 9



CROSSING

#AC=AE,BD & DF
#BF=CE
#BE=CF *AD
4 OUT OF 9



FOURFOLD & CLOCKWISE

#AC=BF,BD,CE,DF& AE
*AD=BE & CF
2 OUT OF 9

B13. FIXED SMITH DIAGRAMS

B13.1. FIXED DIAGRAMS IN GENERAL

Smith Diagrams are frustrating in several ways. Not only are there 2 diagrams for each rectangle with 4 possible drawings for each, but the usual means of drawing them is Unfixed. Although attempts to make rules for fixing them is fraught with problems there are a number of ways of fixing. However an all-embracing system which caters for everything seems to be impossible to find. The next sections deal with 4 distinct types each with advantages and disadvantages. For example the y and + representations only cater for xy and xyz solutions respectively, and some xy solutions defy a y type representation.

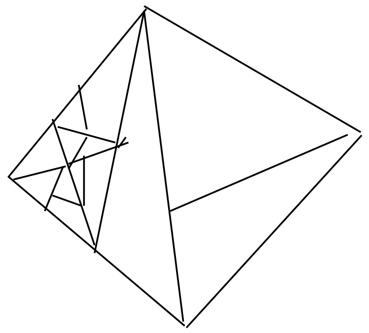
The circular system is a good one but fails in representing symmetric Smith Diagrams in a symmetric fashion.

Ideally we need a system for which each rectangle has one and only one fixed Diagram where the x and y types can have several ways of representing a single solution.

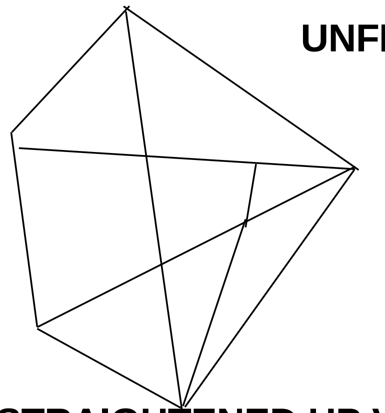
Another problem encountered is that after drawing a Diagram, a part may be rather empty whilst another so overcrowded to be unreadable. There is probably some advantage in having more than one wire in a straight line, that is have as few straight lines as possible. See Below.

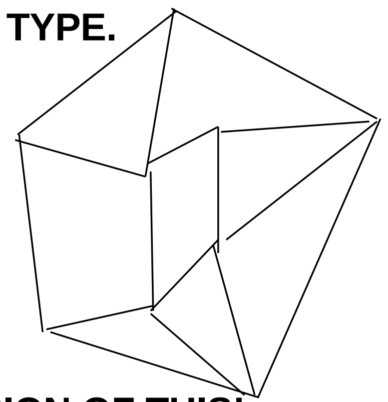
GOOD & BAD FORMATS OF SMITH DIAGRAMS.

UNFIXED TYPE.



OVERCROWDING CAN BE OVERCOME BY REDRAWING.



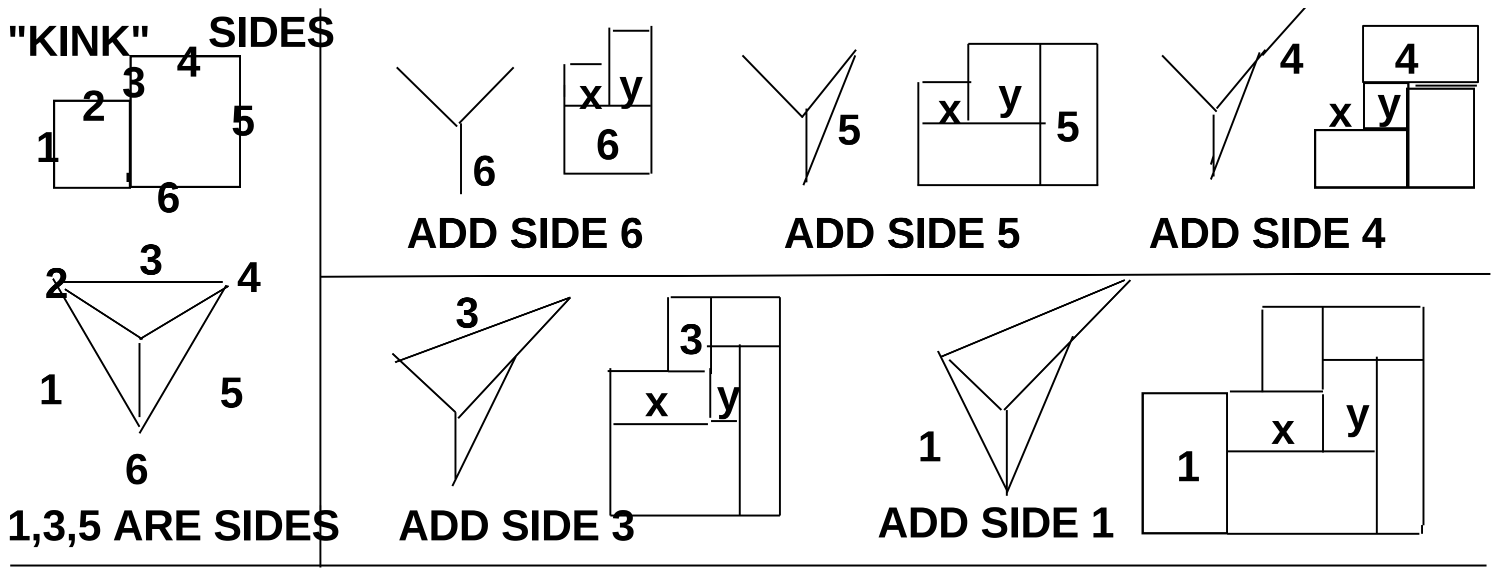


STRAIGHTENED UP VERSION OF THIS! THE FIRST LOOKS BETTER AS WELL.

B13.2. FIXED DIAGRAMS AND SYMMETRIC SMITH DIAGRAMS

Ideally a fixed diagram needs to be found which will look symmetric when it is actually symmetric. This is difficult to achieve.

B13.3. Y TYPE SMITH DIAGRAM REPRESENTATION



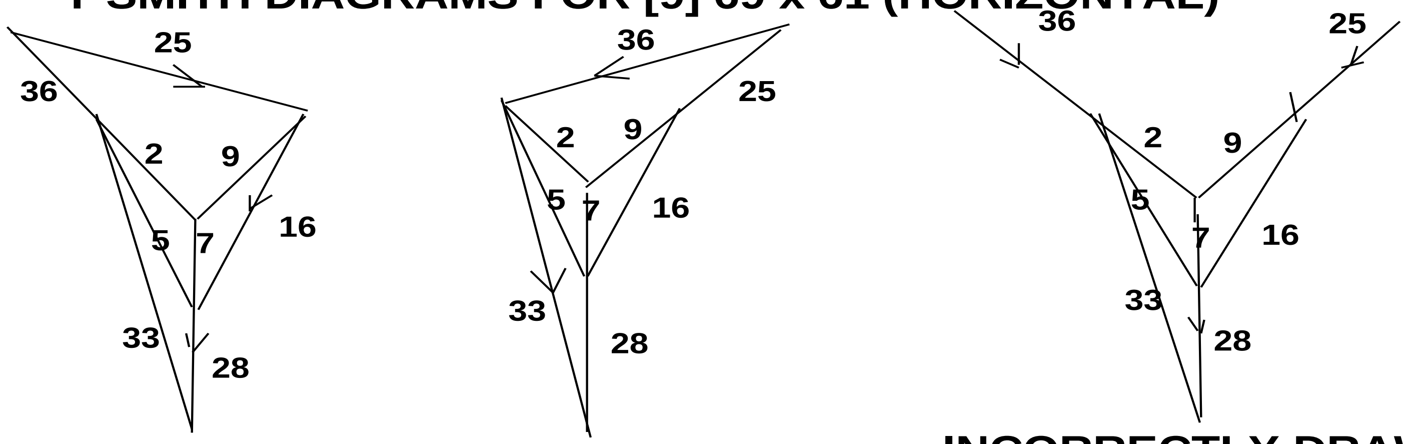
2,4,6 ARE POINTS WHICH FORM THE PRONGS OF THE Y SHAPE, THE SIDES 1,3,5 CROSS FROM ONE PRONG TO ANOTHER.

The type of construction above works only on xy solutions, and is largely explained above. x acts as side 2 a prong “\” and y acts as side 4 a prong “/” Side 6 representing x + y points downwards from the central point. The numbering system is shown (other positions could have been chosen).

B13.3.1. ACTUAL EXAMPLES OF Y SMITH DIAGRAMS

Below are three ways of representing [9] 69 x 61 vertical complexity 61. In completing the rectangle the Elements 25 and 36 can be added as sides 2 and 3, or 4 and 3 or 2 and 4 as shown in Below 1, 2, 3. The last is satisfactory as the upper pole cannot be in two places at once, and is ignored. We will regard the last Element as side 3 the “kink” having been assumed to occur at top left. (The format could have been done several other equally valid ways). In Below x and y are -2 and 9.

Y SMITH DIAGRAMS FOR [9] 69 x 61 (HORIZONTAL)

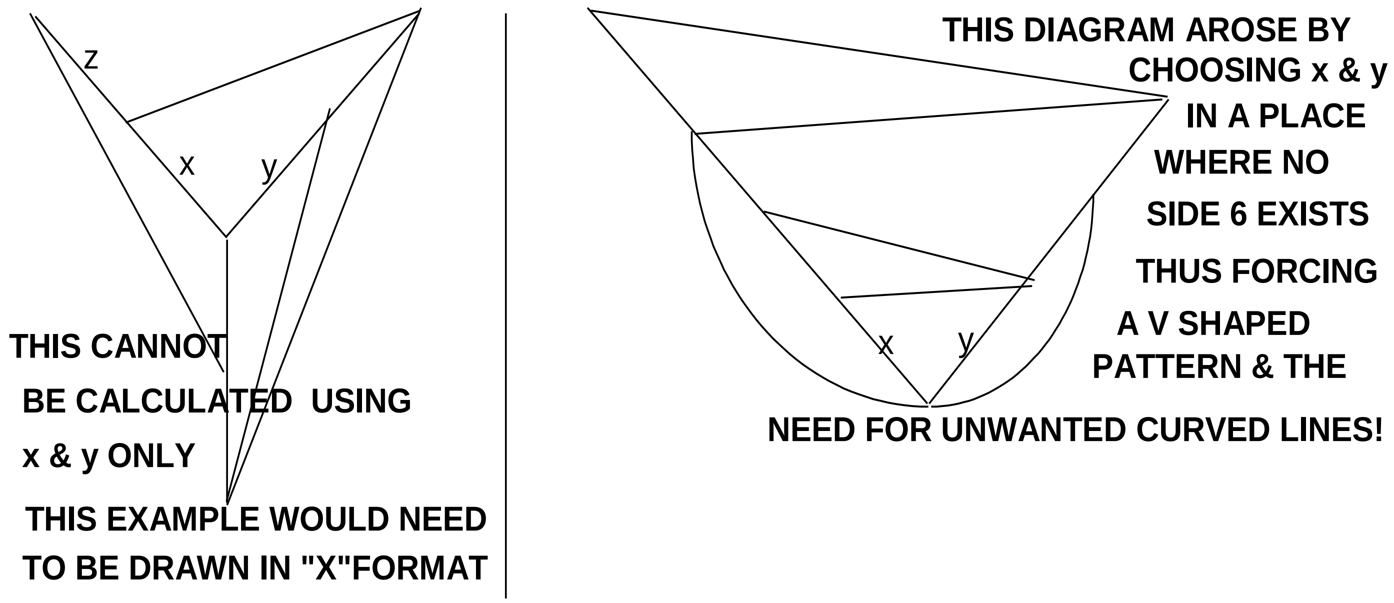


CORRECTLY DRAWN BUT WITH TWO DIFFERENT TERMINATIONS

INCORRECTLY DRAWN

This is not ideal but unavoidable in this case. There are other problems too. Some x and y choices will not have an x + y Element at position 6 available and a “v” diagram results - in such cases sides 1 and 5 may require curved lines to make sense! If y diagrams are drawn randomly and calculation attempted, it sometimes happens that 3 or even more unknowns are necessary to finish it. See Below. Although y diagrams and the related x and star diagram, which are mentioned later, are interesting they do not seem useful enough to adopt!

EXAMPLES OF WRONG DIAGRAMS!



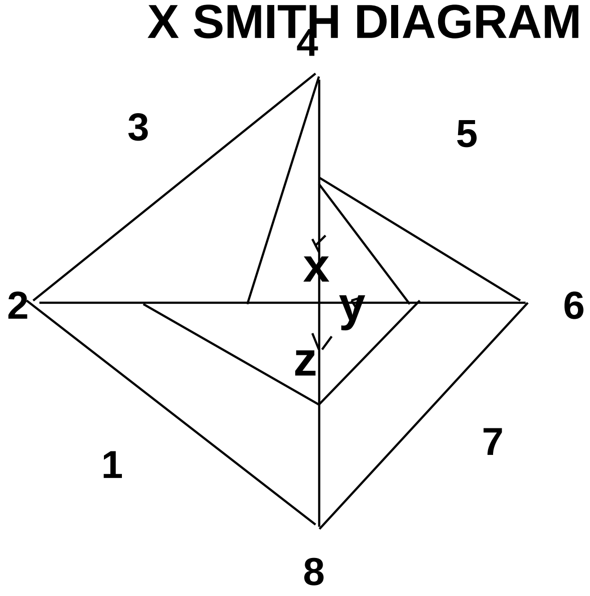
B13.4. X TYPE SMITH DIAGRAM REPRESENTATION

Below shows the format, which requires x y and z to all border an Element at x + y + z. Since any xyz type solutions require z to be detached from x and y this means this representation could not be used for all xyz solutions.

B13.5. STAR TYPE REPRESENTATIONS

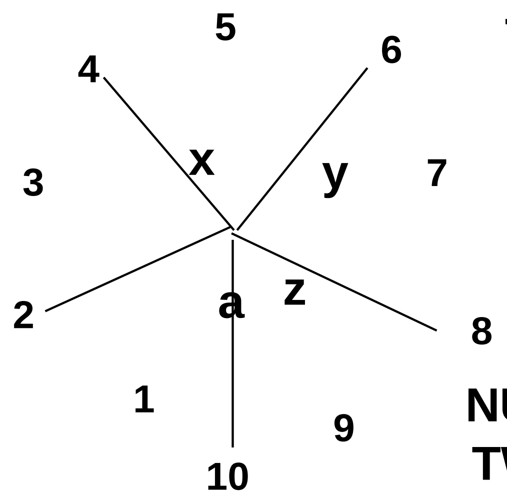
Also shown below is a format for xyza solutions. Again this requires all unknowns to be in line together, whereas z and a will often need to be detached to permit calculation. The idea can be extended to have 6, 7, 8 prongs etc.
These have been shown for reference purposes only as they are not worth using in practice!

X SMITH DIAGRAM



3 UNKNOWNS xyz

STAR SMITH DIAGRAM

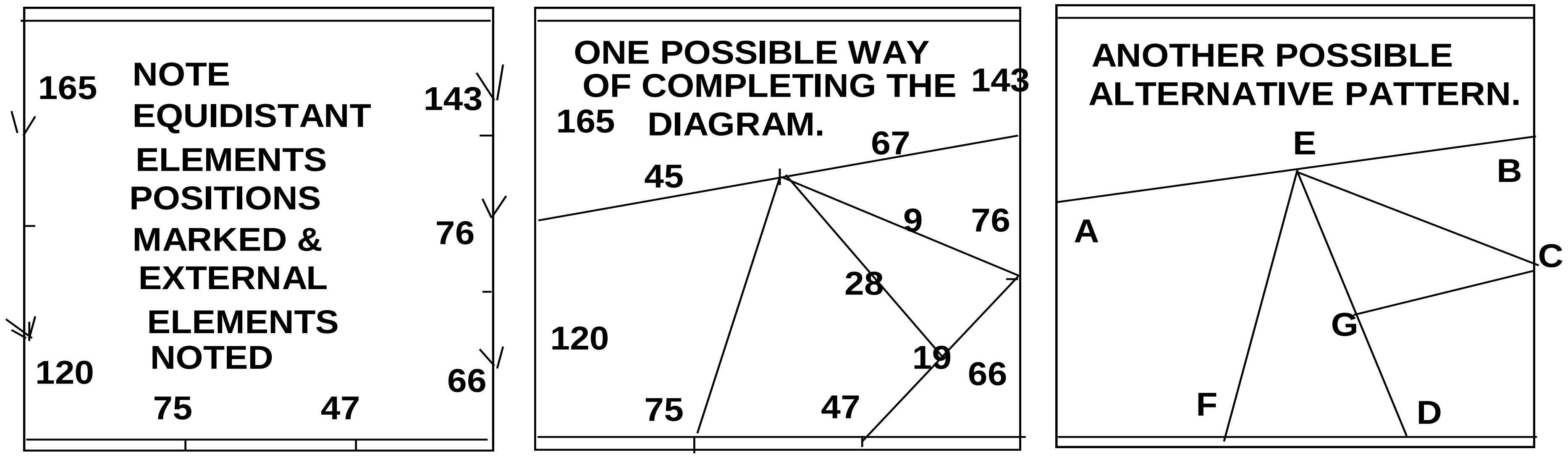


4 UNKNOWNS xyza

THE NUMBERS INDICATE THE SIDED ADDED. AS IN "Y" TYPE EVEN NUMBERS WORK ALONG THE PRONGS & ODD NUMBERS BETWEEN TWO PRONGS.

B13.6. SQUARE SMITH DIAGRAM REPRESENTATION

Variation on the normal Diagram is to replace the poles with a thin line at top of bottom and turn the structure into a square. An attempt is then made to make which will hopefully fix the diagram.



As always the positioning of the internal points causes the main problems. In the Above case where there are only two at E and G E could be on direct line between AB AC or AD and G on line between CD AD DE…

B13.6.1. ADVANTAGES OF BOX SYSTEM

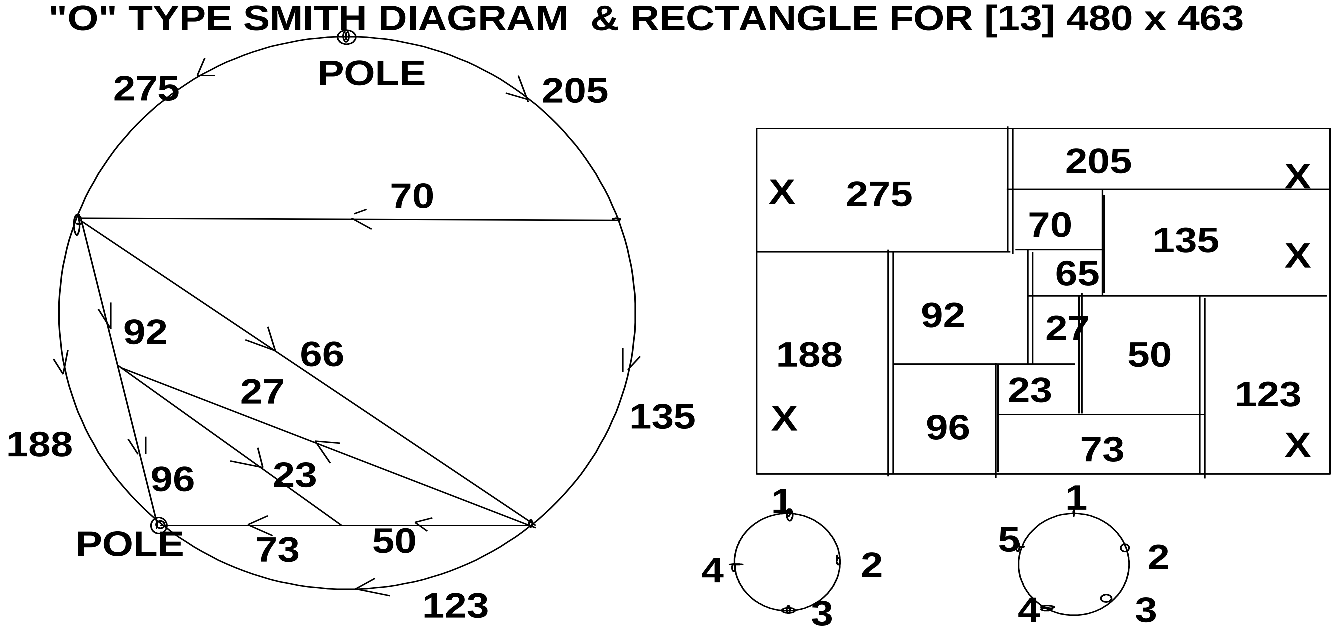
1. Possible congestion at poles avoided.
2. Fixed square shape and equidistant throughout.
3. Poles clearly defined.

B13.6.2. DRAWBACKS OF BOX SYSTEM

1. Difficult to fix as so many combinations possible.
2. Pole positions fixed are so useless as a network which disregards poles.
3. Symmetric patterns will not be obvious. etc...

B13.7. O OR CIRCULAR SMITH DIAGRAM REPRESENTATION

"O" TYPE SMITH DIAGRAM & RECTANGLE FOR [13] 480 x 463



Above shows an O Smith Diagram and the connected rectangle where the external Elements in the diagram are indicated by X's.

B13.7.1. RULES FOR CONSTRUCTING O SMITH DIAGRAM

Firstly arrange the rectangle horizontally so that the largest corner Element is at top left. There is a general rule that any internal lines drawn inside the circle will need to show equidistant points on it if 2 or more Elements are to show along it. Then-

1. Show positive pole at top of circle.
2. Divide the circle into the same amount of points as there are Elements marked x in above 2.
There are 5 so 360° / 5 = 72°. See above 4.
3. Mark the bottom pole position. This will sometimes be at the bottom of the circle, but otherwise somewhere on the left hand side.
4. Show outside Elements with arrows i.e. those marked x in above 2. (B13.6.1)
5. If there are 3 or more Elements along the top of the rectangle go to 6. If only 2 then draw a straight line from the two points nearest the pole. Look to see which Elements connect them. In Above 2 only 70 connects 275 to 205. If more than one Element needs to be inserted divide the line up and insert each, taking care to get the correct arrow directions.
6. Now work round the edge of the circle clockwise from the top pole or point 1. If 5 applied go to the next point 2.

Points 2-3 how is 135 made up? With 70 and 65. But note 70 already inserted so draw a line from points 5-2 and insert 65. Points 3-4 how is 123 made up? 50 + 73 neither number has been used so draw a direct line from p3-4, divide in two and insert 50 and 73. Points 4 -5 how is 188 made up? 92 and 96. Neither number yet used so direct line p4-5 drawn and 92 and 96 inserted. Lastly (in Above example) points 5-1 no action - already occupied.

7. Final stages. It is difficult to give precise rules which can be literally applied for all rectangles. Further lines drawn will connect with the points not touching the circle. In Above example these fall within the triangle p3 - p4 - p5.

Firstly look to see if any vertical groups of 3 are left. Yes 23 + 73 = 96 , so 23 can be drawn as shown.

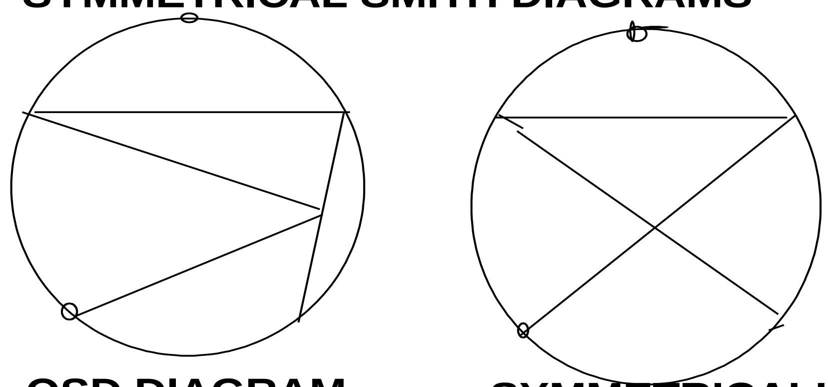
Now 27 borders 50 and 23 (or 65 and 92) and can be inserted. Finished! In tricky cases it may seem that bent lines are necessary but with care this can be avoided. Try driving a line through the middle and then fitting in remaining Elements on either side.

B13.7.2. DRAWBACK OF O SMITH DIAGRAM SYSTEM

Although with definite advantages O Smith Diagrams have these snags -

1. In higher orders crowding of Elements will occur - more annoying still where other parts very sparse.
2. The 2nd pole is often not at the bottom of the circle.
3. Ideally solutions with the same complexity should look identical except for the choice of poles, but do not in O DIAGRAMS. 4. Where the diagram is actually symmetric this does not show up, see below.

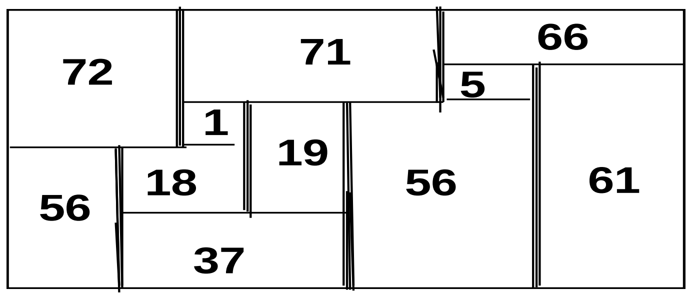
SYMMETRICAL SMITH DIAGRAMS SHOWING THE PROBLEM!



IDEALLY THE SECOND DIAGRAM SHOULD BE PRODUCED

OSD DIAGRAM SYMMETRICALLY DRAWN

TEST FOR SD SYMMETRY



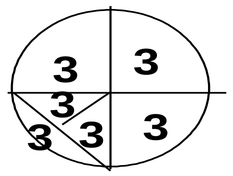
<3
<5
<4
<3

^3 ^3 ^3 ^3 ^3 ^3

[11] 209X127

COMPLEXITY 209
VERT LINES READ 3 3 3 3
PAIRS INTO 33 33 33

COMPLEXITY 127
HORIZ LINES READ 3 3 4
DO NOT PAIR OFF



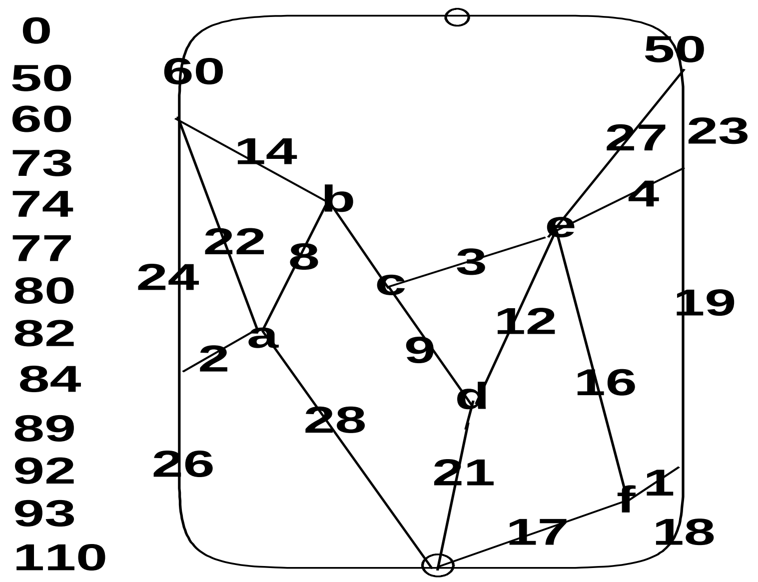
COMPLEXITY 209

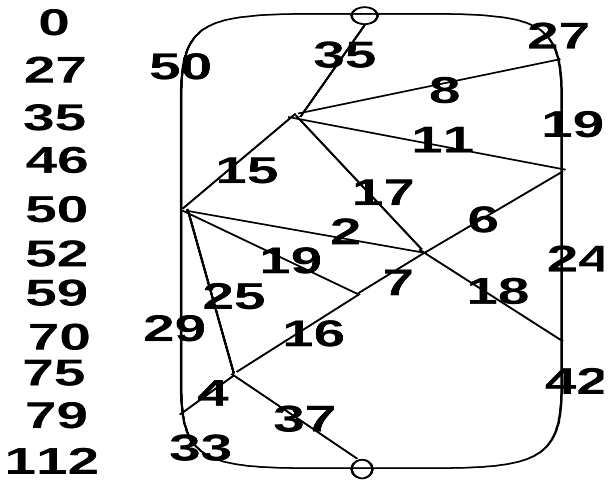
B13.7.3. DETECTING SYMMETRY IN SMITH DIAGRAMS

Although any symmetric rectangle will clearly result in both horizontal and vertical symmetric Diagrams, there are many asymmetrical diagrams which result in symmetric Diagrams.

Whenever symmetric Diagrams exist it means that at least two (sometimes more) solutions are identical despite different choice of poles. Thus poles at AB and CD Below give the same solution, namely [11] 209 x 127.

B13.8. GRID SMITH DIAGRAMS - OR G SMITH DIAGRAMS





[22] 110 x 110 and [21] 112 x
GRID SMITH DIAGRAMS

Two examples of GRID Smith Diagrams are shown. They have been constructed as follows - for each case:-

1. The numbers to the left represent the distances between the top of the Rectangle and each of the horizontal lines occurring within the Rectangle. The bottom number representing the bottom line is of course the second dimension of the rectangle.

2. A curved rectangle has been drawn enclosing each.

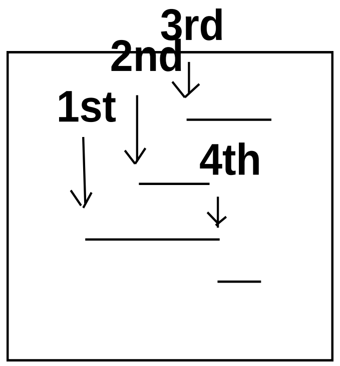
3. The normal arrows have not been shown as in this type of construction the arrows always point downwards (except for Element 0 which is shown as a horizontal line).

4. Note that the internal points a to f in the first Rectangle are drawn vertically equidistant between the two vertical sides i.e. | a b c d e f |.

5. So where exactly are the points a to f put? Every SR contains some horizontal lines which do not touch any sides and it is these lines which show as internal points in Smith Diagrams. In this example the vertical positions are indicated by 1st 2nd 3rd 4th i.e. are the order the lines appear when viewed from the left. Now let suppose the line “32 down” is actually the 4th horizontal line in the Rectangle out of 12 in all. With two vertical sides and four positions (1st to 4th) we have 6 verticals on which points will occur.

The point at 32 down is put 3 along and 4 down in a rounded rectangle representing 6 along and 12 down.

6. For better or worse the two poles are shown half way along the top and bottom lines.



0
32 down (say)
58 down (say)
61 down (say)
83 down (say)
115 down (say)

B13.9. ADVANTAGES OF G SMITH DIAGRAMS

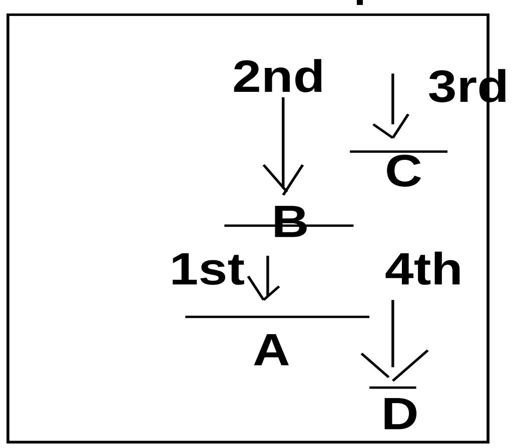
1. Except for Element 0, all lines point downwards in the direction of the arrow.
2. There is no need to show arrows at all!
3. The horizontal Distance Series of numbers need not be shown - alternatively -
4. If the Distance Series is shown, then there is no need for any Element numbers to be shown!
e.g. in above 1. The line running from Horizontal 89 to horizontal 110 will be 21 that is 110 - 89.
5. In theory at least all patterns might be illustrated by a Rounded Square of the same size, providing the horizontal divisions are made equal, and likewise the vertical divisions.

B13.10. DISADVANTAGES OF G SMITH DIAGRAMS

1. Look at Element 18 in [22] 110 x 110 solution. If drawn as a straight line, see what happens!
This is why Rounded Squares have been employed.
2. Look at above 1 and observe what happens when these lines are viewed from the right. We get 3rd 4th 1st and 2nd which is not 1st 2nd 3rd and 4th in reverse, as would be helpful! So as proposed, the SD for the solution with the largest Element at top left is not always an exact mirror image when the largest is at top right.

B13.11. REVISING THE PATTERN

This second problem can be overcome however by considering the midpoints of the line from left to right as shown here-



But it can be annoying in practice to calculate these midpoints from the left-hand vertical line!
There is another problem. Suppose in above the midpoints for 1st and 2nd, and for 3rd and 4th happen to be identical!
These options may be considered:
1. Let the one higher up take precedence i.e. order B A C D, or
2. Put both in the same vertical, i.e. A A C C .
3. Let the longer line take precedence i.e. A B C D. Option 3 is rejected since many lines happen to be the same length.
In Option 1 the upside-down solution SD would give the order of A B D C which is but B A C D in reverse, so should be rejected.
Option 2 is the best one. The cramping effect of using fewer verticals may be a slight problem however.
******* I NEED TO CHECK IF THERE ARE ANY CRAMPED UP PATTERNS, ALSO DO SYMMETRICAL SOLUTIONS GIVE SYMMETRIC SD'S . ALSO DO SYMMETRIC SD'S ACTUALLY APPEAR SYMMETRIC (VERY UNLIKELY

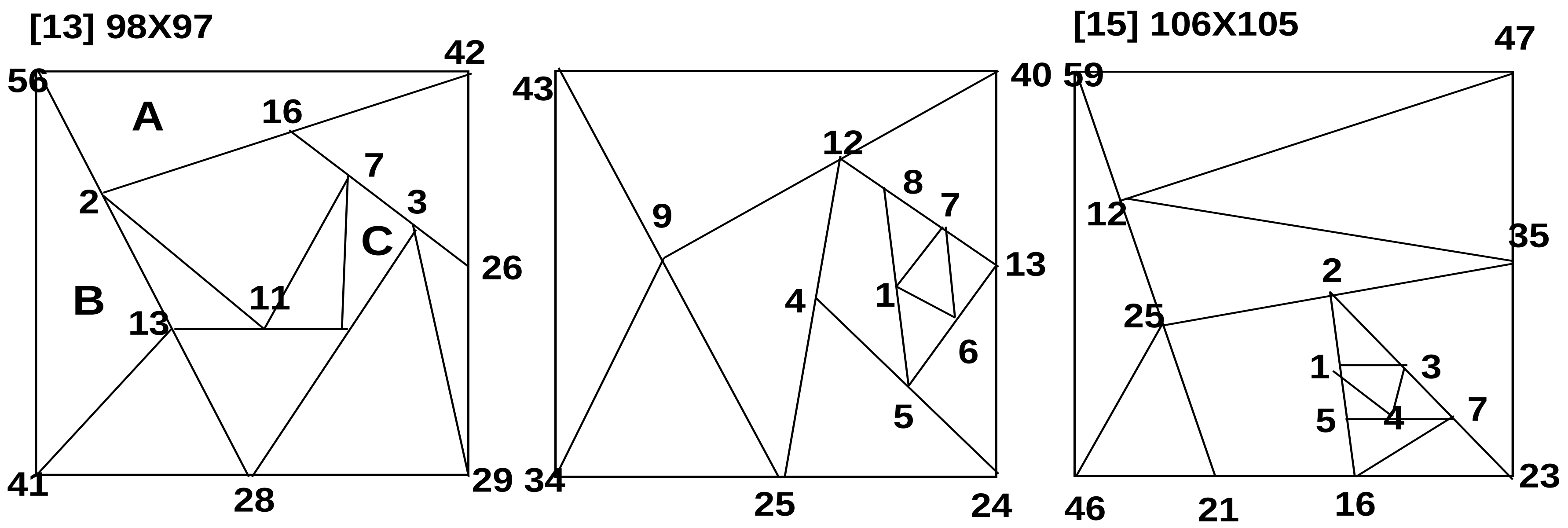
B14. POINT & MOSS DIAGRAMS

B14.1. DEFINITION

A POINT DIAGRAM is a representation of a Squared-Rectangle at first resembling a Smith Diagram but quite different in practice. Some examples are below, and their construction follows a pattern.
If each square Element in a Rectangle is replaced by a numbered dot and the adjoining Elements linked by lines, a similar effect to a Point Diagram is obtained, except that in a Point Diagram a square is always drawn and a segment from top left to 1st from left at bottom drawn. The largest corner is always shown at top left.

B14.2. CONSTRUCTING A SQUARE POINT DIAGRAM

1. Show the corner Elements as corner dots e.g. 56 42 41 29 in [13] 98 x 97.
From now on all dots (Elements) are shown equidistant along the lines.
2. Show side Elements 26 and 28 by dots.
3. Draw a line from top left to the first dot after the bottom left hand corner (28), i.e. 56 to 28. 4.
Divide this oblique line by dots / Elements bordering 56 and 41 (namely 2 and 13).
5. Working clockwise around the rectangle, draw lines to indicate adjacent Elements and space out dots.
If done correctly, a fixed position will always apply to each dot. (NB these instructions brief) ******etc******
6. Examining various segments a. 56 2 16 42 b. 56 41 13 2 c. 7 4 3 and a point indicates those Elements which border it. Each segment has numbers surrounding it which add to 0 if certain numbers are made negative, e.g. 56 + 2 = 42 + 16
and 56 + 2 - 42 - 16 = 0 and 16 + 2 = 11 + 7. The positive numbers surrounding each segment must total an even number and half of this number represents the line. e.g. 56 + 42 + 16 + 2 = 116. Half is 58. Here are some examples of Moss Diagrams.

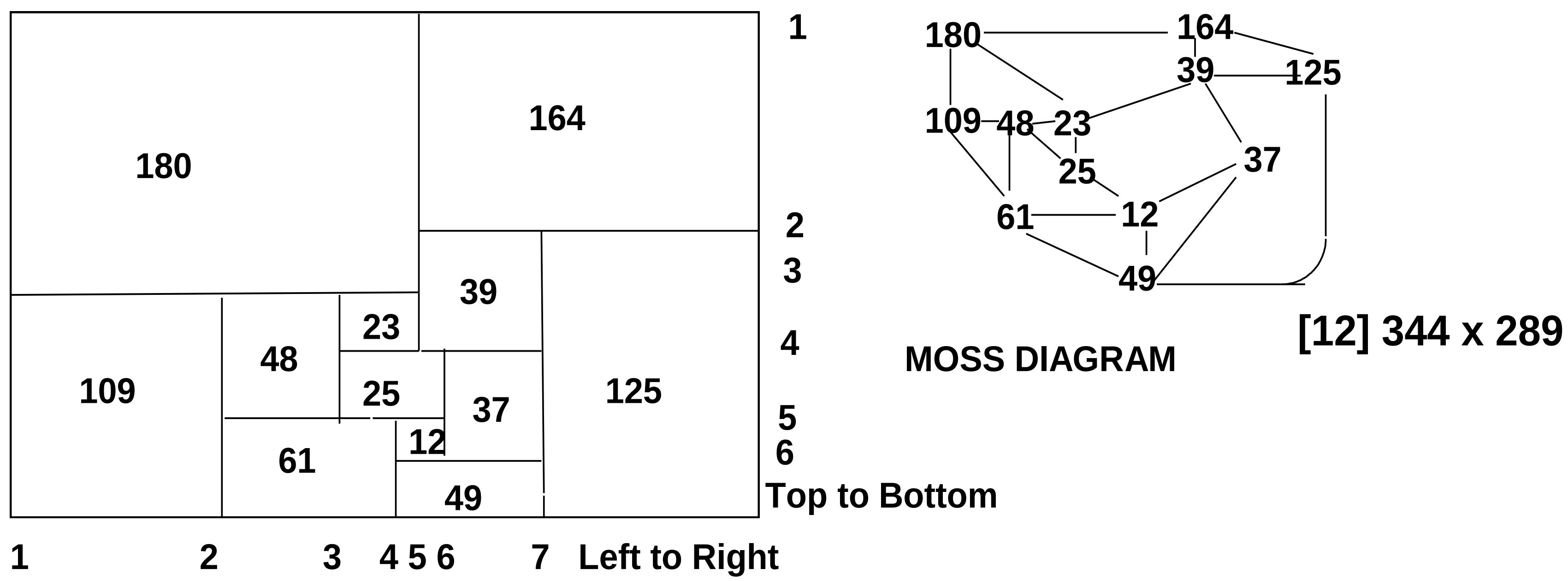


Whether these are useful in practice remains to be seen. For certain solutions problems arise which alter the rigid system above. Therefore these diagrams cannot be regarded as absolutely “Fixed” which is a pity. Something better is needed -

B14.3. THE MOSS DIAGRAM

The Writer has now realized that a better type of construction can be made based on the layers (top to down) and the columns (left to right). Each Element is still represented by a point (not a square as in Rectangles, and not a line as in Smith Diagrams). The square format is not satisfactory.

The Elements in the Moss Diagram can be presented in other similar formats, but probably the best is this:-



The construction always has a line at the top adjoining one on the left. The Elements are accurately shown in rows and columns - following the Grid pattern. (In this case 6 rows & 7 columns).

The Moss Diagram is a rehash of the Grid Squared-Rectangle with points replacing squares for each Element.

In this construction Poles don't exist and there are no arrow directions.

This construction is nearer to a Rectangle Diagram than a Smith Diagram. It differs from an SD in that direction arrows are not shown, and the Elements are denoted by <u>points</u> not lines. Although it might be mistaken as a Smith Diagram, it is quite different and the number of connecting lines being a lot more than the Order number 12 (The SD has 12 lines). In any of the Sectors formed, the surrounding Elements equate to zero, e.g. 180 + 23 - 164 -39 = 0. Likewise, 180 +23 = 164 +39. (203).

The advantage here is that Moss Diagrams are absolutely fixed (unlike SD's). Another advantage is that there is one pattern for each solution, not two as in SD's. (Ignoring up-side-down solutions which will of course give a very different Moss Diagram). Unlike SD's, the Moss Diagram (MD) is the same (but sideways) when the rectangle is turned 90°.

Many Moss Diagrams can be successfully drawn entirely with straight lines. But in the above example a straight line drawn between 125 and 49 would run to the left of the 37 which is incorrect! So a right-angled curved line has been shown instead. Inspection suggests that problems only occur with the Element at the south-east corner, or one touching the bottom line. If so, a right-angled curved line will always solve this problem.

B14.4. PRACTICAL USE OF MOSS DIAGRAMS

They may prove useful but there don't seem to be any advantage compared to Grid Patterns.

B15. INTERESTING SD PATTERNS

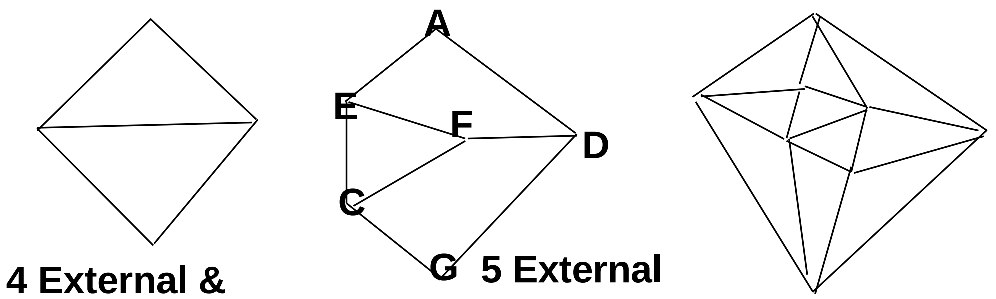
B15.1. IRREGULAR & SYMMETRIC SMITH DIAGRAMS & DIAGRAMS WITH INTERESTING PATTERNS

Already briefly mentioned the subject is considered elsewhere as section b is concerned with basics.

From now on in this book, Smith Diagrams are mentioned as and when required within various subjects since it is inappropriate to consider them in isolation and disregard other features.

B16. CODES IN SD PATTERNS

B16.1. INTERNAL AND EXTERNAL POINTS IN SMITH DIAGRAMS



4 External & 0 Internal Points

5 External & 1 Internal Point

4 External & 4 Internal Points

CODINGS
P4:0 P5:1 P4:4

Look at above where each Network shown has a number of External Wires or Elements which always equal the number of Points i.e. 4, 5 and 4. Internally there are no points in the first and 1 and 4 in the others.

We shall Code these as P4:0 P5:1 and P4:4 respectively.

Next observe that the Order (that is, the number of Wires) can only be 5 in above 1. But in above 2 with Order 8, wires could be added at various combinations of AF, DE, CD and FG giving Orders of 9 or 10.

In the case of the Maximum possible Order all the circuits are reduced to Triangles only. Above 3 shows an Order 13 Solution the maximum for P4:4 since each circuit is a triangle which cannot be any further divided.

For patterns to be valid:-

1. There must be at least 4 points or 4 wires externally.

2. Each Internal Point must be joined by 3 or more wires (see F Above).

3. External Points with only 2 Wires are situated at top or top and bottom and will dictate where one or both Poles must be situated. In the Case of P4: anything the bottom Pole must be joined by 3 or more wires however (to avoid S2222 solutions).

B16.2. MAXIMUM ORDERS PER P: CODINGS

In testing various network patterns P4:0 to P4:6 have maximum Orders of 5 8 11 14 17 20 and 23 whereas P5:0 to P5,6 have 7 10 13 16 19 22 and 25 and P6:0 to P6:5 have 9 12 15 18 21 24 and 27.

Note the increments of 3, and that each set starts with 5, 7, 9 … with increments of 2.

So for P x : y it is found that the Maximum Possible Order is 2x - 3 + 3y

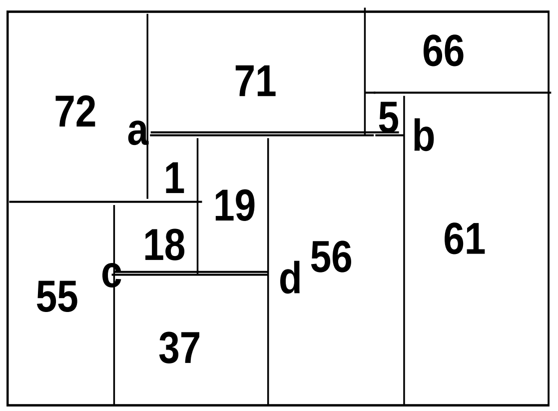
B16.3. MINIMUM ORDERS PER P: CODINGS

As the Table below shows, Values increase by 1 or by 2 as the values of x increases by 1 or y increases by 1.

There does not appear to be any fixed formula.

Thus for P5:3 for instance the Order can be 11 12 13 14 15 or 16. In the Case of Order 16 the Smith Diagram will have exclusively triangular circuits whereas for Order 15 one circuit will be bounded by 4 wires.

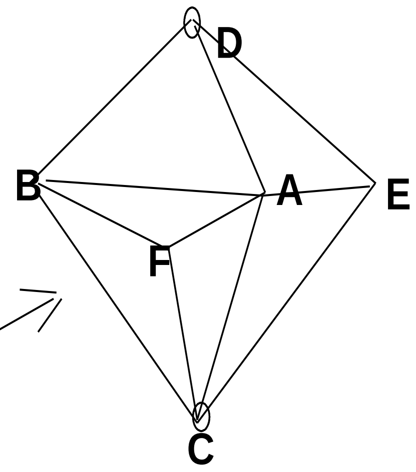
P4:0 5	P5:0 7	P6:0 8	P7:0 10
P4:1 7	P5:1 8	P6:1 10	P7:1 11
P4:2 9	P5:2 10	P6:2 11	P7:2 13
P4:3 10	P5:3 11	P6:3 12	P7:3
P4:4 12	P5:4 13	P6:4 14	P7:4
P4:5 13	P5:5 14	P6:5	P7:5



[11] 209 x 127

Horizontal SD

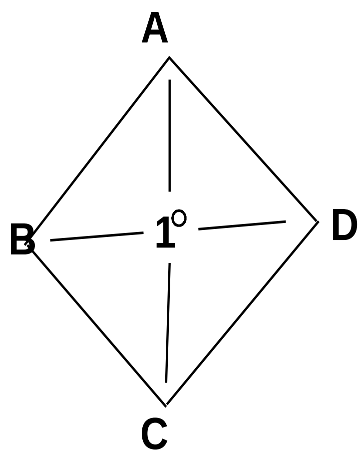
Pattern consists of Triangular Circuits only



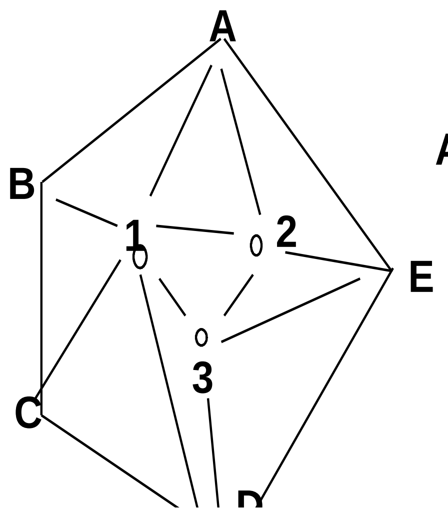
The P Code for the Above is P4:2. Consider the elements at lhs and rhs - there are 2+2 = 4

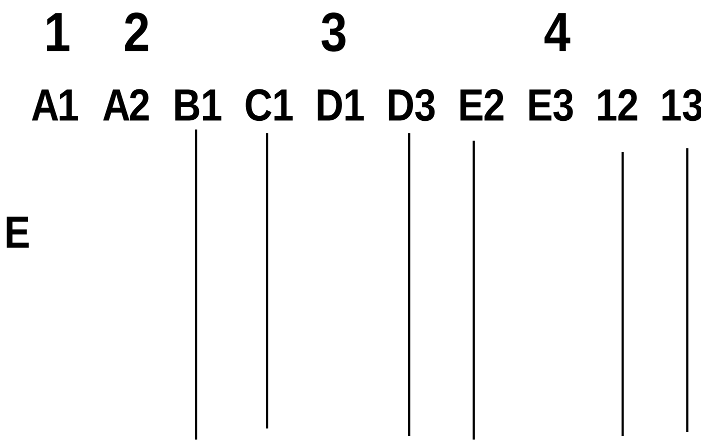
Now look at all Horizontal lines in the <u>Rectangle</u>. Now eliminate the top and bottom ones. Also eliminate those which touch the left or right hand side. We are left with ab and cd = 2 which coincides with the number of External Points in the Smith Diagram.

Thus by observing the Rectangle alone, the Code is P4:2..

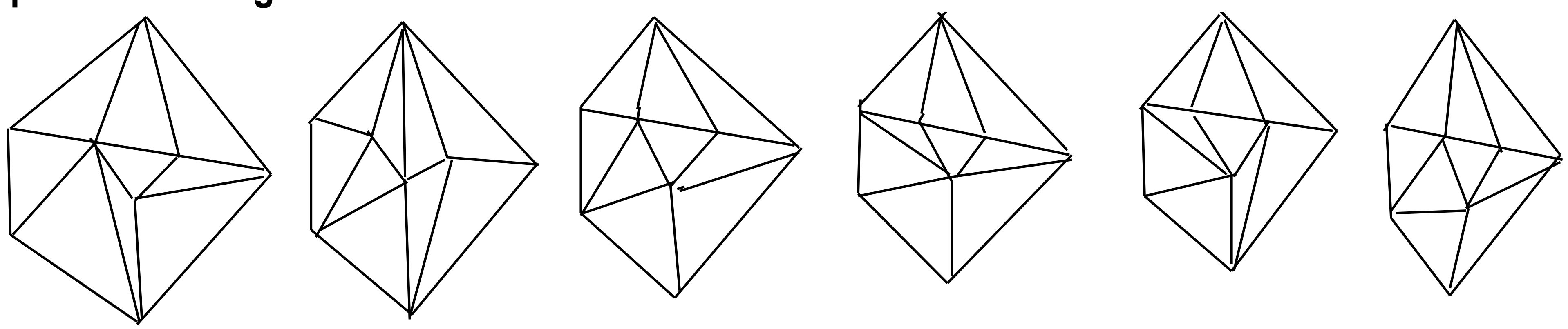


A1 B1 C1 D1





Combinations possible for P5:3 network above. 1234 123 132 213 231 312 321 12 13 14 23 24 34 1 2 3 4 (x) 18 in all, some of which give duplicate Rectangles.

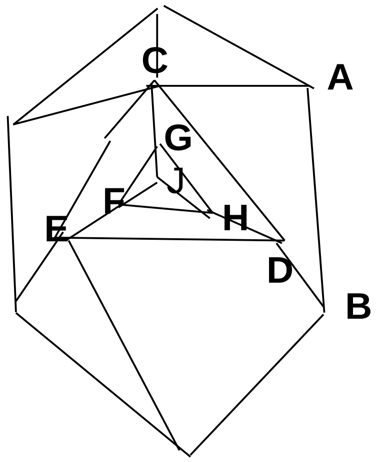


There are many other patterns for Order 16 (the maximum Order for P5:3) each of which can be diminished in a bewildering variety of ways! Just some are shown above. Again, many patterns give duplicate Rectangles.

For example in Above D1 can be replaced with C3, E3 with D2, A1 with B2, 23 with E1, 12 with A3 etc.

Note some combinations of these may be invalid i.e. leave less than 3 wires at any of the points (apart from the Poles).

In each case there are 8 separate triangles and 8 Wires radiating from the External Points each of which must link with the three internal points 1 2 or 3. The number of wires between the three Internal Points is either 2 or 3.



LAYERS - 0 OUTER RING
1 NEXT RING i.e CDE*
2 NEXT RING INSIDE i.e. FGH*
3 NEXT RING INSIDE i.e. J*

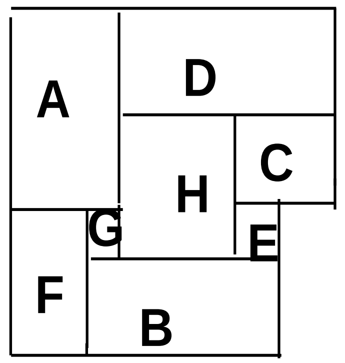
* In this example RIMS 1 & 2 are bounded by 3 wires but can comprise 4 5 or more. The most inner one can be 3 or more - or a single point as shown here.

B16.4. RINGS OF WIRES AND RINGS OF TERMINALS

Every Smith Diagram can be broken down into a series of "Rings", and between each set of rings further wires (Elements) connect them.
1. For instance in above the Outer Ring - Code"0" consists of 6 wires.
2. Inside that are 3 wires bounded by CDE, which is Coded "1"..
3. Inside that are 3 wires bounded by FGH, which is Coded "2"
4. Inside that a single point J which is Coded "3"
But between each Ring and the one inside it, there have to be wires, e.g.
5. AC and BD etc. followed by
6. CG and DH and EF then followed by
7. FJ and GJ and HJ
Thus in above Q:00 = 6, I.E. the Quantity of Wires from Ring 0 to Ring 0 is 6. Q:01 = 6 wires from Ring 0 to Ring 1.
Q:11 is 3 wires from Ring 1 to Ring 1 and so on - Q:12 = 3 Q:22 =3 Q:23 = 3 and Q:33 = 0

B16.5 A CURIOUS RELATIONSHIP

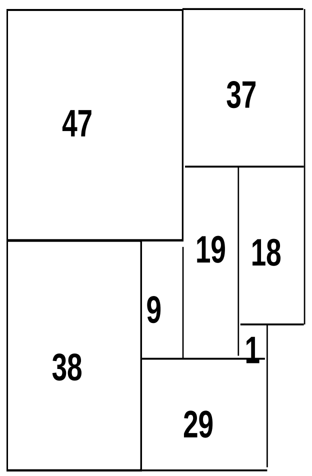
Look at the Pattern shown below. Four relationships (apart from obvious ones such as C + F = H) are found to apply:
1. A = B + C 2. F = D + E 3. C = g x 2 and 4. f = h x 2.
There seems no immediate reason for these relationships. But test it for yourself by choosing any integers for C and E and calculating all the values for A to H.

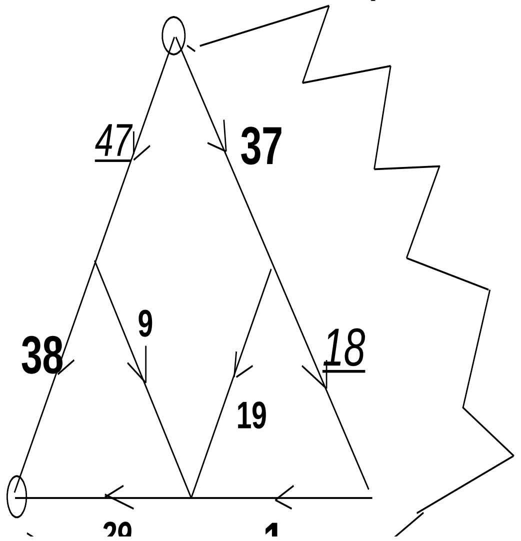


I have has put C as 18 and E as 1 with the result as below 1.
Sure enough 47 = 29 + 18; 38 = 37 + 1; 18 = 9 x 2 and 38 = 19 x 2!
The Values shown below are of course samples and not part of any real Squared-Rectangle of course.

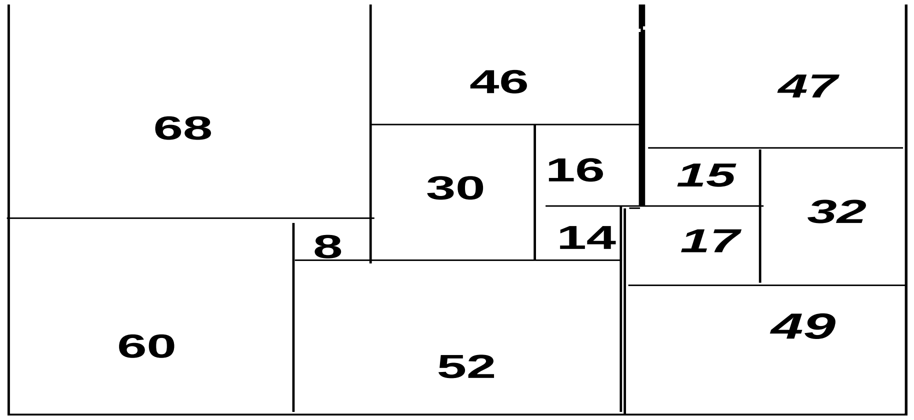
A clue as to why the first two relationships exist is found by drawing the Smith Diagram which is a Triangle with two Elements on each side with two internal extra Elements. Note the positions of the three bold Elements, and that of the three underlined Elements. Also of 38 and 19 and 9 and 18 where one Element is twice linear the other.

This pattern is similar to a ROTOR (described earlier) with the additional wires or Elements forming the STATOR (nor shown).

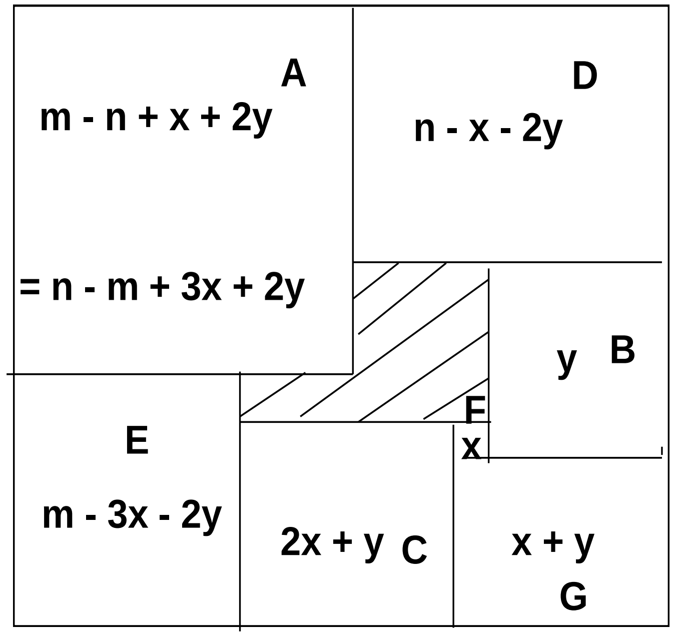


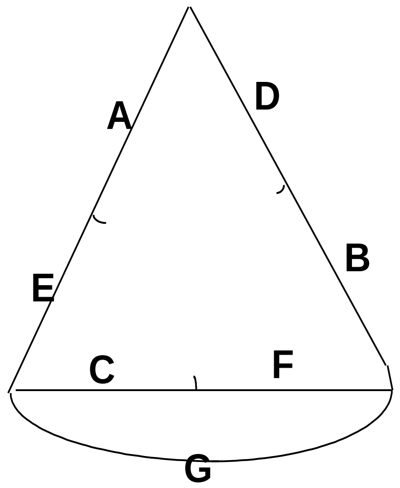


I expected that by taking Algebraic values for the above pattern that these Relationships would become very obvious - but NO!
Now for an actual Rectangle incorporating this pattern (left hand side) - [13] 161 x 128 - once again the four Relationships apply. The right hand side has no special relationships.
16 + 52 = 68, and 14 + 46 = 60, and 30 is half of 60, and 8 is half of 16.

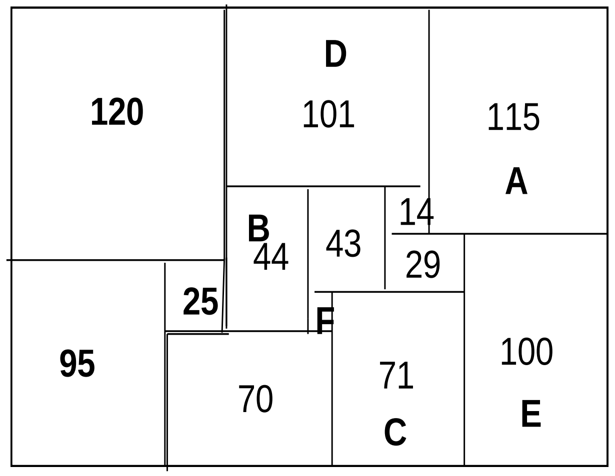


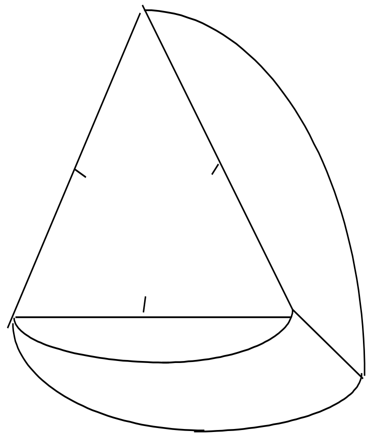
Look at the diagrams below.
This shows a 2233 Solution where the Elements B G and C are connected by a single Element F or x.
Regarding the Dimensions as m x n the Algebraic equivalents have been calculated.
A is found to be equal to both m - n + x + 2y and n - m + 3x + 2y.
Equating these we find that 2m = 2n + 2x so m = n + x so <u>D = E + F</u>.
Likewise A is found to be the sum of B and C, so <u>A = B + C</u>





It does not follow that the remaining Elements within the shaded area in above 1 will always be in sets of 3 where one is the sum of the other two. Likewise in above 2 the pattern inside the triangle may be symmetrical or otherwise.





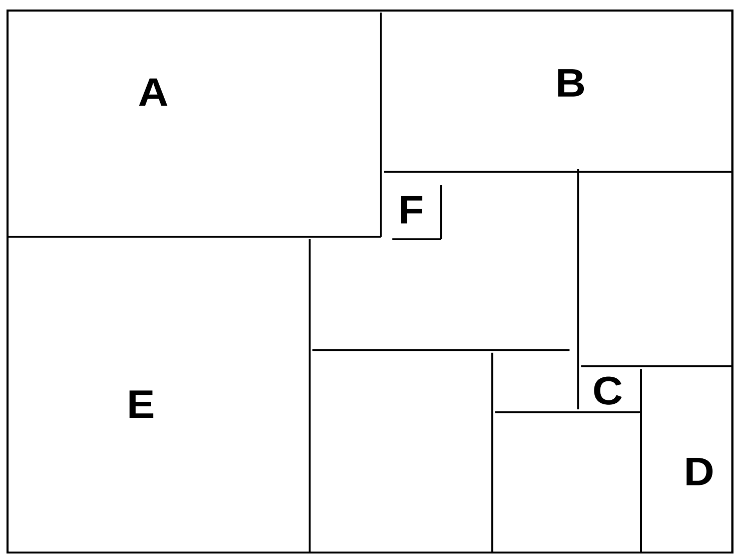
[13] 226 x 215.   When Triad at left is removed the Solution becomes

[10] 57 x 55 of Sides 2233

Now consider above 1. Removing the Triad at left we obtain [10] 57 x 55 of Sides 2233 in which the Elements at A B and C have the relationship of A = B + C as proved above, and also D = E + F also shown above.

So what happens when a Triad is added as Above1? Yes these equations still stand: 101 = 1 + 100 and 115 = 44 + 71. Note that the Triangular pattern in the Smith Diagram is unchanged by the Stator Wires now are four in number in Above 2. In fact if the Triad is replaced by an Octal or Pentad etc. A will always be B + C and D be E + F as only the pattern of Stator Wires changes.

Now look at a slightly different Pattern as Below. Here it is found by inspection A - B =C i.e. A is the total of B and C (Linear), and that E is twice D (Linear). Since A - B is clearly F too (assuming F also forms part of the pattern) it means F and C are equal. So all Rectangles including this pattern can never be Perfect.



In this pattern -

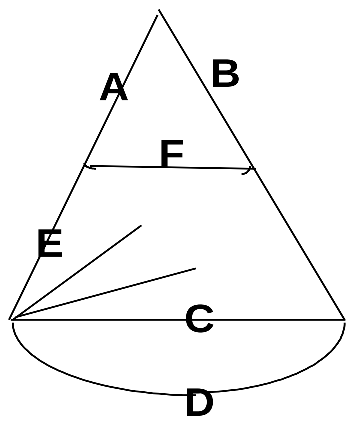
A - B = C

and E = D x 2

and if F is there then

F = C and the Solution

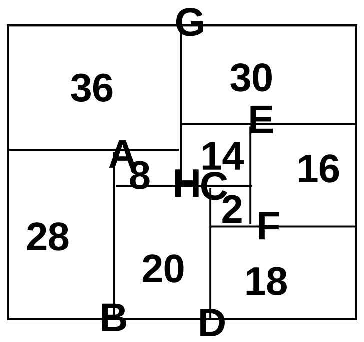
is bound to be Imperfect.

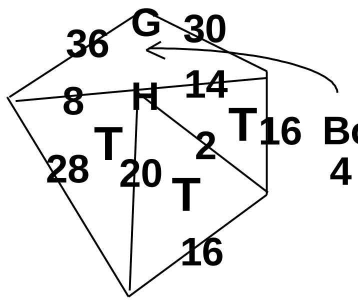


B17 COMPLETING THE TRIANGLES

B17.1. COMPLETING THE TRIANGLES

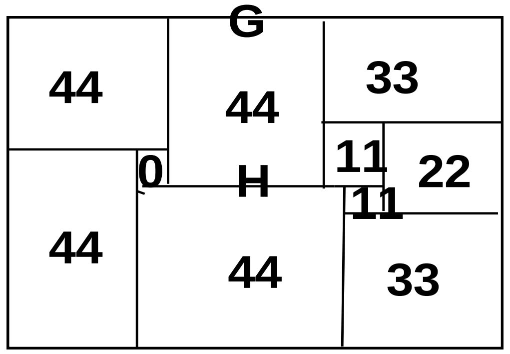
In any Smith Diagram there are always some individual circuits of just three Elements though often there are some with 4 or 5 or 6 or more. Parts with just three Elements naturally form a Triangle in the Smith Diagram.
In any Squared-Rectangle vertical lines exist and often these border just three Elements to left and right of them. Like the above some will have 4, 5, 6 . . . etc.





[9] 66 x 64 (RI 2)

Bounded by 4 Elements



[10]
121 x 88
(RI 11)

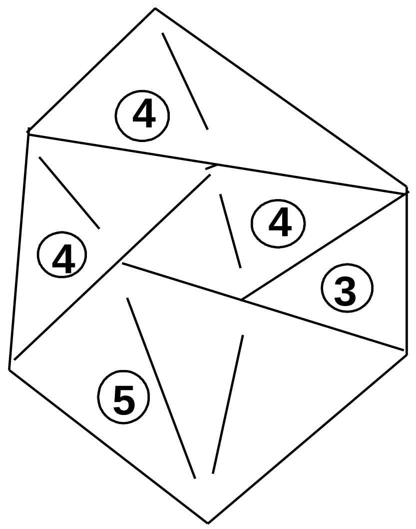
To demonstrate the above, vertical line AB is bounded by 8 + 20 = 28 (three Elements) and the same is true for CD & FE. But at GHQ there are 4 Elements; 36 + 8 = 30 + 14. The Corresponding Smith Diagram shows 3 "Triangles" marked T and an area bounded by 4 Elements. Now if another Element is inserted at GHQ the Solution changes from [9] 66 x 64 (33 x 32) to [10] 121 x 88 (11 x 8) shown at right.
This larger Solution will be called the CTS or Completed Triangle Solution.
The point of all this is this - no matter what Solution is chosen at random - it is always possible to change it to a Solution containing vertical lines with only 3 adjoining Elements throughout and at the same time with individual circuits in the Smith Diagram composed of Triangles only - the CTS

Some Solutions already have this property and no change is necessary.
Others will need several more Elements to Complete the Triangle. Also the same principle can be applied to Horizontal Lines and the alternate Smith Diagram, but the number of Elements to Complete the Triangle may be different.

Actually no Squared-Rectangle has exclusively 3 Elements in every horizontal and every vertical line and this means two CTS Solutions exist for any given Solution (one vertical & one horizontal).

Where there is an individual circuit of 5 or more Elements - several Elements (all of them Vertical) have to be added to Complete the Triangles. For example with 5, two Elements need to be introduced. There is more than one pair possible.

***Work has to be done to determine whether any interesting relationships exist e.g. any rules concerning number of Elements* etc.**



In this Smith Diagram there are three Individual circuits with 4 Elements and one with 5.

To Complete the Triangles note the positions of the 5 extra "Vertical" Elements. 5 happens to be the total of the excess Elements above three. i.e. 1 + 1 + 1 + 2 here.

The number of circuits goes up from 5 to 10 (triangles).

B18.1. MATRIX

Looking at the MATRIX given for an Order 10 Solution it occurs to me that an extra final line can be added as per P6 below.

1. Line P1 (for any Solution) will always have positive ones in it only, whereas the

2. Final P line will always have negative ones only.

3. The In-between P lines (i.e. P2 to penultimate) always have both positive and negative ones. When the actual Elements for these are considered the 'plus' set will always total the 'minus' set. e.g. in P2 the Elements for P23 + P26 = Elements for P12 (asterisked). Likewise P36 and P34 with P13 and P23 and so on.

4. Not surprisingly, the Elements for line P1 will total those for P6 since both are the horizontal of the Rectangle.

12 13 14 15 23 26 34 36 45 56
1 1 1 1 0 0 0 0 0 0 P1 Line A
-1*0 0 0 1* 1* 0 0 0 0 P2 B
0 -1 0 0 -1 0 1 1 0 0 P3 C
0 0 -1 0 0 0 -1 0 1 0 P4 D
0 0 0 -1 0 0 0 0 -1 1 P5 E
0 0 0 0 0 -1 0 -1 0 -1 P6 F

B19.1. ANOTHER SYSTEM OF RECORDING SOLUTIONS - GRID DIAGRAM TABLE

Another system close to that just shown is -

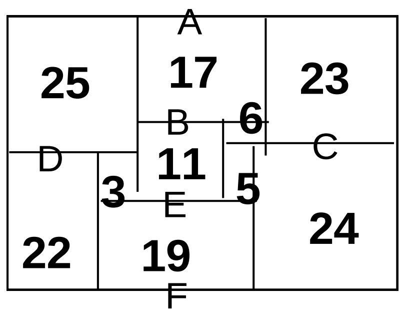
ORDER	AB	AC	AD	AE	AF	BC	BD	BE	BF	CD	CE	CF	DE	DF	EF	DIMS
9	4	5	6	-	XX	1	-	3	-	-	-	6	1	5	4	15x11
9	15	18	-	-	XX	-	7	8	-	4	-	14	1	10	9	32x32
9	33	36	-	-	XX	-	5	-	28	2	9	25	7	-	16	69x61

10	2	3	3	-	XX	1	1	-	-	1	-	3	1	-	2	8x6
10	15	15	-	-	XX	1	-	-	11	1	-	11	2	-	8	30x26
10	27	30	-	-	XX	-	11	13	-	8	-	25	2	17	15	57x55
10	17	23	25	-	XX	6	-	11	-	-	-	24	3	22	10	65x47
10	54	57	-	-	XX	3	7	-	44	4	15	41	11	-	26	111x98
10	55	60	-	-	XX	-	16	-	39	11	15	34	4	23	19	115x94
10	41	44	45	-	XX	3	-	-	38	-	12	35	11	34	23	130x79
11	4	5	5	-	XX	1	1	2	-	1	1	5	5	-	4	14x10
11	90	95	-	-	XX	5	24	-	61	19	25	56	6	37	31`	185x151
11	66	71	72	-	XX	3	-	-	61	1	19	56	18	55	37	209x127

All possible Solutions (Valid) which contain just 5 rows of Elements - 6 lines when the bottom line is included which are termed A B C D E F.

There is always a value for AB and one for the last in the list - here EF. Where no value applies a dash has been shown (NB. these are NOT to be regarded as zeros). Within each "A- set" (AB AC AD...) the Elements go up in size e.g. 17 23 25 and the same is true for the B C D... sets also.

The System is explained below -



GRID DIAGRAM TABLE
Labelling horizontal lines from top to bottom ABCDEF..
25 spans from line A to line D i.e. "AD" Likewise 5, CE 22, DF and so on.

In Smith Diagram terms, 25 in the above solutions lies between lines A & D (AD), 5 between C & E (CE) and so on.

One advantage of GDT is the neat and fixed format which is produced. Another is that it is a Smith Diagram related and to date is the only FIXED REPRESENTATION OF A SMITH DIAGRAM I have been able to find, despite considerable effort.

Disadvantage - although Rectangles are easily converted to a GDT, attempting to draw a Rectangle back from one of these GDT lines can prove a problem!

For a given number of horizontal lines (equating to Smith Diagram terminal points), the Solution Orders will vary a bit.
Also not all solutions for any given Order have the same number of lines (e.g. [10] 105 x 104 is missing in the Above list).

C. ELEMENTS

C1. COEFFICIENTS

C1.1. x AND y COEFFICIENTS - DEFINITION

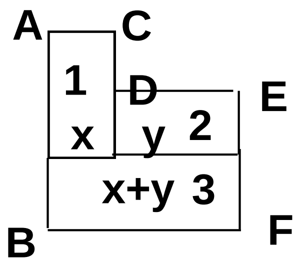
In an algebraic expression say -3x + 6y, the coefficients are -3 and 6 and are shown as - 3~6.

In calculating with x and y Unknowns we obviously start with x and y as positions 1 and 2.

For clarification we will always use x + y as position 3 next. Thus x - y will only be used in addition to x + y and never instead

POSITION 1 x OR COEFFICIENT 1/0-1

POSITION 2 y OR COEFFICIENT 0/1-2



POSITION 3 x+y OR COEFFICIENT 1/1-3

POSITION 4 HAS THREE OPTIONS AT AB CD EF

POSITION 4 AB 2x+y 2~1-4

AT CD x-y 1/~1-4

AT EF x+2y 1/2~4

NOTE: THE FINAL NUMBER IS TO BE REGARDED AS THE MINIMUM POSITION NECESSARY RATHER THAN THE ACTUAL ORDER IN A CALC.

POSITION 5 ONWARDS - THE NUMBER OF COEFFICIENTS ACCELERATES, AND ARE WORKED FROM TAKING EACH POSITION 4

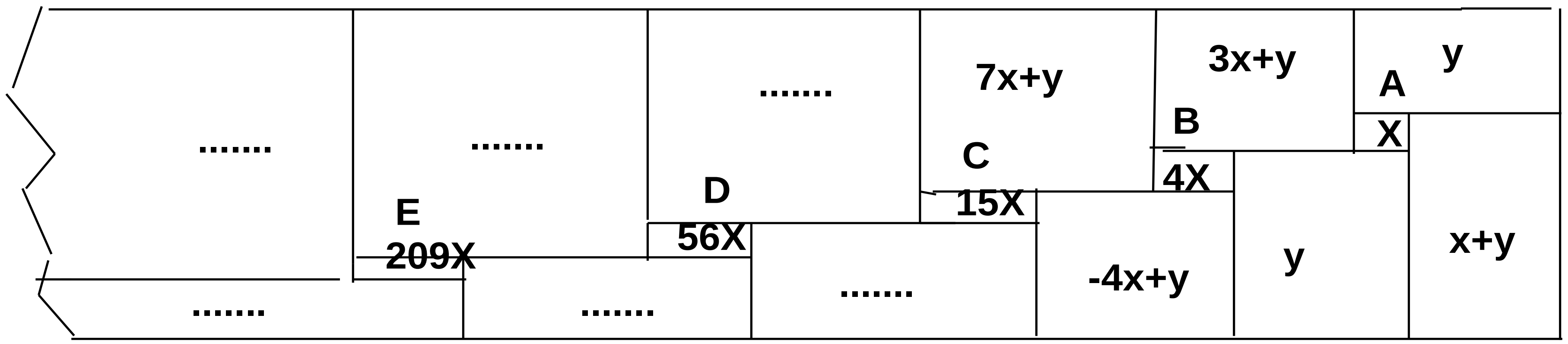
C1.2. INDEPENDENT SETS OF ELEMENTS

In calculating xy solutions some Elements are found to exist in coefficients of x only or y only.

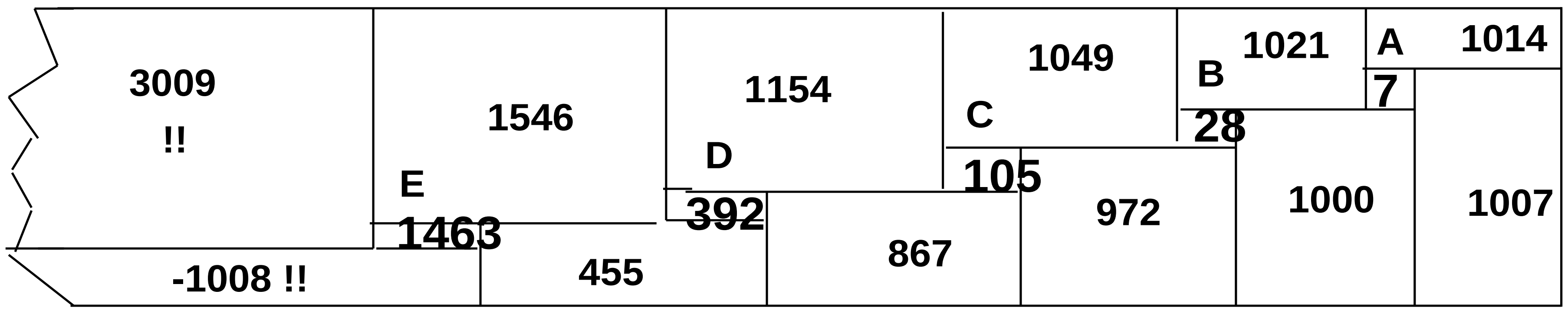
Below shows a common example of this where abcde… are completely unaffected by whatever value is given to y and are found to be x 4x 15x 56x 209x....

Next page>

AN EXAMPLE OF INDEPENDENT ELEMENTS x,4x,15x,56x,209x....

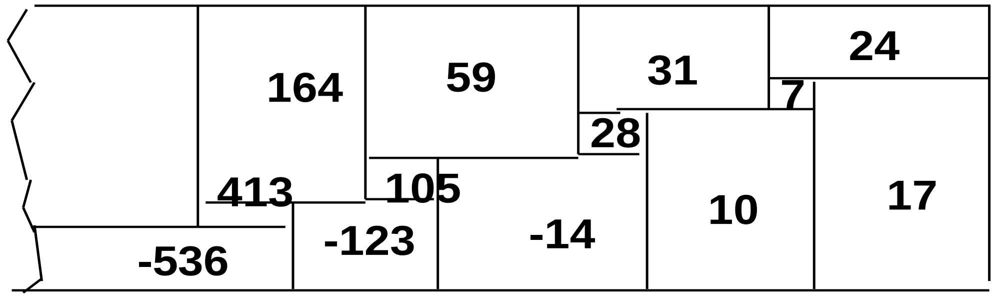


CHOOSING x=7 & y=1000



Clearly in the Above 209x will always be an Element divisible by 209, 56x divisible by 56 and so on. if the Above list continues long enough negative Elements will appear such as -1008 but the diagram can be corrected.

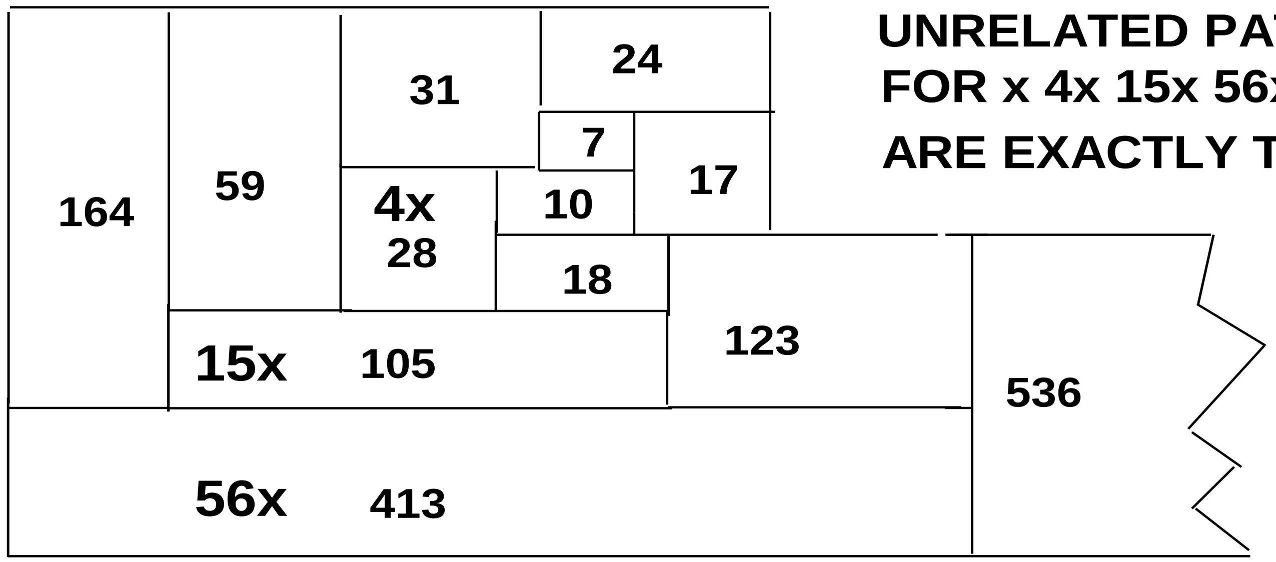
Below is another presentation of really the same series where y happens to be 10 instead of 1000.



INDEPENDENT ELEMENTS ARE CONFUSING SINCE NEGATIVE ELEMENTS ALTER THE PATTEI

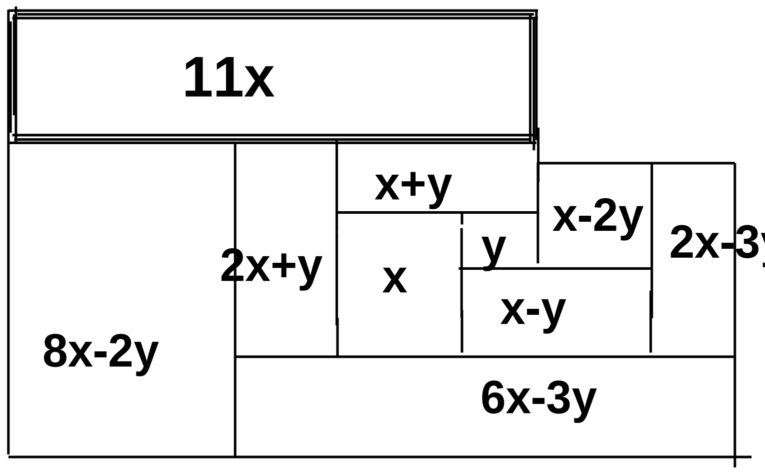
THE CORRECTED DRAWING SHOWS SEEMINGLY UNRELATED PATTERNS FOR x 4x 15x 56x YET THEY ARE EXACTLY THE SAME SERIES.

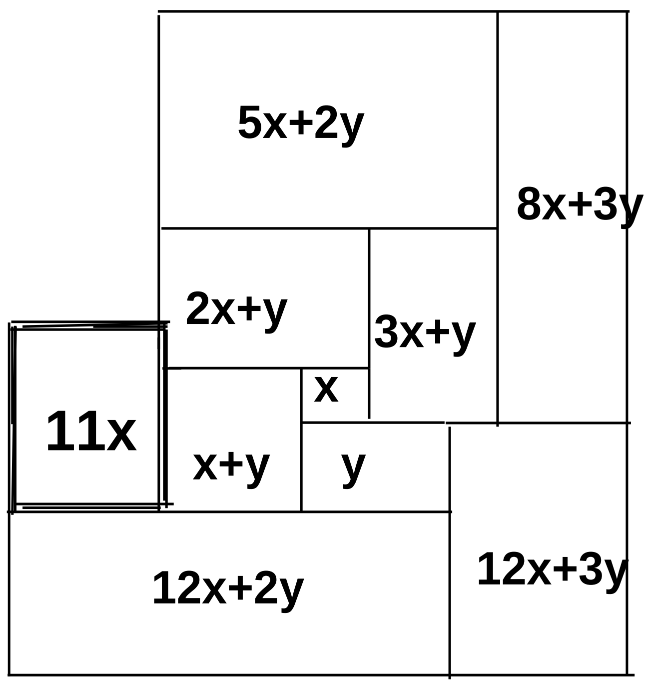
SEE ALSO ABOVE



There are many other Independent Elements possible two of which are Below. It is important to remember that the series of algebraic coefficients are what determines a series and not the pattern which we have seen Above 2 can vary greatly.

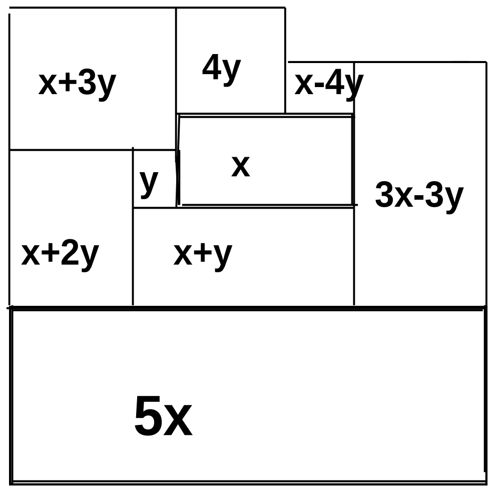
TWO MORE INDEPENDENT ELEMENTS

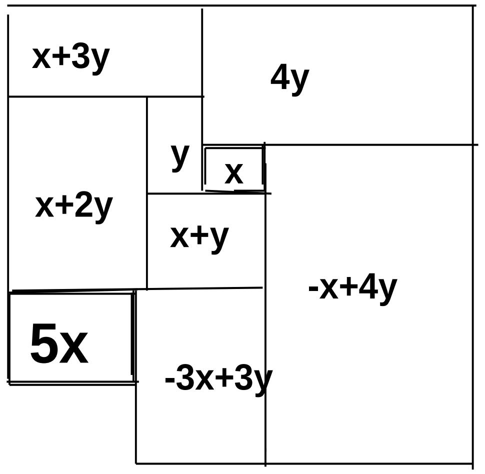




THE ALGEBRA IN THESE DIAGRAMS VARIES SHOWING THAT THE SERIES ARE DIFFERENT (& NOT PATTERNS OF THE SAME SERIES)

The Elements Above are always divisible by 11 and are independent of the value of y.





INDEPENDENT ELEMENTS

NOTE THE VALUE OF 5x.

THESE DIAGRAMS ARE REALLY THE SAME IN PRACTICE!

C1.3. RECORDING INDEPENDENT ELEMENT POSITIONS

Above it is seen how confusing patterns can be in this study for Above 1 and 2 are really the same construction, yet presented differently! As negative elements radically alter the shape it is necessary for a better reference system.

This involves always showing x and y horizontally with x assumed to be larger than y (even if is not numerically in practice), and an element x + y beneath it.

Then additions can be made to sides 1 to 6 as shown so long as adjacent elements are always avoided.

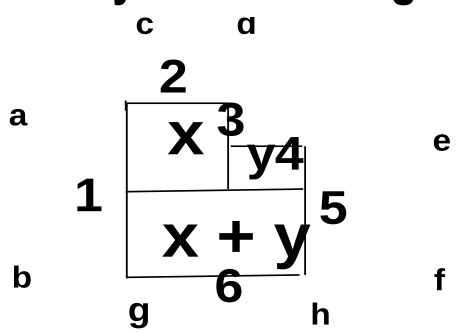
Thus joining lines with ab Below is an addition at side 1 cd at side 2 ef at side 5 and gh at side 6 etc.

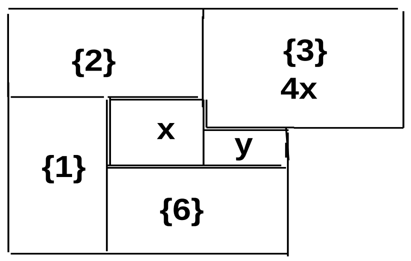
Below 2 shows independent element 4x is obtained by a code "[xy6123}.

The element x + y is always indicated by side 6 so any code will commence with "xy6...".

It is easy to mistake which side is in fact which, once the original "L" shape has changed a number of times! e.g. 4x - y drawn at side 4 in Below 2 will point down whereas in Below 1 it would point up.

In theory it would be good to force the pattern to remain L in shape at all times but difficult in practice.





{-} shows the sides added - side 6 then 1, 2 & 3

The element 4x can be coded by {xy6123}

C1.4. INDEPENDENT ELEMENTS OF 4x - HOW MANY EXIST?

I found three ways of creating independent 4x as shown below with codes {xy6123} {xy6321} and {xy6312} with element 4x appearing on sides 3 1 and 2 respectively.

Strangely, whether these are three really different patterns is a matter of opinion.

Yes they are different, or No they are not according to the way one chooses to look at it!

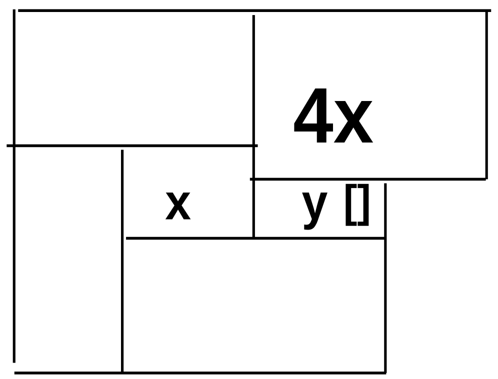
Now whichever pattern is taken with the element x shown in a fixed position, the elements shown as 4x are Independent as already mentioned, and simply cannot possibly be anything else.

So whatever algebraic expression is given to any of the other elements 4x is always the value of the elements shown.

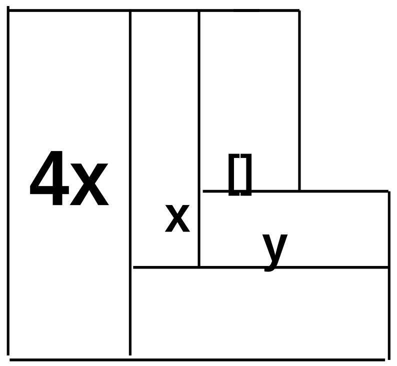
Unlike elements x which must remain in the positions shown , which element is called y is therefore irrelevant - nothing will change the values 4x. Now if the elements [] are given the value y -y and -y respectively, the reader could work out the three patterns and discover them to really be the same as the values of the elements are collectively the same.

However it is confusing, as to make the elements coefficients exactly the same it is necessary to alter the pattern in Above 2 and 3 to eliminate the negatives by making the -y element +y by adjusting the pattern.

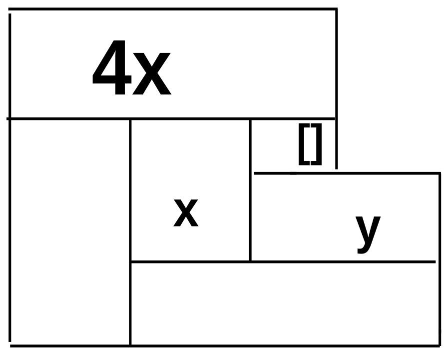
Once this is appreciated, it may well be considered that there is only one real occurrence of element 4x! This reasoning does explain why all other independent elements also <u>seem</u> to be duplicated a number of times. For instance 5x, 6x, 11x, 15x, 19x .... are found to occur several times in patterns which appear to be very different - but when closely examined are really strictly just one if the surrounding elements are not assigned a fixed algebraic value for all patterns at once. It is difficult to explain this without making it seem more complicated than it actually is!



Code {xy6123}



{xy6321}



{xy6312}

C1.5. INDEPENDENT ELEMENTS TO UP TO 12 DIFFERENT ELEMENTS

In experimenting with algebraic patterns of x and y type, it is possible to discover a lot of different Independent Element values. Providing the convention shown is used, no element can ever have the independent value of 2x or 3x or 7x for example, whereas 4x 5x 6x 11x 15x 19x and a host of larger independent elements are all found to be possible.

So far I have no quick way of indicating which values are and which are not possible, but it is true that there is no limit to the size they can be. Values such as 1785x are found with only 22 elements - code {xy612’34234’32123’43434’56}.

Clearly where this final element occurs in a pattern it (1) must be divisible by 1785 and (2) cannot be less than 1785 (assuming x is not zero and 1785x also zero).

Thus finding a very reduced rectangle incorporating this particular pattern would be completely futile!

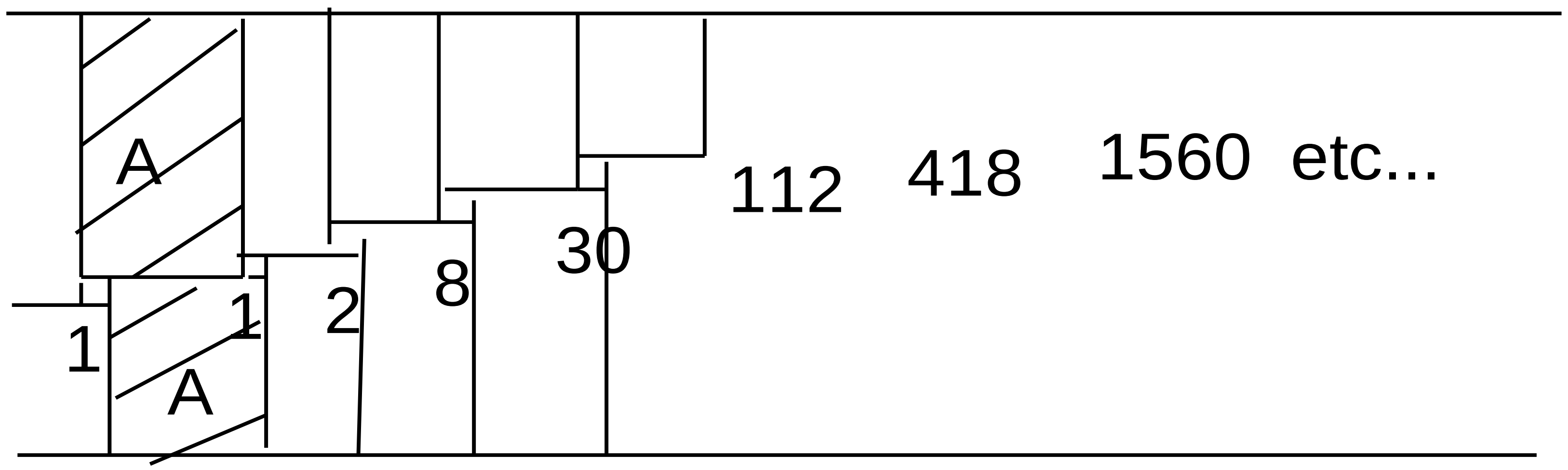
C1.6. MORE TYPES OF COEFFICIENTS

Such values as x + y, 4x + 4y, 11x + 11y and the like, apply to many Rectangles. Subject to checking, I think these may completely pair up with the Independent Elements in the last section of x, 4x, 11x etc.

C1.7. TRIAD NUMBER SERIES

This is the list of Coefficients seen earlier, namely 1, 4, 15, 56, 209, 780, 2911, 10864, 40545, 151316, 564719, 2107560, 7865521 etc... each number being the previous one times 4 less the one before that. (e.g. 15 = 4 x 4 - 1)

Now in some Symmetric solutions the same Element appears top and bottom in the central part shown shaded in Below.



NB. As long as A is large, the size of it is irrelevant.

Now what series is obtained if we commence with 1? We get 1, 2, 8, 112, 418, 1560 etc. Now if this series are halved we have .5, 1, 4, 15, 56, 209 which (if we ignore the .5) is the Triad Series shown above!

C2. INNER AND OUTER ELEMENTS

C2.1. NUMERIC ELEMENTS

Here are some elementary and basic principles. The Reader can easily prove these to be the case.

1. A Squared-Rectangle may contain

(1) More outer than inner Elements, e.g. [9] 69 x 61 with 5 outer, and 4 inner.

or (2) The same amount of inner and outer as in [10] 111 x 98 5 inner and 5 outer. or (3) More inner than outer as in [11] 98 x 86.

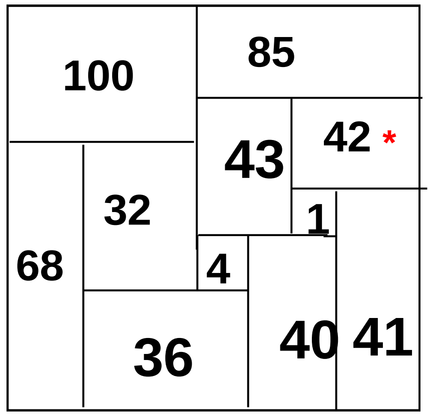
2. The minimum amount of outer Elements possible is 5 with Sides index S2223, if the Invalid solution [5] 2 x 2 with S2222 is disregarded. Thus for Order n (n > 5) there must be a minimum of 5 outer Elements and a maximum of (n - 5) inner.

3. An inner Element must be bordered by at least four Elements.
4. The smallest Element is always bordered by four and only four other Elements.
Often but not always these four Elements form an arithmetic progression like 15 16 17 18 or 4 7 11 15.
5. A corner Element must be bordered by at least three Elements.
6. A side Element must be bordered by at least 4 Elements (unless the solution is invalid).
7. The smallest Element must be an inner Element.
This can be easily shown geometrically by trying to draw otherwise ! Not illustrated.
8. The largest Element can be situated in the corner “Cornex” or on the side “Sidex” or as an inner Element “Centrex” . Centrex solutions are comparatively rare, particularly with low orders where in fact there are none until Order.

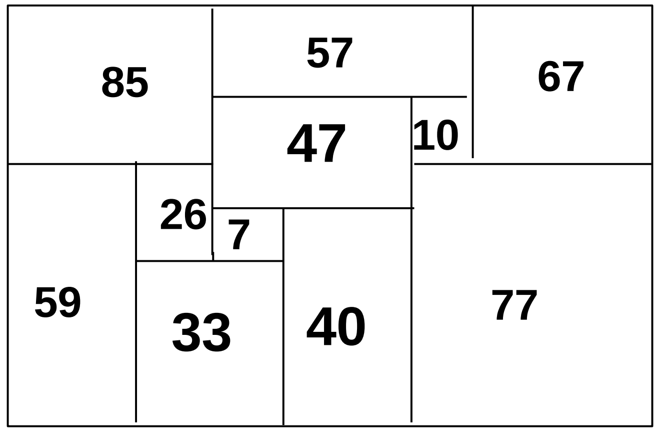
C2.2. POSITIONS OF SMALLEST ELEMENTS

This section is concerned with inner and outer Elements and where the smallest and largest Elements are in relation to the sides index. For instance,
In S2223 and S2323 solutions all inner Elements are smaller than all outer ones are there other sides ratios where this is true? Looking at all solutions to Order 11, only the following did not have all inner Elements smaller than all outer:
[9] 6 x 5 and [10] 6 x 5 (largest inner = smallest outer) and
[11] 22 x 18, 177 x 176, 185 x 168, 191 x 162, 199 x 169, 209 x 144 and 209 x 168 only 9 in the first 51 solutions. sides indexes for these include S2224 S2225 S2233 S2234 and 2324.

SHOWING INTERNAL ELEMENTS CAN BE LARGER THAN OUTER ONES



[11] 185 x 168



HERE THE THREE SMALLEST ELEMENTS ARE INTERNAL

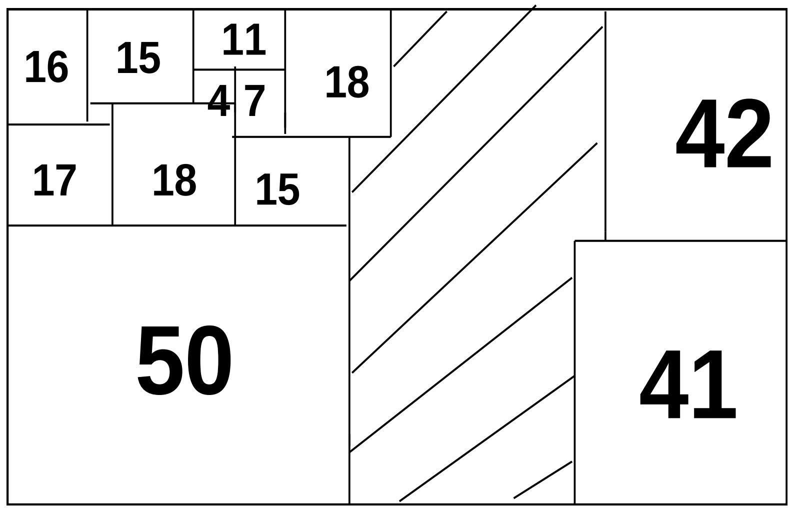
[11] 209 x 144

Above shows how an internal Element can be larger than several outer.
The larger the Order, the far more scope for internal Elements to be larger than outer ones.

C2.3. VARIOUS OBSERVATIONS

The following are given without formal proofs -
1. The smallest Element is always internal

2. ??The 2nd smallest Element is always internal but the 3rd smallest can be external. ** check first part **
3. The largest Element can be internal in some solutions.
4. In S2223 the four largest Elements are the corners, followed by the side Element. All the smallest Elements are internal.
5. In S2323 the four largest Elements are the corners followed by the two sides. All the smallest Elements are internal.
6. In S222# the four largest Elements are corners.



s2333+ SOLUTIONS.
Here is a theoretical diagram to show that the largest element must not be assumed to be contained in in side of 2 elements, although this is frequently true.

C3. CORNEX SIDEX AND CENTREX

C3.1. CALCULATING A CORNEX SOLUTION

Easily done by usual calculation methods from a rough diagram where a corner is clearly the largest Element.

C3.2. CALCULATING A SIDEX SOLUTION

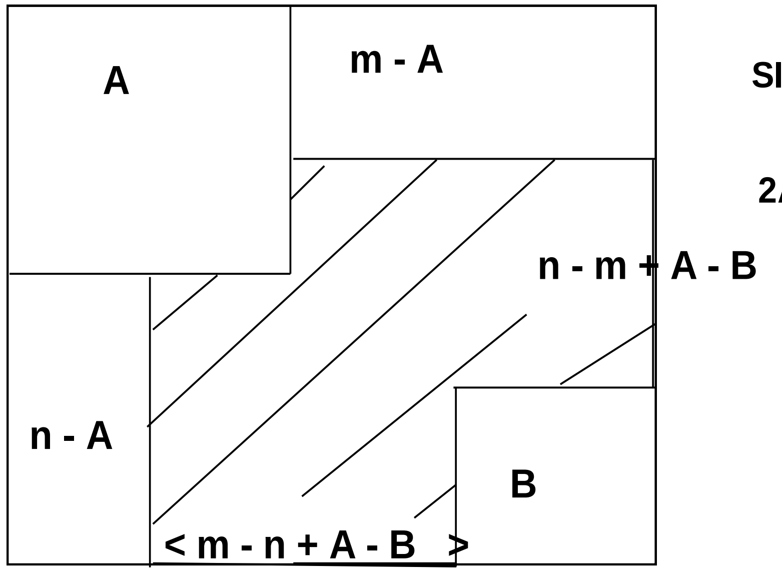
Take a known Cornex solution avoiding one where the largest Element is easily the largest and add a Triad at one end and recalculate.

C3.3. CALCULATING A CENTREX SOLUTION

Take a known Sidex solution a rehash the pattern and recalculate the rectangle by algebra as shown Below.
Occasionally the result is unlucky and careful choice is advised.

C3.4. SIDE ELEMENTS

Consider a solution with sides S2223 where only one side Element applies. It is easily shown that if the dimensions are m by n then the side Element has the value 2m - 2n, a linear amount divisible by two.
Consider S2224, S2225 S2226...
Elementary algebra shows that all the side Elements must total 2m - 2n also, again a linear amount divisible by two. Next consider sides S22## where # is 3 or greater in Below:-



Irrespective of the number of ELEMENTS between n - A and B and m - A and B, as they total 2A – 2B together i.e. divisible by two. Also b - a is always found to be half m - n.

If similar diagrams for sides S2### and S#### with # as 3 or more are drawn using m X n as dimensions and a b c and d as corner Elements, the linear total lengths can be shown to be divisible by two also, namely
2m - 2b - 2c and 2m + 2n - 2a -2b -2c - 2d.

As all solutions are one of these four groups it follows that in any Squared-Rectangle - The total linear length of all side Elements is always an even number!

3.5. CORNER ELEMENTS

The linear total of all four corner Elements may be even or odd. However in solutions S2#2# where # is 3 or greater, the four corners must equal 2n in length and thus always be an even number.

For other sides some will be odd and the others even for any given sides index.

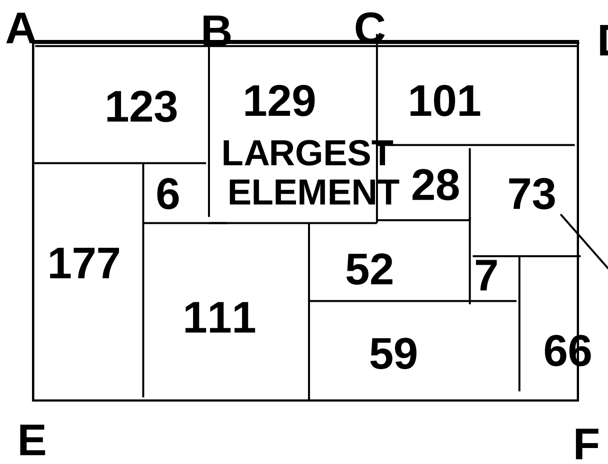
C3.6. INTERNAL ELEMENTS

The linear total of all internal Elements may be even or odd.
There may be some rules as to when the total is odd and when even.

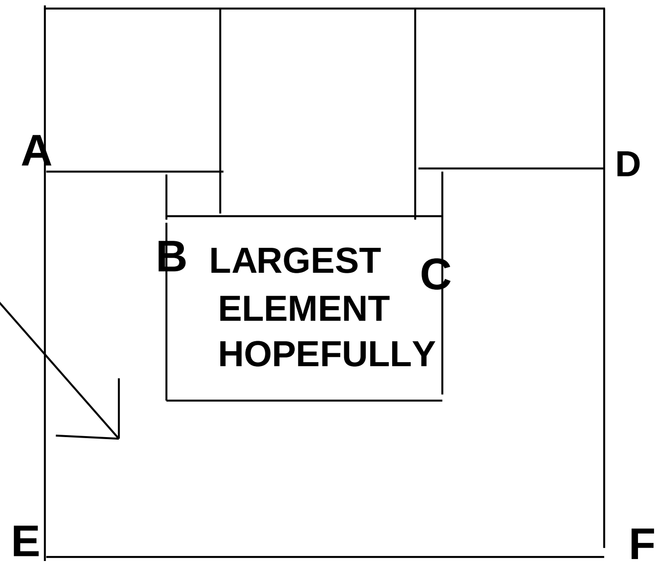
C4. RATIO

C4.1. LARGEST AND SMALLEST ELEMENTS RATIO

This is simply the largest Element in a rectangle divided by the smallest, the smallest always being an internal Element.



ORDER 12 IS THE SMALLEST ORDER FOR THE LARGEST ELEMENT ON A SIDE.



TO OBTAIN A CENTREX SOLUTION TAKE A SIDEX SOLUTION (AS ON LEFT) . REDRAW DIAGRAM WITH B & C INSET AND 5 NEW ELEMENTS ADDED AS SHOWN. THEN RECALCULATE. THE LARGER BC IS IN PROPORTION TO AB & CD IN ORIGINAL, THE BETTER THE CHANCE OF BEING SUCCESSFUL.

C4.2. HIGH RATIOS

In a rectangle such as [18] 1653 x 577 the Elements range from 1 to 315 in size.

The largest divided by the smallest is called the "ratio" and is 315.00. This is a high ratio. The ratio is surprising often an integer, but frequently not. The higher the Order, the higher the ratio which can be found.

C4.3. LOW RATIOS

In such rectangles the smallest Element is an surprisingly high number, surprising as really low ratios Below 8 are unusual.

C4.4. LOWEST POSSIBLE RATIOS

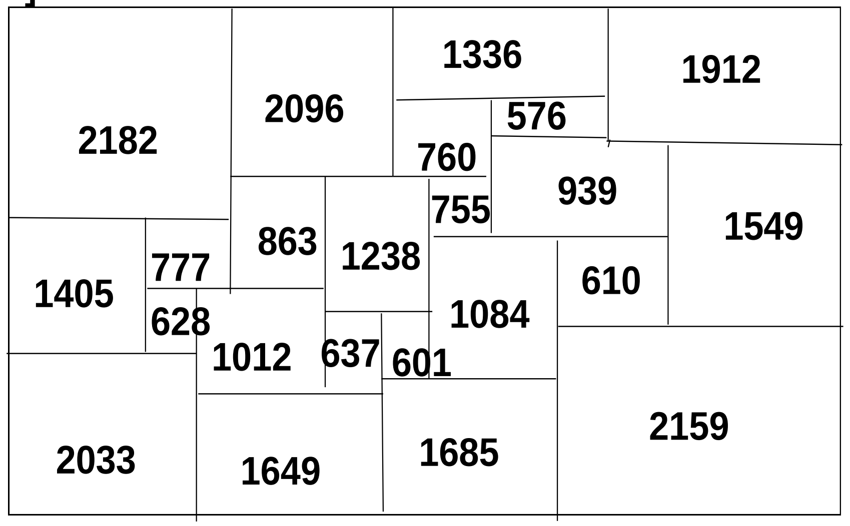
Below 1 is a remarkable asymmetric solution with a ratio of 4.16027 which takes much to beat.

Note how the Elements tend towards three typical sizes of 400, 800 and 1200!

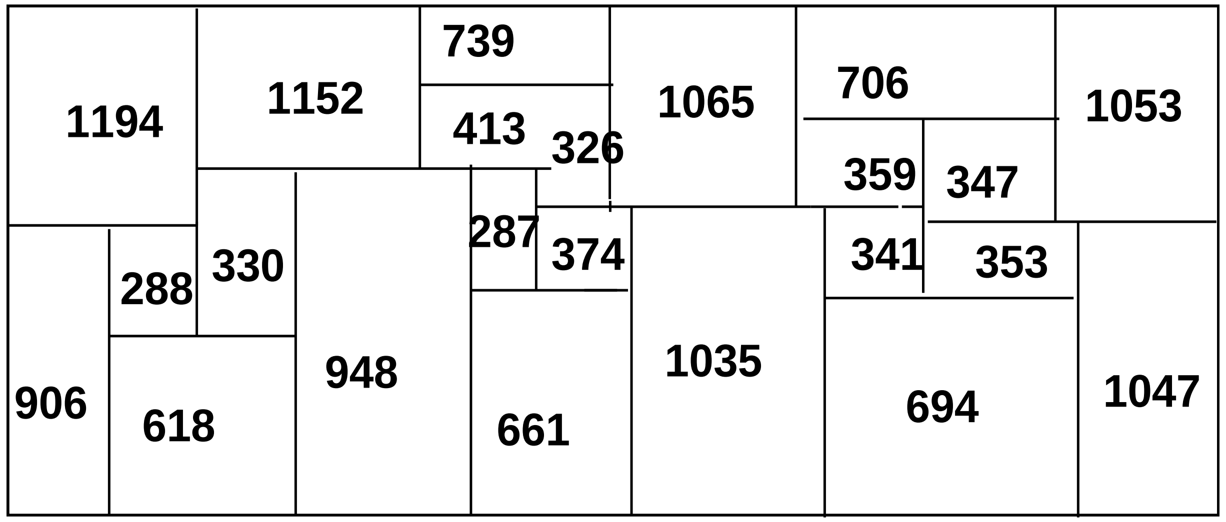
[10] 8 x 6 Non-zero has a ratio of 3.00 but is not a Valid solution.

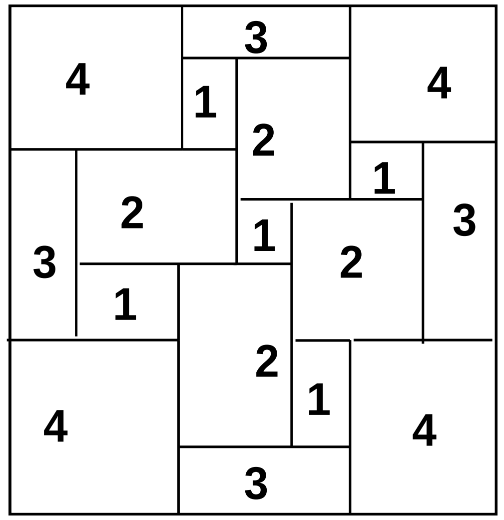
Below is a remarkable solution with a Ratio of only 3.7881944!! 2182 divided by 576

[23] 7526 x 5620



I discovered Below 2 with ratio exactly 4 which is 4-fold symmetric





[23] 5909 x 2100 LARGEST ELEMENT 1194

[17] 11 x 11 R495.

SMALLEST 287. RATIO 4.16027 VERY LOW.

RATIO 4/1 = 4.00 !

I guess that even lower Ratios must exist but whether that solution with the very lowest ratio can be found, is difficult to predict. Even harder would be to prove it was the lowest.

Looking at [23] 5909 x 2100 Above it is seen that the Elements neatly group into three sizes -

287 ... 413 SIZE 1 618 ... 706 SIZE 2 908 ... 1194 SIZE 3.

It is evident that the more these groups are manipulated to be as near as possible to an average size, the lower the Ratio will be. But of course it only needs one rogue smallest Element to ruin the Ratio! Also one highest Element!

C5. PRIME PERCENTAGES

C5.1. PRIME PERCENTAGES

How many primes occur in the Elements of a Perfect Rectangle?

An interesting Solution is [10] 65 x 47 with elements 24 19 22, 5 11 3, 25, 23 6, 17 .

Here 6 out of 10 are prime and PP = 60% Someone testing solutions to Order 16 found the next Solution which bettered this was [16] 179 163,24 139,123 43 13,7 17,11 2,9,19,1,18,80 10 out of 16, PP = 62.5% a particularly high value.

Another Solution is 235 139 151,127 12,163,125 79 31,110 48,46 14 19,211,9 5,4 20,13,10 3,23,165 16,149

13 out of 26 gives PP = 50%

C5.2. ARE THERE 100% PRIME PERCENTAGE SOLUTIONS?

Can all Elements in a Rectangle be Prime?

Most (if not all) Rectangles appear to have several instances where a single Element is bordered by 2 more.

Except for number 2 all primes are of course odd numbers and since 2 odds together give an even.

Any possible Primes only Rectangle is bound to have Element lines of 4 or more only.

It seems very unlikely that such a Rectangle exists.

C5.3. ARE THERE 0% PRIME PERCENTAGE SOLUTIONS?

Can all Elements in a Rectangle be Factorial?

Bearing in mind that only Reduced Elements are being considered, such a Rectangle cannot contain a Highest Common Factor throughout e.g. of 2. Obviously any Elements ending with a 0 2 4 5 6 or 8 are automatically factorial, but those ending with 1 3 7 or 9 might also be factorial in some solutions.

C5.4. RANGE OF PRIME PERCENTAGES

What range of Percentages (PP = Prime Percentages) occur for all Solutions?

The following statements have been proved true (copied from Internet by me) -

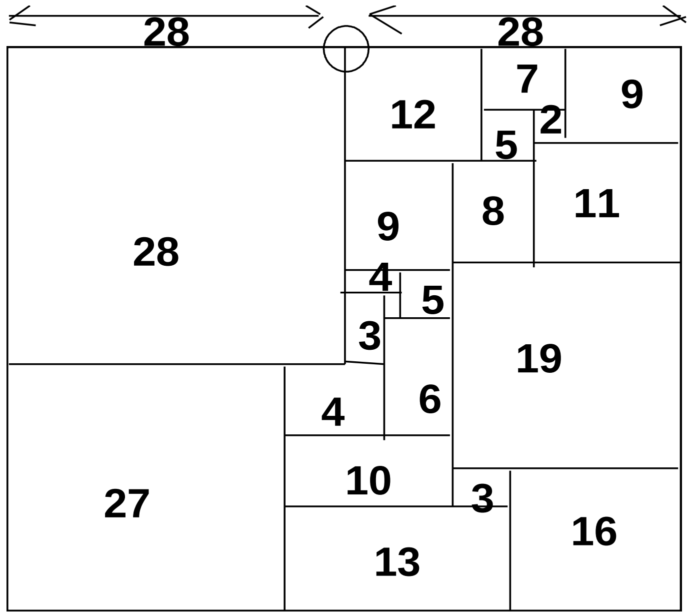
C5.5. VARIOUS FACTS CONCERNING PRIMES

All numbers greater than 188 can be expressed as the arithmetic sum of at most 5 distinct squares. Below 188 there are 31 numbers which cannot be expressed as a sum of distinct squares and only 124 and 188 require the sum of 6 distinct squares. Peabody knows - “All numbers smaller than 100,000 can be written - but greater than 17163 as the sum of at most 16 distinct square primes” so it is conjectured that no number will ever need to sum 17 square primes.

C6. MIDPOINT SOLUTIONS

C6.1. MIDPOINT SOLUTIONS

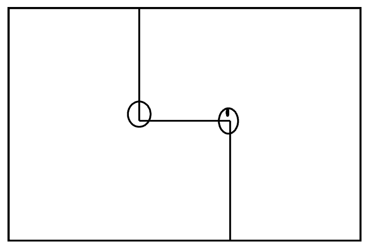
An example of this straightforward but relatively rare happening is shown below.
The Midpoint is shown by a Ring half way along the top horizontal.



[21] 56 x 55 IMPERFECT

SMALLEST FOUND

MIDWAY SOLUTION



I have deliberately not considered Invalid Solutions with two Elements of the same size horizontally together e.g. 28 and 28 which are common and unremarkable.

The Above Solution probably holds the small Dimensions record being only 56 across! Most solutions found tend to be Imperfect rather than Perfect.

Where Orders are concerned [13] 120 x 109 with corner Element 60 is the smallest with 13 Elements.

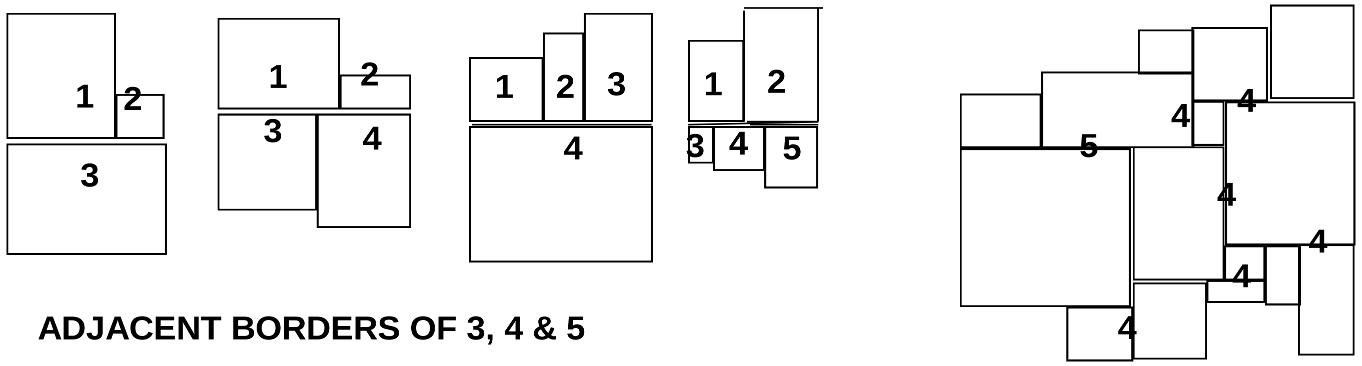
There is an Order 15 116 x 109 solution and solutions for all higher Orders are possible.

By removing the largest Element and putting the remaining Elements in twice as Above2 another solution may always be found - albeit Symmetric only. The resulting Imperfect Solution is of Order [2 x o + 2].

C7. BORDERS

C7.1. ADJACENT BORDERS

Adjacent borders is a term used in this section and shown Below various groupings into 3 4 5.. form part of a solution either horizontally or vertically. Each line is the border of at least three Elements.
1. No solution apart from the Invalid [5] 2 x 2 exists entirely of adjacent borders of three.
2. No solution exists entirely of adjacent borders of four.
Below 1 shows an attempt to achieve this results in increasing chaos.



ADJACENT BORDERS OF 3, 4 & 5

C8. INNER ELEMENT BLOCKS

C8.1. INNER ELEMENT BLOCKS

In many rectangles it is found that all the inner Elements join together as a single block.

Ignoring Invalid solutions, there are just 32 solutions otherwise up to Order 13. Most of these contain 2 blocks with a Triad or Pentad ending, but one [13] 633 x 295 has three.

10	130 x 79	c2	a1	S2324	13	123 x 80	4	a1	S2334
11	209 x 127	d3	a1	S2324	13	140 x 92	d3	b2	S2334
12	29 x 16 sym	c2	c2	S2424	13	152 x 100	e3	c2	S2334
12	46 x 26 sym	c2	c2	S2325	13	195 x 141	4	a1	S2333
12	92 x 60	f3	a1	S2334	13	211 x 144	4	a1	S2334
12	353 x 207	f3	a1	S2325	13	322 x 171	4	a1	S2424
12	353 x 232	c2	b2	S2334	13	585 x 343	4	a1	S2325
12	353 x 240	f3	a1	S2334	13	585 x 358	5	a1	S2324
12	353 x 255	4	a1	S2333	13	593 x 335	d3	c2	S2325
12	368 x 225	4	a1	S2324	13	593 x 342	4	a1	S2325
12	377 x 231	4	a1	S2324	13	593 x 392	4	a1	S2334
12	386 x 207	g3	a1	S2424	13	608 x 335	d3	c2	S2424
13	72 x 44	5	a1	S2324	13	608 x 377	5	a1	S2324
13	76 x 44	4	a1	S2325	13	633 x 295	21	a1	S2425
13	112 x 75a	4	a1	S2334	13	633 x 382	5	a1	S2324
13	112 x 85	4	a1	S2343	13	663 x 352	4	a1	S2424

8.2. ENDINGS FOR BLOCKS

The coding Above denote different endings which are shown Below

C9. RECORD SIZE ELEMENTS

This section is concerned with the largest and smallest Elements found in rectangles.

The Elements considered here are the full size ones, not reduced ones.

What ranges of larger and what ranges of smallest Elements are possible per Order?

Do the ranges vary much for Invalid, Imperfect and Perfect solutions? C9.5

How does the minimum amount of Unknowns affect these ranges? C9.6

Does the elongation of rectangles have a bearing on the values? C9.7

Are there formulae or rules to be found connecting the values in different orders?

Are there other features that radically affect the orders?

Do the results found give definite rules, general tendencies or are they just coincidental?

As values which are primes can only occur where the Reduction Index is 1 and all others must be factorial, it is not surprising that factorial numbers will occur more often than primes and evens more often than odds.

Many rules found are generalized rules relating to tendencies rather than absolute rules relating to every instance.

For the time being no reference is made to the dimensions of actual solutions containing the record sizes, but it must be stated that records often shared by several solutions rather than just one.

These solutions may be differing types e.g. Invalid and Valid, Perfect and Imperfect etc..

C9.1. ABBREVIATIONS LLE SSE…

Clearly a given solution has a largest and a smallest Element (even if more than one exist). But within a LL solutions for a given Order there are a largest and a smallest of such largest Element and ditto for smallest Element, a little more confusing.

“LLE10” means the *largest #largest Element actually possible for Order 10.

“SLE11” means the *smallest #largest Element for Order 11.

“LSE12” means the *largest #smallest Element for Order 12.

“SSE13” means the *smallest #smallest Element for Order 13.

“LLE14?” means the largest# largest Element found so far using a computer by me.

Some are believed to be the record possible, but for many better values are undoubtedly possible.

Note the question mark used where the best value is unconfirmed.

note carefully that the words marked * relate to all solutions for an Order and # to individual solutions.

C9.2. THE RANGES FOR ORDERS TO 13 CALLED “SLE” and “LLE”

From my list of rectangles which excludes duds and Compound solutions but includes Invalid and adjacent

solutions the following ranges for smallest and largest larger found -
Order 7 12 to 12 Order 8 20 to 20 Order 9 30 to 36 Order 10 44 to 60
Order 11 56 to 105 Order 12 120 to 180 Order 13 164 to 336
SLE13 = 164 and LLE13 = 336, quite a considerable range whereas SLE7 and LLE7 have no range at all.

It is easy to see the range definitely widens as the orders increase - this is always true for any given Order.

The fact that all values shown are evens is purely coincidental! No rule here.

In view that there are so few solutions for low orders it is not surprising that values found are less reliable when looking for rules, than in higher orders.

Sometimes values do not even exist for low orders - as in xyz for Order 9 and Imperfect solutions for Order 10 and 11.

C9.3. SMALLEST SMALLEST ELEMENT - CALLED "SSE"

It was seen elsewhere that smallest Elements are always internal ones.

Since all Orders have Reduction 1 solutions containing an Element 1, clearly the values for Valid rectangles are simply 1 for any given Order.

However, in Invalid solutions many of which contain zeros, 0 is clearly the SSE is 0 for any given Order.

C9.4. LARGEST SMALLEST ELEMENTS - CALLED "LSE"

These are less easily found and will occur where the largest/smallest Element ratio is very low.

As such solutions are relatively very few indeed, it is difficult to find reliable values even by computer where millions or trillions of solutions exist for the Order.

The actual values for orders 5 to 13 are - [5] 0 [7] 1 [8] 0 [9] 5 [10] 7 [11] 16 (invalid) and 13 (valid) [12] 32 (invalid) 26 (valid) [13] 39 (invalid) and 46 (valid).

Only orders 5 and 8 have LSE of 0 and higher orders have much larger LSE's than these.

Excepting orders 5,7 and 8 the series definitely always increases as the Order increases.

C9.5. RECORD ELEMENTS VERSUS INVALID, IMPERFECT and PERFECT

There is some variation in record size Elements where Invalid, Imperfect and Perfect solutions are considered.

C9.6. RECORD ELEMENTS VERSUS x, xy, xyz … SOLUTIONS

I was amazed to discover how much the amount of minimum Unknowns does affect record sizes, as the differences in the low orders are not that remarkable, and Orders 7 to 9 have no xyz solutions anyway.

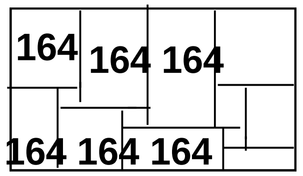
But in higher orders differences are stark, e.g. LLE28? = 818451 for xy but LLE28? = 1208790 for xyz!

In this case any Elements 830000+ must have 3 or 4 Unknowns.

Subject to checking xyza and xyzab solutions probably have a still greater LLE.

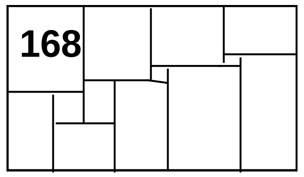
In the case of x solutions - all of which are duds lle's are considerably lower than xy ones.

SMALLEST LARGEST ELEMENTS



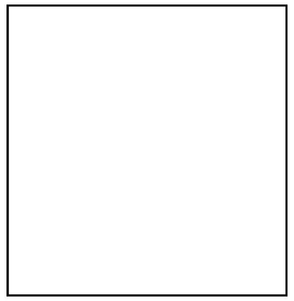
ORDER 13

[13] 615 x 328 xyz
INVALID

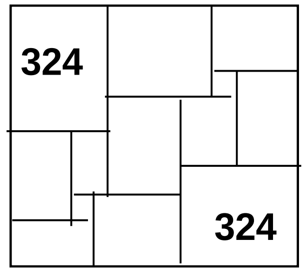


[13] 633 x 295 xy
PERFECT

LARGEST LARGEST ELEMENTS



[13] 576 x 576 xyz
INVALID



[13] 576 x 512 R32 xy
INVALID

C9.7. RECORD ELEMENTS VERSUS DEGREE OF ELONGATION

As it would not be expected for long thin rectangles (very elongated) to have the largest possible Element for an Order, elongation has clearly a large influence on record size Elements.

Generally, the less elongated the solution the larger the largest Element *tends* to be, and the more elongated, the smaller the largest Element *tends* to be. Having said this, some 70% elongated solutions have a larger largest Element than a 95% one. Extremely elongated solutions on the other hand will tend to have very small largest Elements, and probably contain the record sizes.

C9.8 LLE FOR xy and xyz SOLUTIONS COMPARED

Below is a table of largest Elements found so far for orders 7 to 30.
Note that better statistics actually exist for the larger orders than shown (i.e. higher):-

xy solutions (others possible)	Order	xyz solutions (others possible)
24 x 21 Invalid only possible 12	7	none possible
40 x 35 Invalid only possible 20	8	none possible
69 x 61 36	9	none possible
105 x 104 60	10	48 128 x 96 Invalid only possible
199 x 169 105	11	112 194 x 192
318 x 315 180	12	194 338 x 334
576 x 512 Imperfect 324	13	336 576 x 576
1017 x 795 534	14	568 1031 x 1016...
1808 x 158 920	15	1,004 1732 x 1726
2778 x 2671 1.514	16	1,733 3034 x 2867
4711 x 4571 2,590	17	3,068 5232 x 5216
8217 x 7555 4.420	18	5,315 9123 x 9061

14193 x 12033 7,431	19	9,319 16321 x 15484
24009 x 22650 12,687	20	15,825 27278 x 25393
36882 x 36459 20,910	21	27,416 46704 x 46451..
67680x56601 35,340	22	48,988 84635 x 80404
116193x98372 61,021	23	83,988 71657 x 63308
192546x191775 104,754	24	144,025 255489 x 252640
157627x133133 171,106	25	256,925 477863 x 438834
590817x515813 295,515	26	432,060 777566 x 760156
888591x867628 496,939	27	740,441 1307443 x 1268401
1512465x1245856 818,451	28	1,208,790 2104948 x 1974349
2590810x2574649 1,440,090	29	2,230,926 3957264 x 3941086
4145575x4100986 2,287,622	30	3,721,373 6687728 x 5876048

In Above, it is interesting to see that each amount is very roughly 3 times that of that two before, e.g. 20910 x 3= 62730 against 6102. Or each amount roughly square root 3 or 1.732… times that of the previous.

Also each is usually a bit more than the total of the previous two e.g. 1733 + 3068 = 5011 against 5315.
The gap widens in favour of xyz solutions as the Order increases - in Order 30 3721373 is 67% higher!

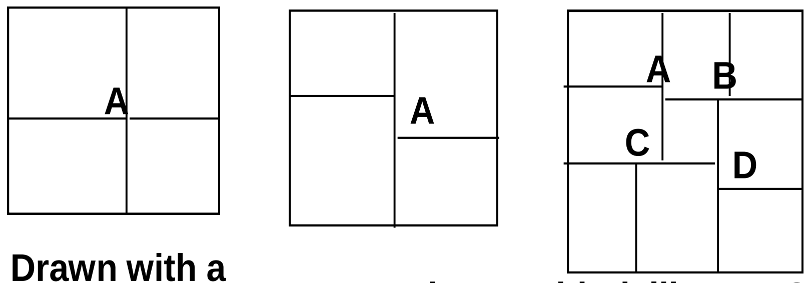
C10. CROSSOVERS

C10. CROSSOVER POINTS

Referred to as a "CROSS" on the Internet these are described briefly in Section A, a Crossover is a comparatively unusual occurrence when compared with all solutions within a range. It appears at first sight to occur as a sheer coincidence.
Also like so many aspects of Squared-Rectangles it seems to be a difficult feature to find good explanations for, and this is not helped by that Theory suggests one thing and Practice something different!

C10.1. TYPES OF SOLUTIONS CONTAINING CROSSOVERS

Crossovers occur from worthless solutions to the best ones. Starting with the worst -
1. Dud Solutions always contain at least one Crossover, e.g. [4] 2 x 2 having a box of four elements of 1.
In calculating this category of Rectangles by algebra using a roughly drawn diagram, a Crossover may be clearly in the diagram (as Below 1) or seen to exist on further inspection (as Below 2). But even where a Crossover is not so obvious (Below 3) the Crossover may be seen as occurring by design and not by coincidence.



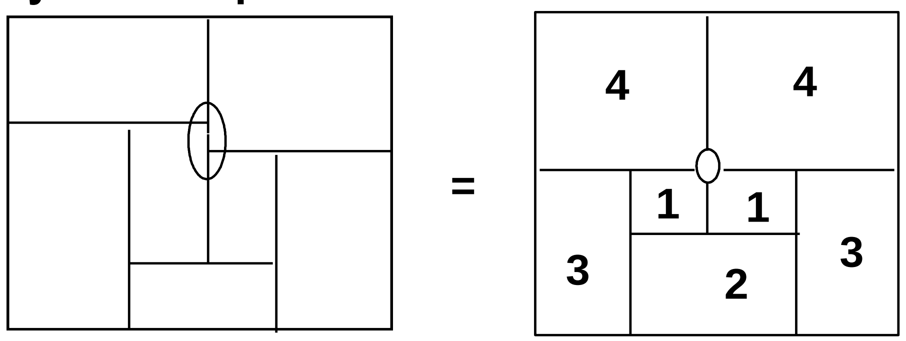
2nd & 3rd Diagrams are not drawn deliberately with Crossovers however with a little reasoning it will be seen that at A B C & D the lines more logically will join at single points and are CROSSOVERS

Drawn with a Crossover present. Not drawn with deliberate Crossovers.

2. Zero Solutions usually have no Crossovers in theory, but in practice may have one for every Zero. If the Zero Element is not drawn at all, or indicated by a dot only, a Crossover Point always occurs in practice (though not in theory).

3. Some Non-zero Solutions also have Crossovers in practice (but not in theory).

A symmetric pattern as below indicates that the lines at " O " actually Cross in reality.



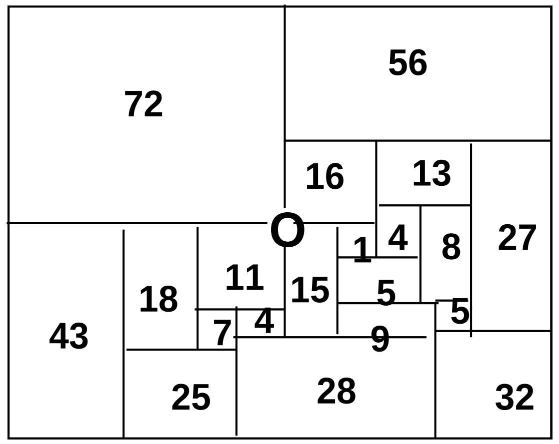
Crossovers will be indicated with " O "

4. Normal Perfect or Imperfect Solutions which contain Crossovers are a much more interesting set and the term "CROSSOVER" from now on should be regarded as applying to these alone.

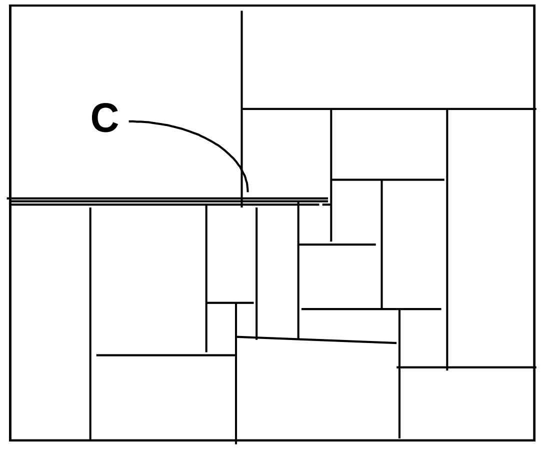
C10.2. CROSSOVERS WHICH ARE APPARENTLY COINCIDENTAL

Below a good Crossover example is shown - [20] 128 x 115. If we had randomly chosen Pattern A for calculating a solution by algebra, then we would find [20] 128 x 115 and that by coincidence * that the piece of line at C disappears, being of zero length. But suppose Pattern B had been randomly chosen! We would again find that the piece of line D does not exist either in reality! However the solution is [20] 128 x 115 - not a Twin, but the same solution.

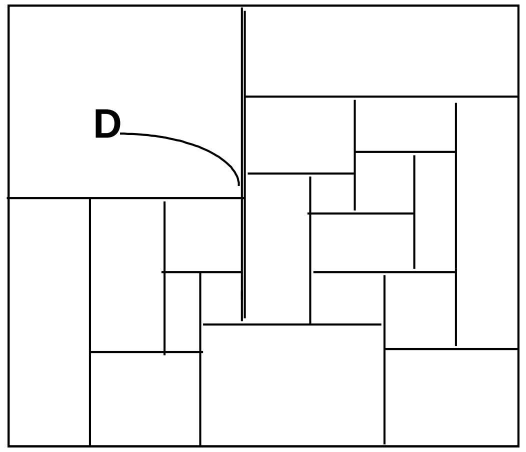
* It still has to be found whether this is coincidence or not.



[20] 128 x 115 RI 172 Imperfect
CROSSOVER AT O



Rough Pattern A



Rough Pattern B

C10.3. THE PARADOX OF ONE PATTERN OR TWO!

Readers will say that Patterns A and B are clearly different, and I agree. But the resulting Solutions are the SAME!
For ages I believed that different Patterns must always give different Solutions.
Note that many badly drawn patterns will change shape when an internal element or more proves to be negative -
in such cases the patterns are beyond doubt the same - but this is different!
Of course when drawing out Patterns, these are *theoretical* Patterns in which we assume the following:-
1. That all squares drawn will have positive values.
2. That none will be Zero.
3. No two horizontal lines will effectively become one and likewise-
4. No two vertical lines will effectively be one.
These are usually true, but any or all of these can prove to be wrong!

C10.4. CROSSOVERS AND REDUCTION INDEXES

Could a single Crossover solution have two differing R.I.' s and two lots of full dimensions at the same time, coinciding with the two patterns - horizontal and vertical?

Are most or all Crossover Solutions highly reduced rectangles? If so do the Full Dimensions always vary?

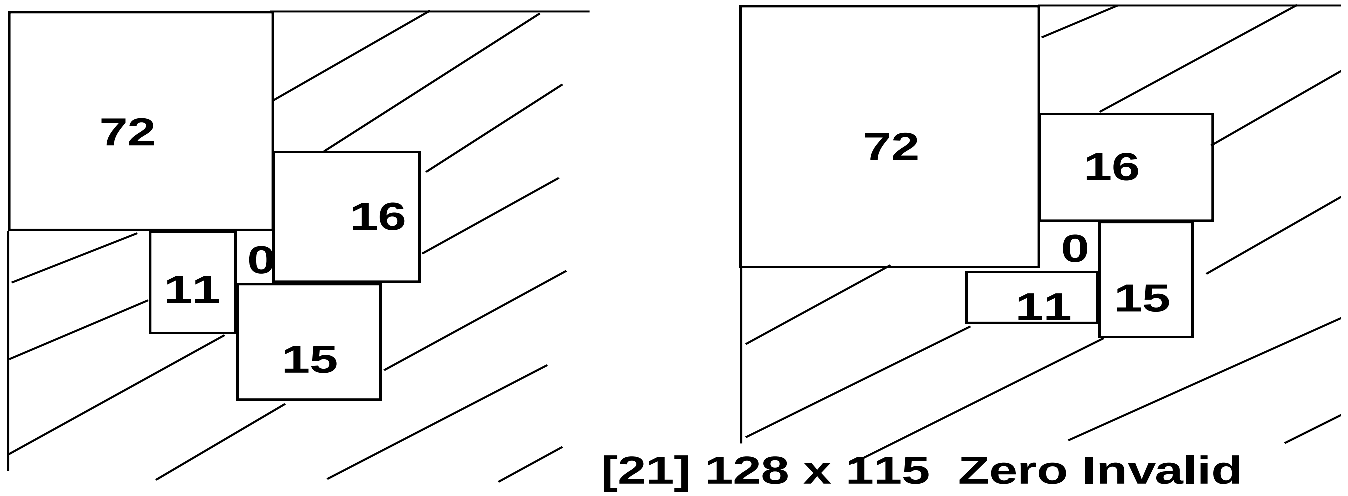
Much to be done. 1. Are there real reasons why they occur?
2. Can complexities vary with the 2 patterns - very likely yes!
3. Are all Crossover Solutions Imperfect? NO!!

C10.5. WHAT CAUSES CROSSOVERS TO OCCUR?

This seemed very puzzling until I realized a simple fact that

<u>An extra Element of zero size can always be drawn at the point of Crossover thus increasing the Order by one!</u>

Thus [20] 128 x 115 Above can be made into Zero solution [21] 128 x 115 Below. The Zero Element can always can be put in one of two places as shown. No problem in this. Clearly the Reduction Index for the bigger Order will be greater.



[21] 128 x 115 Zero Invalid

So once suitable Zero Solutions have been found (most being unsuitable) a Crossover Point automatically occurs for the Order below, once the Zero has been removed, provided that the resulting solution is Valid.

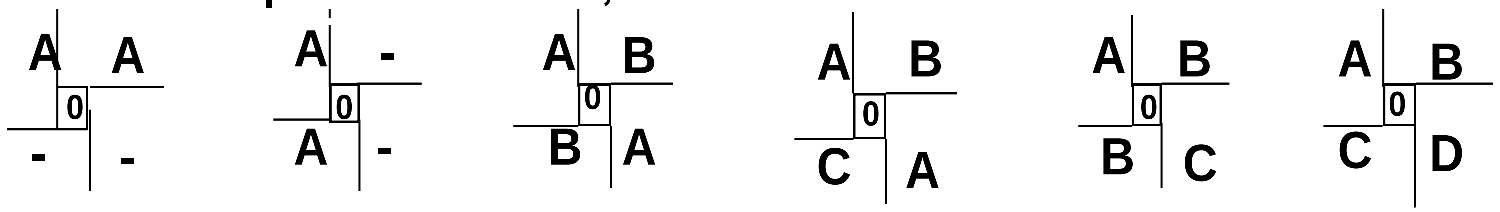
In many instances Duds occur when the Zero is removed. Occasionally a Non-simple solution arises (see 2 below).

C10.6. LIST OF GOOD CROSSOVER SOLUTIONS FOUND SO FAR

1. [19] 60 x 42 r270 imperfect s2324 xyz
2. [19] 118 x 117 r116 imperfect s2234 xyz is Compound containing [9] 33 x 32!
3. [20] 128 x 115 r172 imperfect s2234 xyz
4. [20] 149 x 146 r147 imperfect s2233 xyz

C10.7. ZERO SOLUTIONS FOR GOOD CROSSOVERS

To establish on inspection whether a known Zero solution can be changed into a Good Crossover solution, look at the four Elements bordering the Zero. If the same number occurs in horizontal pairs or vertical pairs as Below 1 and 2, Dud solutions will occur, but anything else is OK, and these include the same number on opposite diagonals (either once or twice).
A A denote repeated numbers, whereas A B C D are deemed all different numbers.



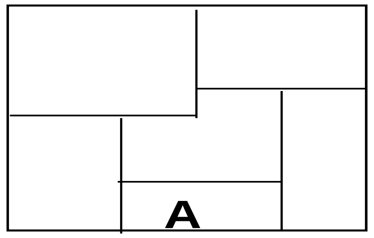
No Good - give DUDS

Acceptable patterns for Good Crossovers.

C11. ELEMENT SIZES DETERMINED BY DIMENSION SIZES

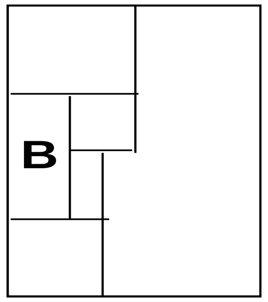
C11.1

The following are mentioned in Section E in connection with Links, but the purpose of this Section is to concentrate on individual Elements whose values are fixed by the actual Dimensions of the Rectangle.

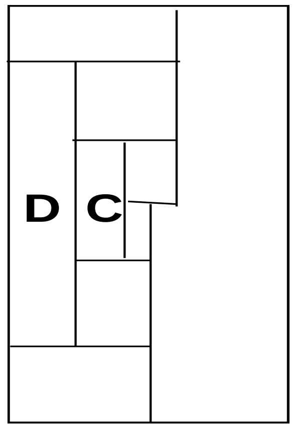


A = 2m - 2n

* if placed horizontally.



B = ¼n or ¼m*



ELEMENTS A B C D
can be expressed in
of the Dimensions m
as shown.

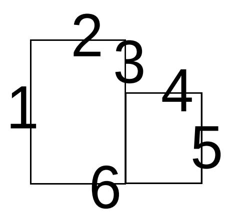
C = 1/11n or 1/11m*
D = 4/11n or 4/11m*

Suppose in Above 1 the Dimensions were 69 x 61. Using m = 69 and n = 61 it is evident that A is fixed at 16.

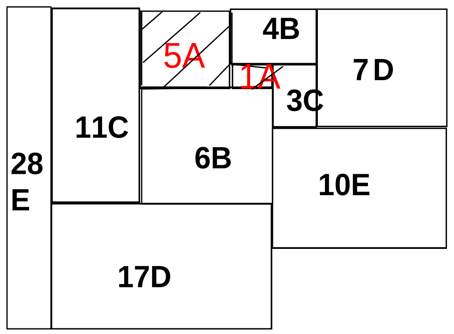
C12. AREA RELATIONSHIPS - *SEE SECTION R*

C12.1 AREA RELATIONSHIPS

Below a set of Elements has been put together starting with 5 and 1
(any combination could have been used).
5 and 1 have an Area of 25 + 1 = 26 = 26 x 1.
By adding at sides 3 & 6, 4 and 6 have an area of 16 + 36 = 52 = 26 x 2.
By adding at sides 1 & 4, 11 and 3 have an area of 9 + 121 = 130 = 26 x 5.
By adding at sides 3 & 6, 7 and 17 have an area of 49 + 289 = 338 = 26 x 13.
By adding at sides 1 & 4, 28 and 10 have an area of 784 + 100 = 884 = 26 x 34.
The coefficients of 26 are alternate numbers in the Fibonacci series, 1, 2, 5, 13, 34, etc. provided we add alternately at sides 3 & 6 and 1 & 4!



SHOWING SIDES
CODES 1 TO 6



The Reciprocal arrangement applies no matter what pair of numbers we commence with, e.g. If we started with 6 and 7, 36 + 49 = 85 and the B pairs twice this, the C pairs 5 times this, the D pairs 13 times this ...

As this subject links with Reciprocal Pairs, it is dealt with fully in Section R of this book, which see.

C13. HORIZONTAL AND VERTICAL LINES (AND LEVELS)

(The following may be better placed elsewhere in the text)

C13.1. QUANTITY OF HORIZONTAL LINES IN A SOLUTION

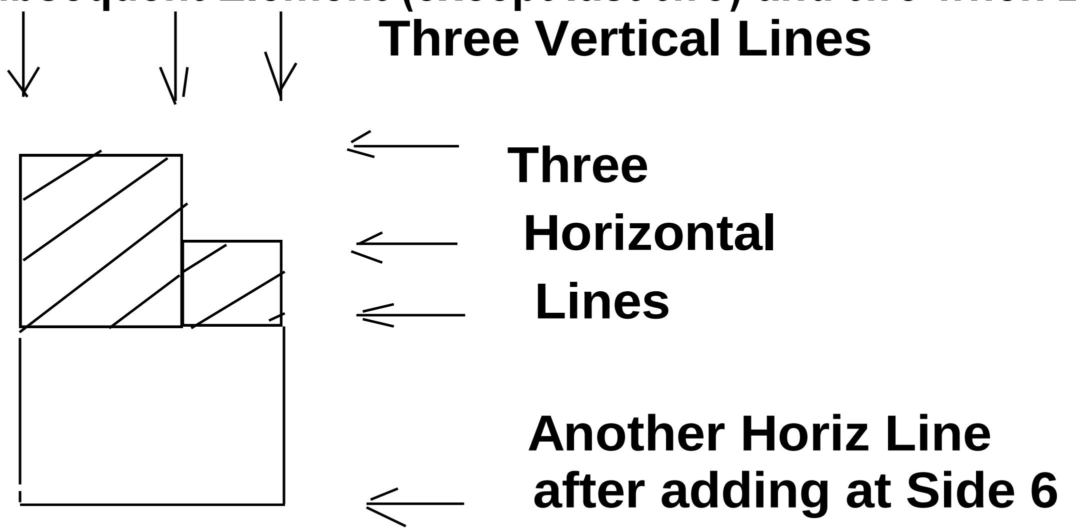
The first two Elements in an xy Rectangle construction produce three horizontal and three vertical lines as Below.

Now when Elements are added at sides 1 3 or 5 it is evident that one new vertical line is created (the two horizontal lines already exist). Likewise at 2 4 or 6 one new horizontal line is created e.g. as in side 6 Below.
(The two vertical lines already exist).

However in the final Element at Side 3 (or side 4 if preferred) the bordering lines are already in existence.

So in an Order 9 solution we may expect there to be 12 Horizontal and Vertical lines altogether made up as follows:
1st and 2nd Elements - six. 3rd one. 4th one. 5th one. 6th one. 7th one. 8th one 9th nil. This is proved correct. So -
IN AN ORDER [O] SOLUTION THE TOTAL QUANTITY OF LINES (VERTICAL AND HORIZONTAL) IS [O + 3]

But what about xyz (not detached) Solutions? It is found that the first three Elements (x, y, z) give 8 lines and each subsequent Element 1 more except the last two. Total O + 3 as before. Xyz (detached) Solutions give 4 lines for the first two Elements (x and y), 1 for each subsequent Element (except last two) and two when z is introduced. Total O + 3 again!

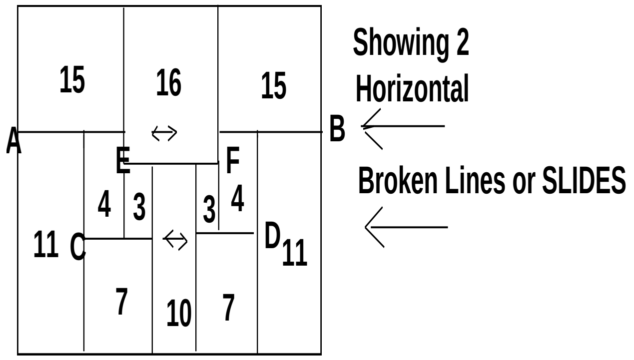


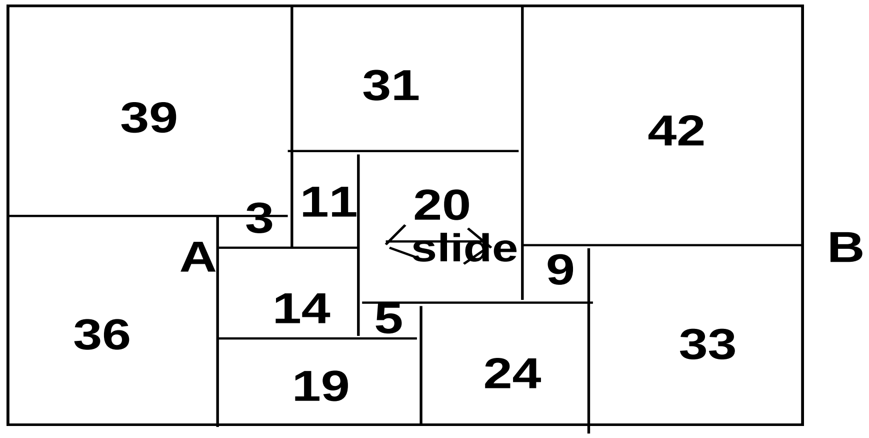
Inspection shows that there in any Solution whatever the total Horizontal and Vertical Lines is always the Order plus three, regardless of how many algebraic Unknowns are needed for the Solution.

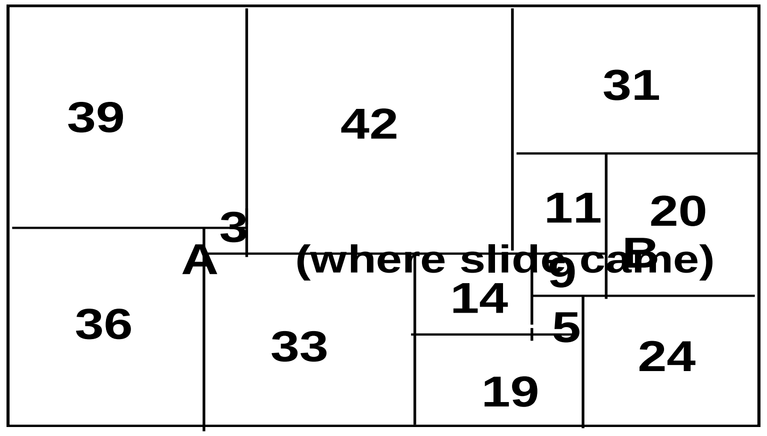
Of course 4 of these lines (2 Horizontal and 2 Vertical) form the borders so there are always O - 1 internal lines in any solution. However the proportions of Horizontal to Vertical ones varies of course.

C13.2. UNBROKEN AND BROKEN LINES CALLED SLIDES

Many of the smaller Solutions contain a Broken Line or more. Many Symmetric Solutions have them - Below lines AB and CD are broken though on the same horizontal level whilst EF etc. are Unbroken.
Although this Solution [12] 46 x 26 contain 7 Horizontal Lines, it has only 5 Horizontal Levels.







[13] 112 x 75 (1) & (2) with Identical Elements

Above 1 shows a Broken Line AB and Above 2 an Unbroken Line or SLIDE AB.
The first has 8 vertical and 8 horizontal lines = 16 the second 9 vertical and 7 horizontal lines = 16.
Where Broken Line solutions exist there is sometimes an associated solution which has a related but Unbroken Line.
Twin Solutions with Identical Elements often follow this pattern of Unbroken and Broken.

C13.3. ROWS AND COLUMNS (COMPARED WITH HORIZONTALS & VERTICALS)

In the Grid System of recording Elements by rows and columns it is clear that the number of rows r, and number of columns c is one less in each case than the horizontals and verticals. $h + v = o + 3$ but $r + c = o + 1$ (N.B. Slides ignored)

C13.4 VARIATIONS PER ORDER IN QUANTITIES OF ROWS & COLUMNS

The three Order 9 solutions each contain 5 rows and 5 columns of Elements. (This is according to the convention of showing Solutions horizontally).
Most of the Order 11's have 6 rows and 6 columns. But 185 x 151 and 209 x 127 have 5 rows and 7 columns - not really surprising since these are so elongated.

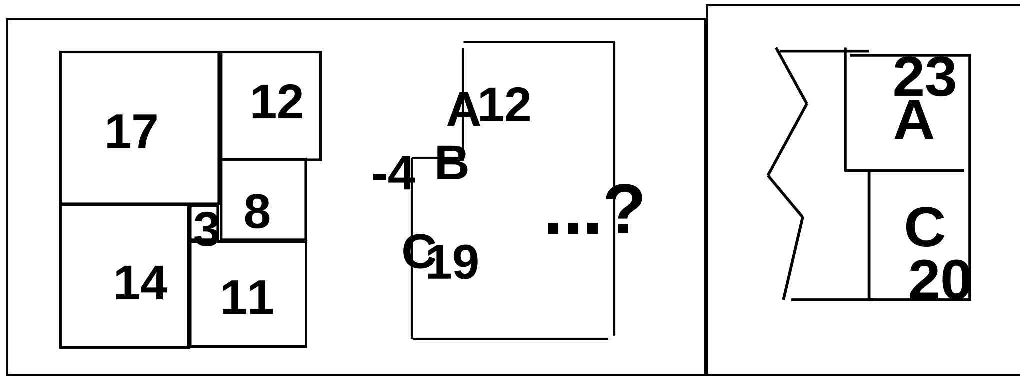
In even Orders the rows and columns always vary, their sum being an odd number. But in Order 10, 5 rows and 6 columns is more frequent than 6 and 5. In Order 12 we find most have 6 and 7, the remainder having 7 and 6 (rows & columns).
In the case of larger Orders the variations are much more than just 2.
e.g. Order 32 has a maximum of 18 rows but may have only 14 rows in very elongated solutions. Order 33 has also a maximum of 18 rows.
<u>FOR ODD ORDERS THE MAXIMUM NO OF ROWS POSSIBLE IS (O+3) /2</u>
<u>FOR EVEN ORDERS THE MAXIMUM NO OF ROWS POSSIBLE IS (O+4) /2</u>

C14. DISTRIBUTIONS AND TOTALS OF ODD ELEMENTS see section S

C14.1. TRIAL & ERROR ENDS

The following may seem pointless to the Reader but its use will be explained later!

In Below 1, two random numbers have been selected 17 & 14 and other Elements added. At any stage of construction three lines exist which will be called "A B and C" of which B will sometimes be regarded as negative (turning to left) otherwise positive. At the point of the theoretic trial solution Below A = 12, B= -4 and C = 19. Let us suppose some remaining pattern gives rise to an actual Solution. If true then the Ratio 12:-4:19 will be the key. Then if another random pattern gave rise to say 84:-28 and 133 (which has the same Ratio 12:-4:19) then the pattern to the right will fit even though it is often found that in order to remove possible fractions Up-rating of part or both parts may be necessary.
Note that in this Section the importance is not actual *value of the numbers* - but the actual fixed ***<u>RATIO of numbers</u>***.



For this DIAD ending to apply,
A must equal B + C
(still true if B is negative)

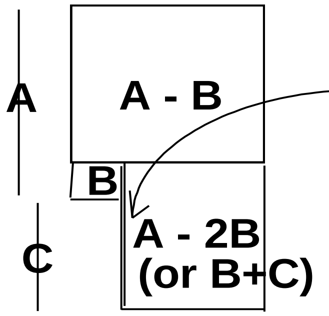
C14.2. THE SIMPLEST TRIAL & ERROR END PATTERN RATIO - DIAD

Look at Above 2 showing a theoretic Diad End with any values we choose, say 23 and 20. Now *if* our trial & error pattern (from the left) just happened to end with A = 23, B = 3 and C = 20 then a Diad ending will give a proper rectangle. It is readily seen that the two Elements must be A in value at top) and C in value at bottom - also that A = B + C. This relationship will be termed the RATIO CODE. Without attempting to give reasons - two important facts are found:

1. No matter what Ending (i.e. Pattern of Elements) is drawn on the right hand side, each and every individual Element may be expressed in coefficients of A B and C alone.
2. There is always a RATIO CODE which is also in terms of A B and C alone . This RATIO is absolutely fixed. In Above 2 this is *A = B + C*. (Note that although the Ratio 3A = 3B + 3C also stands true here, it is sensible to cancel down the Ratio as far as possible). From now on dummy Element Numbers will be omitted since they could be virtually any set of numbers.

C15.1. RATIO CODE FOR TRIAD

Studying Below it is easy to see the value of the Elements in terms of A B & C. At the double line an equation has been formed from which is found the RATIO CODE of *A = 3B + C*. Thus say in a trial diagram we arrive at A = 21 B = 2 & C = 15 this Ratio is found to agree which means that a Triad end is possible. In both the DIAD & TRIAD patterns it is never necessary to UP-RATE all the left hand Elements by some integer. But in other patterns this frequently applies - though not always.



At this line B+C (at left) must equal A-2B (at right)
So A = 3B + C

C15.2. IS THERE ANY POINT IN CALCULATING RATIO CODES?

There would be not much point in calculating these - except for the purpose of Trial & Error Computer programs!

These programs are based on luck so in theory are not worth while. However in practice they do work well if and when a great number of ending options are tried. Also in working out one numbered Element at a time and testing each ending in turn the possibility of finding Rectangles by chance is again greatly improved.

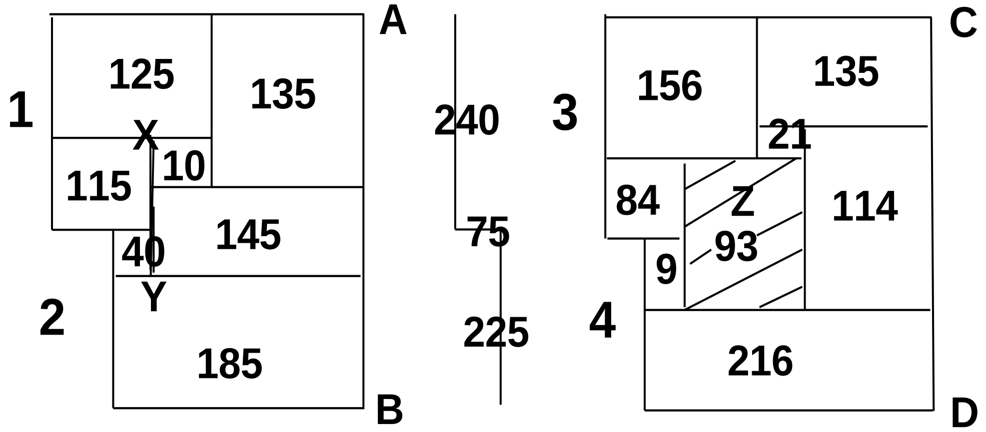
C15.3. MORE PROFITABLE PATTERNS FOR RATIO CODES

Although Ratio Codes for Diad, Triad, Doubles & Triples can be successfully, they have a tendency to produce Symmetric solutions. Also the Solutions are restricted to “ENZ” solutions (mentioned elsewhere). So it is better to use patterns which do not have a single Element at A or at B or at C. The simplest of these is the Pentad Ending with five Elements followed by a few with six Elements.

C15.4. TWO PATTERNS WITH THE SAME RATIO CODE

Both the patterns Below happen to have the same Ratio Codes of *5A = 7B + 3C*! But the first has 7 Elements and the second 8. The second is obtained by adding an Element at XY (called Z here). Z happens to be three-fifths of XY as explained below.

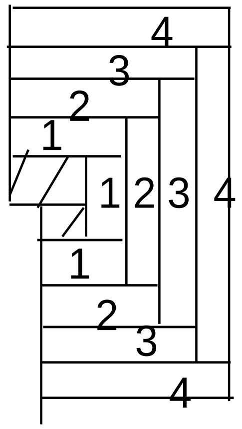
In working the Algebra out it is found that AB has to one-third of XY and CD must be one-fifth of Element Z (i.e. 155 is 1/3 of 465 and 93 is 1/5 of 465). In the dummy Element numbers shown (up-rated to make A B C 240 75 and 225 in each case) those in the first are divisible by 5 and those in the second by 3. This means that Z has to be three-fifths of XY in order to fit.



So these patterns are interchangeable - subject to various Up-rating and recalculation!

C15.5 RATIO CODES SERIES FOUND BY ADDING CLAWS TO SYMMETRIC PATTERNS

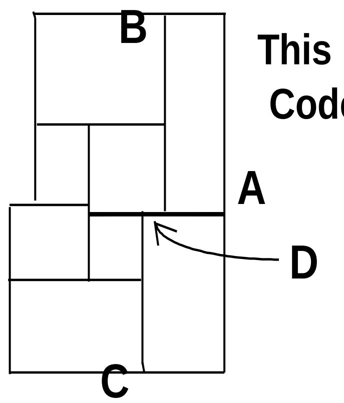
The Ratio Code for the Diad ending is A = B + C. The Code for Diad plus a Claw i.e. Pentad is A = 2B +C and the Code for Octad (i.e. Pentad plus a claw) is A = 3B + C. Yes there is a definite series found and the next (Undecad) has the Code A = 4B + C and if a Claw is put round this (Fourteen-add) A = 5B + C and so on . . . ! Knowing this removes much algebra.



<< A = 5B + C
<< A = 4B + C
<< A = 3B + C
<< A = 2B + C
<< A = B + C

DIAD PLUS CLAWS

The CLAWS are numbered 1 2 3 4 . . .



This has a Ratio Code of 3A = 5B +3C

Lets look at the Double pattern and add Claws to it - the Series runs 4A = 3B +4C then 4A= 7B + 4C then 4A = 11B + 4C and 4A = 15B + 4C and clearly the coefficient of B rises by four each time!

The Triple pattern has a series of 15A = 11B + 15C, 15A = 26B + 15C, 15A = 41B + 15C, 15A = 56B + 15C a rise of 15B each time.

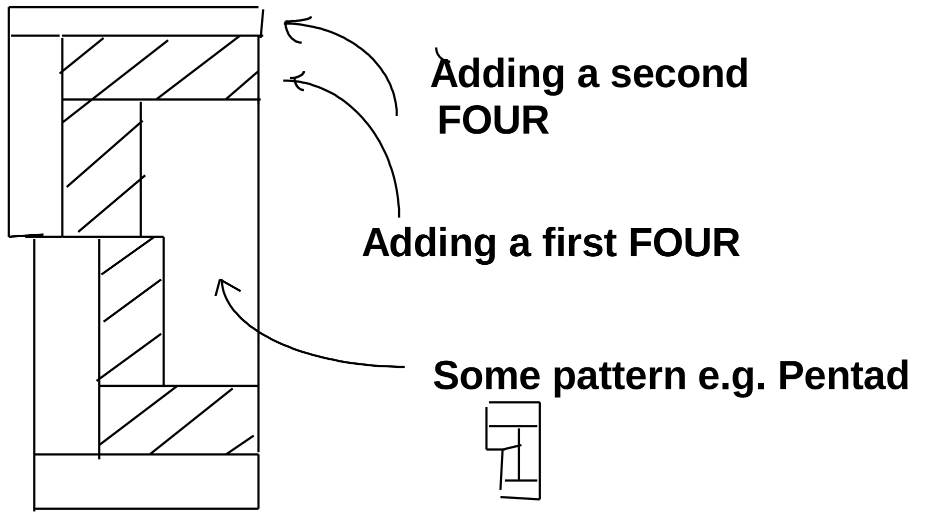
The Quadruple pattern starts with 56A = 41B + 56C with increments up of 56B each time.

Now look at Above 2 a more complicated design of 8 Elements. This pattern has a series of 3A = 5B + 3C, 3A = 8B + 3C (with one claw added), 3A = 11B + 3C (with 2 claws added), 3A = 14B + 3C etc. It is interesting to note that if the Element at A is omitted the same Codes apply. For instance adding Elements at B & C only give the formula 3A = 8B + 3C. If an Element D is added in-between the bold line the Ratio Code is unaltered at 3A = 5B + 3C. If a Claw is added or just Elements at B and C only, the Ratio Code is 3A = 8B + 3C the same as before. However it is important to realize that although the Codes are the same the individual Elements require total recalculation. Their coefficients in terms of A B & C will vary.

What if the pattern commences with an 'Added' Element or contains one somewhere in the pattern e.g. any pattern ending with a Triad? Well it was mentioned elsewhere that an Added Element does NOT change a symmetric pattern into an asymmetric one. It remains Symmetric. See C19.10.

C15.6. SYMMETRIC PATTERNS WITH FOURS ADDED

We have seen the effect on Ratio Codes when CLAWS are added.
What happens if FOURS are added? These are sets of Elements added - four at a time - as shown Below -



The Code for a Pentad is A = 2B + C.
When 1 Four is added the code becomes 3A = 5B + 3C. With 2 Fours the Code is 8A = 13B +8C

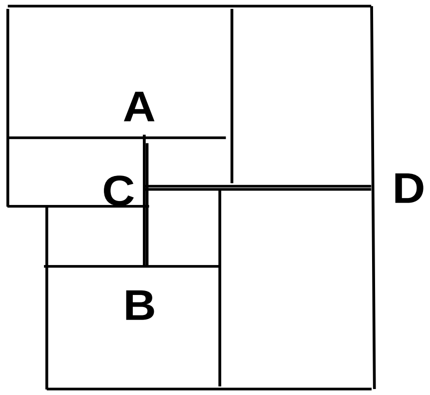
A simple relationship is soon found to work in all cases -
1. For the new coefficient of A add the present coefficients of A & B i.e. 1+2 = 3, 3 + 5 = 8 and so on.
2. Double the present coefficient of B and add the coefficient of A, i.e. 2 * 2 +1 = 5, 5 * 2 +3 = 13, and so on.
3. The coefficient of C is always that of A.

C15.7. SYMMETRIC PATTERNS WITH INITIAL PLUS ADDED

For a Diad the Code is A = B + C and a Triad A = 3B + C. In adding a PLUS the Codes change as follows:
1. Coefficients of both A and C stay the same.
2. The Coefficient of B increases by 2 times A. i.e. by 1 * 2. 1 + 2 = 3

C15.8. SYMMETRIC PATTERNS WHEN MIDDLE ELEMENTS ARE ADDED

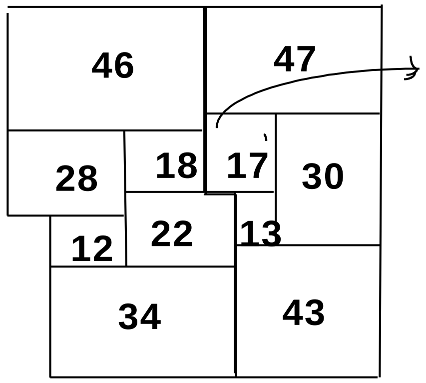
MIDDLE Elements are those which can be added one at a time without spoiling the symmetry, and cross the half way down position. Whereas some patterns do not have one, others have several. In Below 2 an Element can be inserted at vertical line AB. Or a Element can be inserted at the horizontal line CD. Or both AB and CD added together.
For all such combinations the Ratio Code simply stays the same! However it is again stressed that the individual Elements do change & require total recalculation.



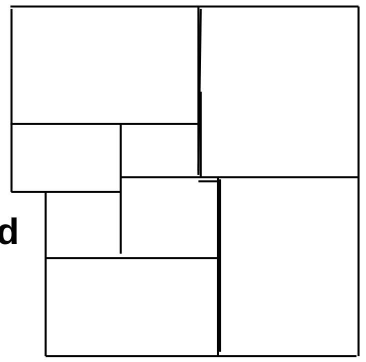
ADDING OF 'MIDDLE' ELEMENTS AT AB OR CD OR BOTH AB & CD DOES NOT ALTER THE RATIO CODE!

The last four sections mean that the finding of Ratio Codes is fairly easy. Also it is surprising how often different patterns have the same Code. Refer to Section E LINKS which uses this fact.

C15.09. FURTHER INTERESTING PATTERNS



The right hand side of this pattern is a PENTAD but the effect of adding an Octad or Triad instead is mentioned below.
Ratio Code 4A = 7B + 7C



Ratio Code
3A = 5B + 3C
Right hand side a DIAD

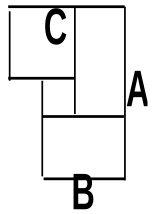
Bearing in mind the Codes for the series for Diads, Pentads, Octads etc. on the one hand and the Code for the Above pattern but with a Diad on the right, I was able by guesswork to obtain the Ratio Code of 4A = 7B + 4C! On inserting some dummy numbers (see Above) this formula agreed. Now what happens when Claws are added? As always only the coefficient of B changes. The series then runs 4A = 7B + 4C, 4A = 11B + 4C, 4A = 15B +4C ... by increments of 4B.

Compare this with the Diad ending which has the series 3A = 5B + 3C, 3A = 9B + 3C, 3A = 13B + 3C...
Subject to checking it appears that an Octad ending with the above pattern will give the Code 5A = 9B + 5C etc.
i.e. 3A = 5B + 3C for Pattern plus Diad, 4A = 7B + 4C for Pentad, 5A = 9B +5C for Octad, 6A = 11B + 6C for Undecad and so on - Now as Octads can be replaced by a Triad 5A = 9B +5C will also apply to this (however the Elements vary totally).

So these patterns are interchangeable - subject to various Up-rating and recalculation!

C15.10. RATIO CODES FOR ASYMMETRIC PATTERNS

The writer hoped that only the coefficient of B would alter if other patterns not of a symmetric nature have claws added to it. No such luck! Take the pattern below which has a Ratio Code of 4A = 3B + C.

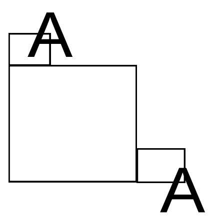


With one claw * added the Code is 4A = 8B + 3C, with two 41A = 130B +38C and three 149A = 633B + 146C. If there is a relationship it is hard to find even with the inconvenience of having to calculate four types. Also the coefficients are terribly large compared to a Symmetric design with similar numbers of Elements!
* i.e. Elements added at A B & C - again if B & C only are added and Claws added to this thereafter the same Codes apply but the individual Elements require total recalculation.

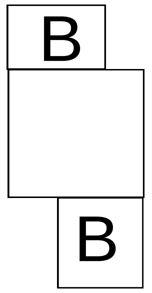
C16.1. IMPERFECT SOLUTIONS HAVING A SINGLE DUPLICATED ELEMENT

There are many examples of Imperfect Solutions where only one Element repeats, for example 9 in Above in [15] 107 x 79, nothing surprising about that. However, the Writer was surprised to find many Imperfect solutions (but not all) where the duplicated Elements happen to be either left and right of a vertical line, or top and bottom of a horizontal line. Often the line borders a more than average number of Elements - 17 14 9 14 & 9 44 in Above.

In Above, clearly 44 = 17 + 14 + 13 though of course 44 does not directly border 17 14 & 13.
But there are many exceptions to this rule - typical of this study! Some of these follow other patterns. Complete SR's will not shown for these but the reader is assured the theory is sound.



PATTERN 2



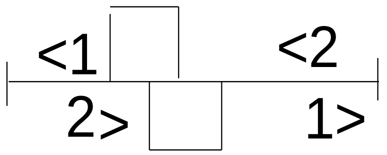
PATTERN 3

SOME SINGLE DUPLICATED ELEMENTS
OCCUR WITH ONE ELEMENT INBETWEEN

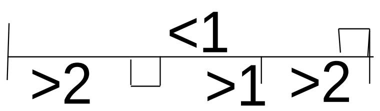
Note that these patterns may be rotated, and the relative sizes are irrelevant as they vary wildly in size & proportion. Thus the only duplicated Element may be situated as A & A, or as B & B (or similar).
At the present time the Writer has no explanation for these Patterns occurring.

C16.2. PATTERN 1

I have discovered that Pattern 1 occurs in some Imperfect Solutions where the Reduced Dimensions are both divisible by 4 e.g. 752 x 588. Where they do occur, it is found that the duplicated Elements occur every third Element in a group of 6 - as illustrated twice.



Three above & three below



Two above & four below

The line shown (which could also be vertical) has 6 Elements bordering it. If the Element shown as a box is Imperfect count to three in circular motion from it - doesn't matter whether left or right - and the 3rd Element will be the same size!

I have seen cases where the above is true with five Elements e.g. [16] 689 x 411, the reason for this is that neither 689 nor 411 are divisible by 4.

C17.1. FEATURES OF FORMULA LISTING

There are a number of rules which apply to Bouwkamp listing as modified by me, i.e. arranged in layers with + denoting corner Elements and - denoting Side Elements.

A. First Line. ++ +-+ +--+ +---+ +----+ etc.
First and last Element +_ with any in-between -_.
At least 2 Elements - no upper limit.
Largest Element is never last in the line.
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
IN LATER LINES EXCEPT THE LAST - At least one of the Elements is ._ Unless all Elements are ._ no line may contain more than one +_ or -_ .-. is possible where the - Element occurs on the bottom, line, usually 3rd or 4th Line from the end. But .+. is never possible.
xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx
B. Second Line.
First Element is ._
All Elements are ._ where First Line more than ++ *and* smallest Element -_.
Otherwise Last Element is +_ or -_ and all others ._
C. Line two from last.
Maximum three Elements
D. Line one from last . . - + -. +. .- .+

Maximum 2 Elements.
Is either 1 Element, +_ or -_,
or 2 Elements one of which is ._
<u>E. Last Line. - or +</u>
There is one Element only, sometimes +_ and sometimes -_, never ".-".

C18. VARIBLOCKS

Although this section involves Elements, it also involves Twin Rectangles and is therefore dealt with under Section H12 which see.

D. CALCULATING BY ALGEBRA

D1. xy SOLUTIONS

D1.1. CALCULATING USING ALGEBRA xy

(See also the counterpart procedure for calculating from Smith Diagrams).

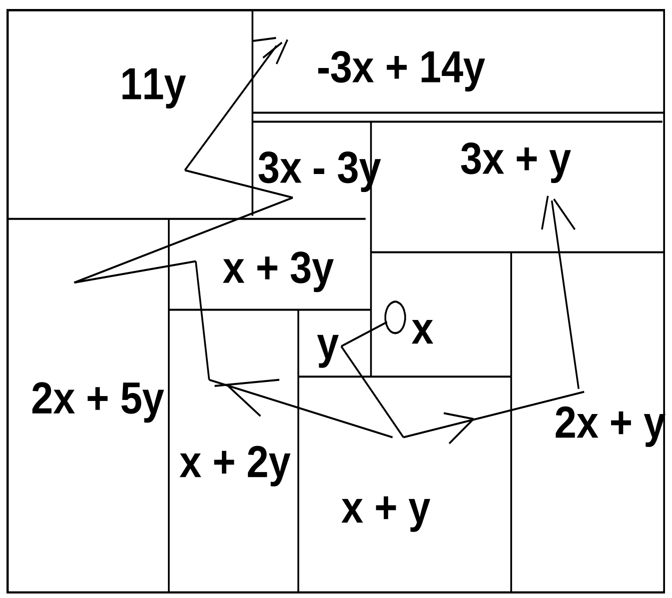
The purpose here is to show how the values are calculated for a squared- rectangle chosen at random. many rectangles will require a greater amount of Unknowns than two, in fact there is no greatest possible number.

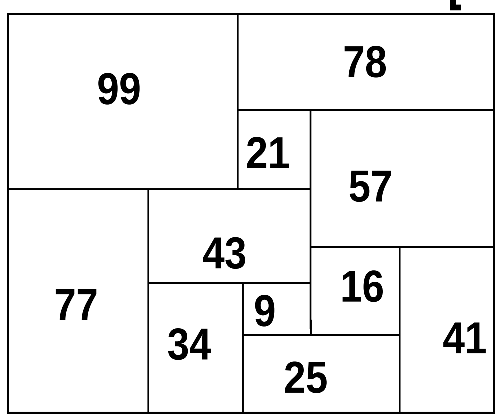
Below is calculated with x and y only. Though wordy to describe, it is not that difficult.

1. Draw a rough sketch of a rectangle divided into rectangles as though badly drawn squares.
2. Choose two inner adjacent Elements for x and y. Often a good choice is two of the smallest internal Elements, but a little practice and experimenting is needed for good choices, and no easy fixed rules can be given.

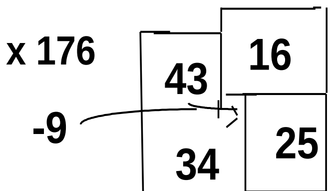
Always keep the Unknowns down to the minimum necessary.

3. Write down the values of adjoining Elements in terms of x and y until all Elements are valued (or as far as possible), remember the dimension of each Element works both horizontally and vertically.
4. There is a vertical or horizontal line to be found where an equation can be formed. Below in AB it shows -3x + 14y (top) must equal 3x - 3y plus 3x + y. -3x + 14y = 6x - 2y 16y = 9x which is satisfied by putting x = 16 and y = 9.
5. Using a similar Order to the algebra, the numbers are worked out one by one. 6. check the verticals and horizontals that they agree. In Below, 99 + 78 does equal 77 + 34 + 25 + 41.
7. Calculate the Order and dimensions, to conclude. Below is [10] 177 x 176.





[10] 177 x 176



NOTE THE TRAIL LINE SHOWING THE CALCULATION OF NUMBERS ROUTE

Sometimes a negative internal Element may occur in the calculation. See Above 3. In such cases a Simple adjustment of the pattern can be made to correct. It is not such a problem as it first appears. Compare Above 2.

D1.2. CALCULATING OF xy SOLUTIONS AND REDUCTION INDEX

In the typical equation of x and y we have something like 5x = 8y or -4x = 16y etc. where x is the coefficient of y and y the coefficient of x. That is x = 8; y = 5 and x = 16;y=-4 in the Above equations.

In the case of x = 16 and y = -4, y can be regarded as +4 with the pattern adjusted and so negatives can be disregarded. here both can be divided by 4 as x = 4 and y = 1 also satisfies the equations, but reduces all Elements to ¼.

The highest common factor of x and y, or the factor by which they can be cancelled down, if you prefer, is the Reduction Index. In Above 2 [10] 177 x 176 was 16 and y was 9. As these are prime to each other the Reduction Index is one.

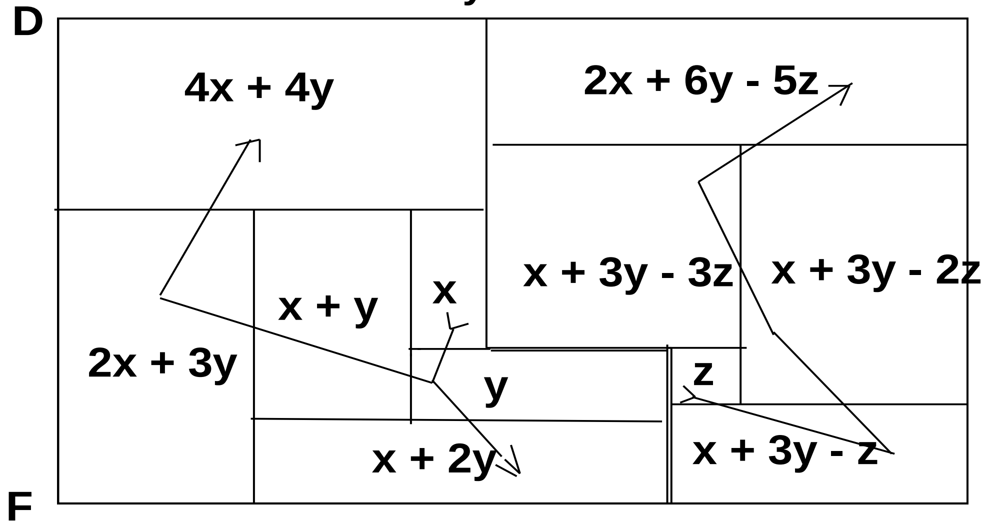
D2. xyz SOLUTIONS

D2.1. CALCULATING USING ALGEBRA xyz. FIRST METHOD

Calculating using 3 Unknowns is more complicated than xy only, and is to be avoided by any who hate simultaneous equations! Although x and y must be clearly adjacent to each other, z and further Unknowns required, may be detached, and often are.

In Below 1 x and y have been employed as previously, but the calculation is ‘stuck’ by a Gulf until a suitable choice for z is made. Then two equations have to be found in the form kx + ly = mz for the relative values of x, y and z to be calculated.

AN EXAMPLE OF xyz CALCULATION



NOTE: THE CHOICES FOR x, y & z ARE NOT LIMITED TO THIS EXACT PATTERN.
NOR ARE THE TRAIL PATTERNS UNIQUE.
IE. THIS SOLUTION CAN BE CALCULATEI IN A VARIETY OF WAYS.

In Above the Element at corner g + z must equal y + (x + 2y). it is therefore x + 3y - z.

All Elements are similarly calculated.

D2.2. FINDING EQUATIONS AND CALCULATING

Since DE = FG and DF = EG two sets of equations can be calculated, but there is no guarantee the sides of the equations will always be different. If not, other lines will need to be chosen.

Now DF = 6x + 10y -5z and EG = 4x + 8y - z so 6x + 10y - 5z = 4x + 8y - z which reduces to x + y = 2z---(1).

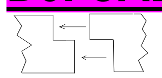
Likewise DE = 6x + 7y and FG = 4x + 12y - 8z which finally results in 2x + 8z = 5y----(2). from (1) and (2) may be calculated that x=2, y=12 and z=7. I have not shown the actual rectangle as this can be calculated readily.

D2.3. CALCULATING USING ALGEBRA xyz. SECOND METHOD

The previous method fails in that determination of the Reduction Index is difficult or impossible ***to be concluded ***.

D3. xy + xy SOLUTIONS

D3. CALCULATING ADDING TWO xy CHUNKS



An alternate way of calculating rectangles which have a single step is to create a formula from the equations Above. I have found this useful in some computer programs, but found that some of the resulting solutions could be calculated on x and y only, and the rest on x y and z, when it appeared that xyza solutions would arise!

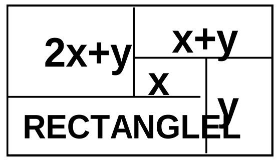
D4. COMPLEX SOLUTIONS

D4.1. CALCULATION OF COMPLEX COMPOUND SOLUTIONS

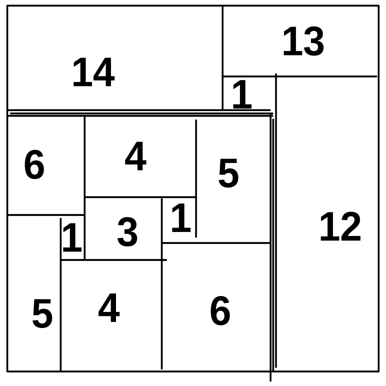
The smallest Order for Valid Complex solutions is 13. This is because the smallest ‘cover’ around the smaller rectangle is 4, and the smallest Order for Valid rectangles, 9.

Below shows the algebraic form from which it can be seen that the differences in dimensions in the old rectangle have to be 4x. as 33 - 32 in [9] 33 x 32 is not divisible by 4 all Elements have been multiplied up 4 times. Construction is easy.

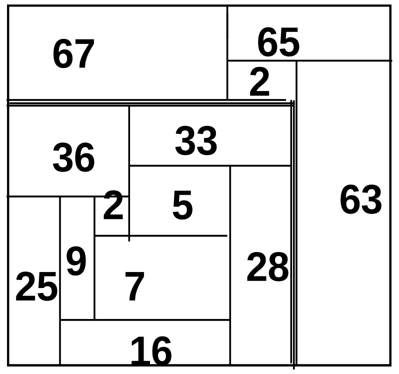
Note there are 10 solutions in all, as there are 4 for 132 x 128 , four for 261 x 259 and two for 27 x 25 according to the rotating of the original rectangle. It is obvious that twins will exist for all solutions, usually sets of 4 unless the original rectangle is symmetry 2. The full sizes for the Above are 540 x 500, 528 x 512 and 522 x 518 all having Semi-perimeters of 1040 being 8 x 130 the original Semi-perimeter.



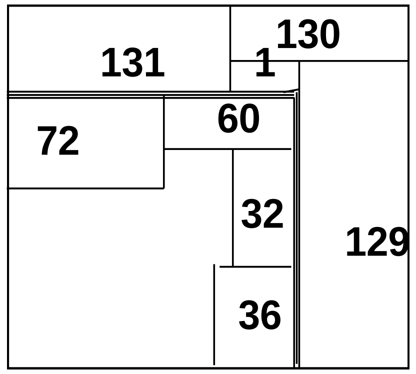
ALGEBRA OF 4 ELEMENT COVER.
NONSIMPLE COMPLEX SOLUTIONS.



[13] 27 x 25 (1) NS



[13] 132 x 128 (1) NS.



[13] 261 x 259 (1) NS

D5. UNKNOWNS THEORY

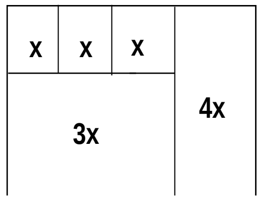
D5.1. AMOUNT OF UNKNOWNS NEEDED

The more complicated the Squared-Rectangle and the greater the amount of Elements, the more difficult the calculation becomes, and the likelihood that more Unknowns may have to be used. In practice a rectangle may be calculated using more Unknowns than the basic minimum, but as this always means much more calculation, it is sense to check and keep the Unknowns to the minimum.

Clearly the smallest amount of Unknowns for non-Dud Rectangles is two, x and y, and for these to be usable x and y must be adjacent, with a single line connecting both.

For solutions up to and including Order 13, many can be calculated with xy only, and the rest xyz.

On the whole Compound and Invalid solutions tend to attract more Unknowns compared to Imperfect and Perfect ones.



Dud Solutions may in theory often be calculated using unknown x only. However as x is always 1, this is just mentioned in passing.

D5.2. UNKNOWNS THEORY (xyz ETC.)

The greater the Order used, the more complicated the calculation of the rectangle tends to become, and also the likelihood of more Unknowns needed. However xy solutions do exist for all Orders even to Order infinity!

In practice it is always possible to use more Unknowns than the minimum actually needed for calculating.

Up to Order 13 many solutions can be calculated using xy, and all others using xyz. xyza is not necessary until higher Orders are used. There is a tendency for Compound and Invalid rectangles to need more Unknowns than Perfect and Imperfect ones.

Even though x y and z may be necessary to calculate a particular solution …

It is always possible to define every Elements in terms of x and y only in any Squared-Rectangle.

For a value of z in terms of x and y can always be found, and z can be removed wherever it occurs and replaced in terms of x and y only. This was touched on previously). The statement above is still true however many Unknowns are necessary for calculation.

D5.3. THE MAXIMUM UNKNOWNS NEEDED IN EACH GIVEN ORDER

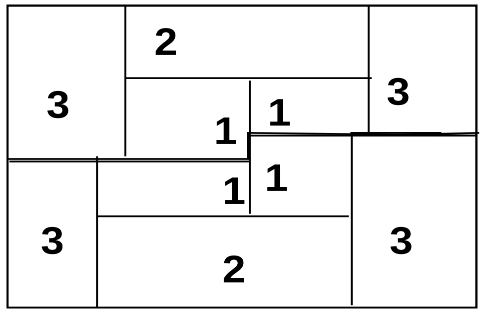
Which solutions require the most Unknowns in any given order? Referring to the solutions catalogue, the smallest solution needing three Unknowns xyz is [10] 8 x 6 non-zero. See Below.

In this solution the line AB acts as a barrier, and Elements on the other side of the barrier cannot be calculated without introducing a new Unknown z.

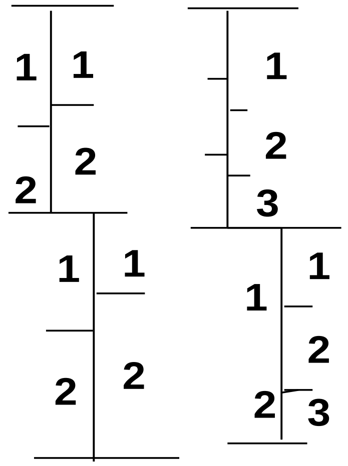
Such a line is termed a Gulf and examples are shown in Below 2.

There must be at least two sets of two Elements on both sides for that line to be a True Gulf.

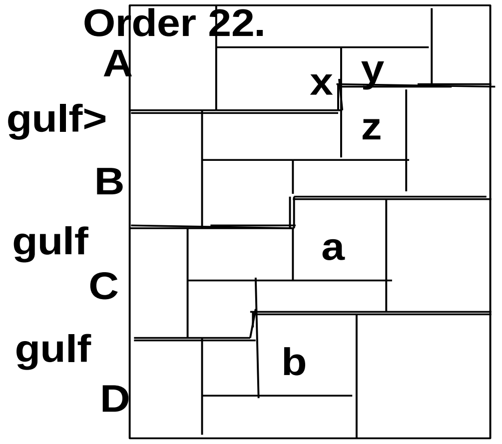
Now by introducing repeated Gulfs in a diagram as Below 3 it is possible to create rectangles requiring exceptional amount of Unknowns. Starting with sector a x and y are clearly required for the whole Pentad to be calculated. the next 6 Elements In sector b are calculated using z. ditto next 6 using a and the final Pentad using b.



[10] 8x6 the smallest rect. requiring xyz unknowns



EXAMPLES OF TRUE GULFS



This Requires FIVE UNKNOWNS xyza & b.

The Order 22 solution is actually non-zero with 1's 2's and 3's only. It happens to be the one and only Order 22 solution requiring five Unknowns, in the same way Above 1 is the only Order 10 solution requiring three Unknowns.

A table can now be shown-

Orders 7 to 9 xy maximum Orders 10 to 15 xyz maximum

Orders 16 to 21 xyza maximum Orders 22 to 27 xyzab maximum… and so on.

The Order range increases by 6 each time. Note that the lowest Order in each group contains only one Invalid solution.

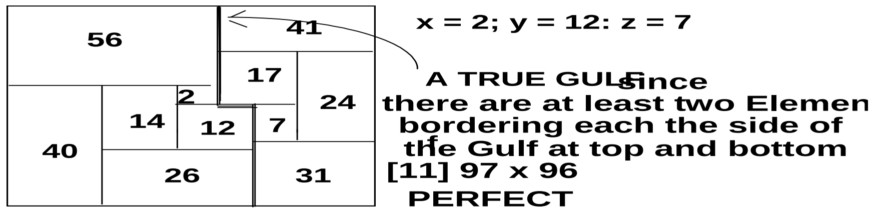
D5.4. TABLE OF ORDERS AND UNKNOWNS

Order 7	Order 8	Order 9	Order 10	Order 11	Order 12	Order 13
xy 1	2	1	4	5	9	24
xyz 0	0	0	1	6	9	27
Valid	shown	Below			Invalid	Above
rectangles	xy	3	6	21	62	167
	xyz	0	0	1	14	80

Observations:-

1. xyz Valid Solutions hardly start until Order 12 there being just one for Order 11: See [11] 97 x 96 Below.
Note the Pentad end which acts as a Gulf.
2. Note the startling higher proportion of xyz Invalid solutions compared with Valid.
3. Note the acceleration of xyz Valid solutions.

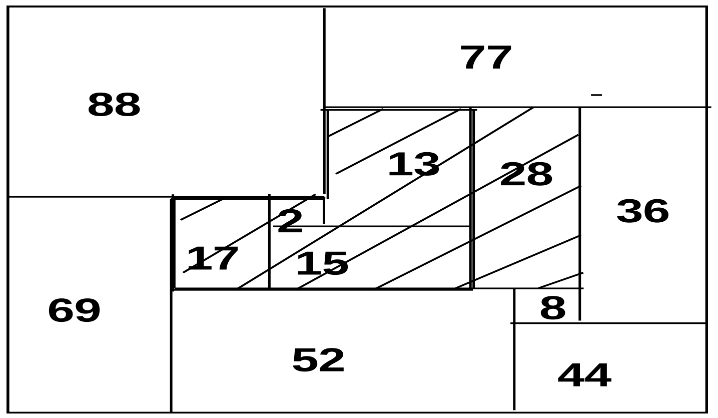
THE ONLY VALID ORDER 11 xyz RECTANGLE



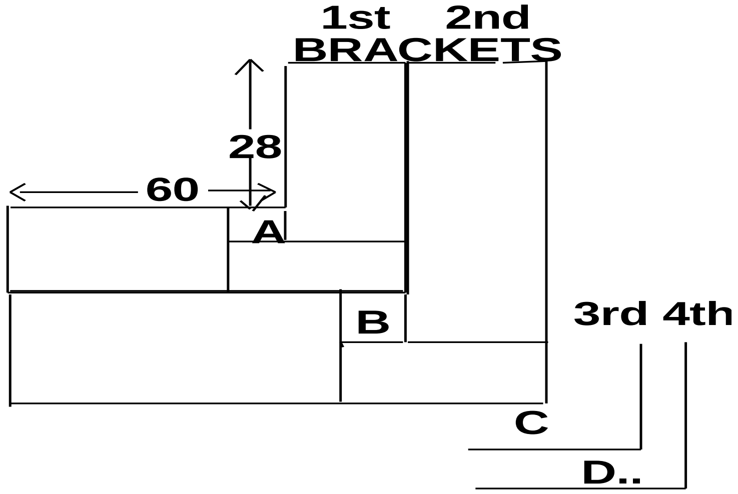
[11] 97 x 96 PERFECT

D5.5. BRACKETS AND THEIR EFFECT ON NUMBER OF UNKNOWNS

See Section F18 for more on this.



[12] 165 x 157



Look at Above 1 which contains a "Bracket". The Bracket is the L shaped part of four Elements at bottom left (8, 36, 44 & 52). Above 2 shows two Brackets. There can be many repeated Brackets within a Rectangle.

Had the Element numbers been unknown and Algebra used to calculate the Rectangle, the shaded area would only need unknowns x and y but the rest would *ordinarily* need the third unknown z. But, we can dispense with the need for z..

The Element A with a bit of inspection is always found to be a quarter of the positive difference between the horizontal & vertical lines. So in this instance, (60 + 28) ÷ by 4 = 8.

If a second Bracket is added as in Above 2 then B will be a quarter of 96 (52 + 44) less 80 (36 + 44) that is 4.

If there was a Third Bracket then C would be found to be 2, and the fourth D would be 1.

Although A, B, C, D seem rather like the Independent Elements mentioned in Section C the set up is somewhat different. However D C B and A have the ratios of 1 : 2 : 4 : and 8 which is interesting.

Where the whole rectangle can be calculated using x and y only, irritating fractional coefficients of both x and y will occur. However these can be eliminated by calling the initial Elements x and y, as 2x and 2y (or 4x & 4y, 8x & 8y... as appropriate)

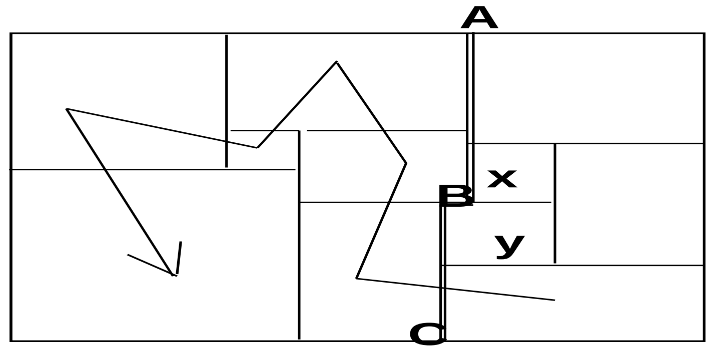
(NOTE : As the values in the shaded area do NOT stay the same when further Brackets are added the actual values of Elements calculated here do NOT fit into proper Rectangles)

D6. MORE ON UNKNOWNS THEORY

D6.1. PENTAD SOLUTIONS AND UNKNOWNS

Does the Pentad part of Pentad solutions always form a Gulf, and are all Pentad solutions xyz or greater?

No - look at Below and choose x and y as the two left corner squares - the Pentad acts as a barrier, and z appears necessary.



< THIS DIAGRAM SHOWS THAT THE EXISTENCE OF A PENTAD DOES NOT ALWAYS MEAN 3 UNKNOWNS ARE NECESSARY

But with x and y chosen <u>inside</u> the Pentad the whole rectangle can be calculated following the trail line!

This can happen due to a single element bordering left of BC , or left of AB or both.

This means some Pentad solutions may be calculated with xy only. In Above 2 the Pentad does not form a Gulf since the barrier is in one direction, not both. But most Pentad solutions require at least three unknowns for full calculation.

D6.2. ARE xyz SOLUTIONS MOSTLY SMALLER ONES?

It is interesting to observe that in Order 12 where 94 Valid + Invalid solutions exist, 47 of which are Reduction 1 that there are just three xyz Reduction 1 solutions, 297 x 296, 313 x 280 and 353 x 280 (all with Pentads).

There are 11 others with Reductions 2+.

But in Order 13 almost half (38 of 80) xyz solutions are reduction 1 even including the Invalid. Although xyz are not mostly smaller ones, there is a tendency for highly reduced Perfect rectangles to be xyz rather than xy.

The reason for this is made clear in Section E.

D6.3. PROBLEMS CHOOSING BEST SITES FOR x AND y

1. Where any Rectangle pattern is drawn at random choosing the best or most suitable sites for Unknowns x and y is often a problem and a careful inspection is necessary.

Many solutions which apparently require three unknowns xyz can be calculated with xy on closer inspection when better choices of x and y are seen.

As all xy solutions can also be calculated using three or even more Unknowns, it should be made clear that xy strictly refers to two being the <u>minimum</u> amount of Unknowns.

Xyz, xyza also refer to the <u>minimum</u> amount of unknowns, when in practice more could be used to calculate the solution.

2. A further complication is that the choice of x and y in any given solution is not fixed.

Whereas in some solutions suitable sites for x and y are very restricted, others will have a large number of acceptable sites. Unlike z, a, b, c.. x and y obviously have to be chosen adjacent to each other.

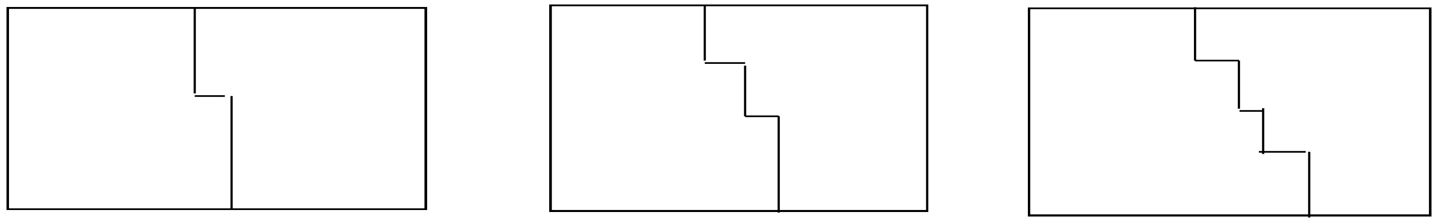
z and further unknowns can be sited next to x and y in many solutions, but frequently are separated from x y and one another.

D6.4. STEPS AND GULFS AND THEIR EFFECT ON xyz

Shown Below

D6.5. ONE STEP, TWO OR THREE.. . .

DIVIDERS EXAMPLES OF STEPS.



NOTE- A GIVEN RECTANGLE MAY HAVE SEVERAL EXAMPLES OF THESE IN VARIOUS PLACES. THE RELEVANT FEATURE IS THE MINIMUM NUMBER OF STEPS IN A RECTANGLE. STEPS MAY ALSO BE LEFT OR RIGHT.

In many solutions the solution can be divided by a single step, as Above 1. Often this occurs many times in a solution.

In some solutions these steps can also act as a Gulf. Where the three lines making up the division are not bordered anywhere with just one Element - and that means on both sides of the division, not just one - i.e. in 6 places, then the division is a Gulf.

In such cases x and y can only be chosen on one side (or other) of the Gulf, and it is found impossible to calculate Elements on the other side of the Gulf since this forms a barrier. However choosing z as an Element on the other side of the Gulf sometimes is sufficient to calculate the whole rectangle.

It is important to observe in xyz solutions that x and y can be chosen one side of the Gulf and z the other, (i.e. x and y chosen from left and z from right side or x and y from right and z from left side).

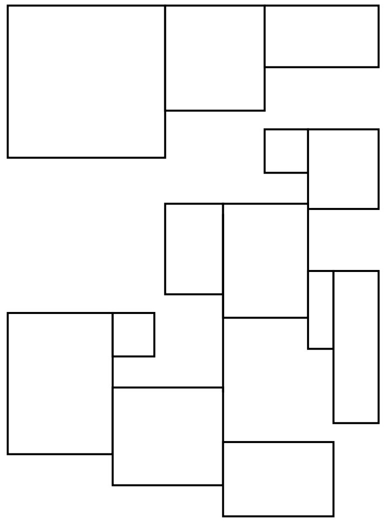
Some single step divisions form a barrier from one side but not the other!
These are One-way steps. if x and y are chosen on the correct side of the step any other Unknowns found necessary will not due to this type of Gulf. Steps divide into three types

1. Solution [16] 503 x 403 this can be calculated thus - 1 34 35 36 37 4 41 45 86 69 54 77 140 163 240 263 with 1 and 34 being x and y respectively. But 77 and 86 chosen as x and y connect with 163 and 240 only, whereas 54 and 77 connect with nothing. In this Rectangle any adjacent pair from 1 34 35 36 37 4 41 45 86 69 can be chosen as x and y whilst adjacent pairs selected from 54 77 140 163 240 263 would require a third unknown to calculate the whole solution.

Valid pairs for xy are 1 34,1 35,1 36,1 37,4 37,4 41,4 45,34 35,35 36,36 37,37 41,41 86,41 45 and 45 86- 14 in all.

In the pattern of x y x + y 2x + y x + 2y and 3x + y this pattern can proceed with further Elements none of which can be used effectively as x and y. 9 pairs on 6 Elements are possible, which would seem to be the least possible.
On the whole internal Elements will more often work as x and y; two side Elements are less likely.

The Elements 1 34 35 36 37 4 41 45 86 69 in the Above solution is termed the <u>xy-Block.</u> Another example is Below.



a

Above [13] 638 x 465 is shown. The solution can be calculated using x & y only as long as the choice of x and y fall within the 5 Elements in the SE Corner.

D7. ENZ SOLUTIONS

D7.1. ENZ SOLUTIONS

In ENZ solutions the xy-Block includes the entire rectangle.

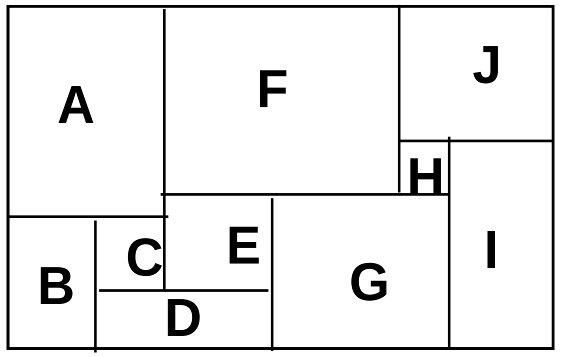
Any xy solution can be calculated using two appropriately selected inner Elements for x and y, but it is possible in a relatively few xy solutions for x and y to be the two end Elements. These are called ENZ solutions.

ENZ solutions can be Invalid or Valid, symmetric or otherwise, Perfect or Imperfect. Symmetric solutions are frequently found.

The Sides Index is always of the form S2#2# and both ends of ENZ solutions are Triads.

I devised a computer program to calculate all possible solutions to a given Order, and uses all three types of Add-ons later shown. Its only drawback is that same solutions can be produced up to 4 times.

D7.2. EXAMPLE OF ENZ SOLUTION



IF A & B ARE CHOSEN FOR x & y, THE WHOLE SOLUTION CAN BE CALCULATED IN TURN FROM A TO J

[10] 130 x 79 AN ENZ RECTANGLE

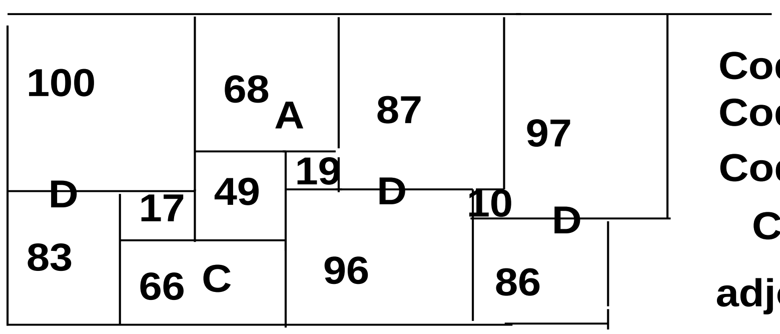
If A and B above are chosen for x and y, the whole rectangle can be calculated in terms of x and y only. Likewise I and J are suitable sites for x and y, but not all adjacent pairs can be sites for x and y without the need for z, such as E and F.

D7.3. OTHER ENDING TYPES SIMILAR TO ENZ SOLUTIONS

I adapted the computer program by replacing the Triad end with a Pentad, Septad, Octad … In the Octad set an interesting feature emerged - an Octad solution can always be converted into a Triad solution!

See E6.1. for full details.

D7.4. CLASSIFYING ENZ PATTERNS



Code A for Elements at top
Code C for Elements at bottom
Code D for Element at top &
CODE DCADD..
adjoining Element at bottom (ie A + C

Regard the pattern above as any ENZ pattern randomly drawn, the top left Element being larger than the bottom left Element. Ignoring the Internal Elements and looking at the combinations of external Elements between them, we have 3 possibilities:

1. At Top only which will be called A.
2. At bottom only which will be called C. and
3. Adjoining Elements at Top and Bottom (e.g. 87 and 96, or 97 and 86) which we will call D (rather than A & C).

We now consider the relative sizes of 100 and 83 at the left. 100/83 gives 0.83 exactly. Different ratios give, not surprisingly, different codes.

DCCC 62% 63% 64% 65% DCCD 66% 67% DCCA 68% 69% 70% 71% DCDC 72% 73% 74% DCDD 75% 76% DCDA 77% DCAC 78% 79% 80% 81% DCAD 82% 83% 84% DCAA 85% 86% 87% 88% DDCC 89% 90% 91% DDCD 92% 93% 94% DDCA 95% 96% 97% DDDC 98% 99% DDDD 99.5%

D8. CALCULATING OUT ALL SOLUTIONS

D8.1. FINDING ALL EXISTING SOLUTIONS FOR A GIVEN ORDER

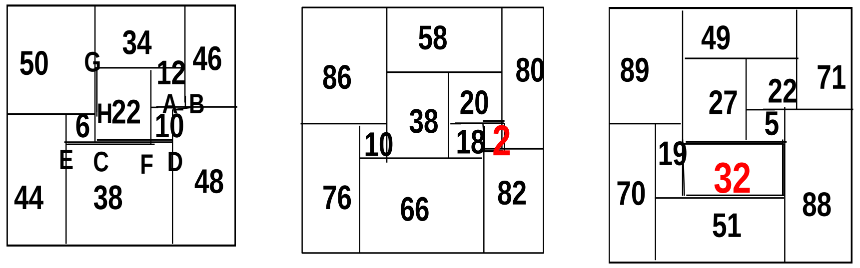
This Section discusses the calculating all possible Squared-Rectangles for a given Order, and the ways it might be done.
Is there a way of being sure that all possible Rectangles have been found?
Are there any shortcuts in doing this? Well, yes but is not easy!

There seems no way of producing all Perfect and Imperfect solutions without Invalid types such as Zero and Non-zero arising. It is logical to start with the lowest Orders first and work upwards.

There are so few Order 9 and 10 solutions that these are easily found.
It seems sensible to amend existing known solutions by adding a single Element in various ways and recalculate the rectangles formed, and adopting this idea I found all Order 11, 12 and 13 solutions that are Perfect, Imperfect Non-zero or Zero by the Add and Deduct rule.

D8.2. ADDING AND DEDUCTING

ADDING INTERNAL ELEMENTS & RECALCULATING



[10] 130 x 94 (65 x 47) [11] 224 x 162 (112 x 81) [11] 209 x 159

NOTE THAT ELEMENTS CAN BE ADDED AT AB & CD
ALSO AT EF & GH (NOT SHOWN)

Look at above 1 where the lines ABC DEF and GH indicate positions where a single Element can be added to create a new rectangle up an Order. Two of four possible have been shown above 2 and 3, having been previously calculated.

Note that the lines have been carefully selected so they are not adjacent with any single Element, e.g. Line GD will cause a Compound solution.

Although each resultant rectangle has to be laboriously calculated, the advantage is that the results are mostly different every time a different Order 10 solution is used, but duplications do arise. Many but not all Order 11 solutions are produced this way. The remaining solutions can be obtained by using non-adjacent lines which touch the sides of the rectangle.

Suitable examples of these are ST, UV and WX in above.

D8.3. THE ADD AND DEDUCT RULE

The Lines referred to above will be referred to as “Add-Lines”

When an Element is added in this way the value of the Element can easily be found if full dimensions are considered.

The length of the non adjacent Add-Line equals the size of the added Element.

See below where Elements added in various places take on the value of the line.

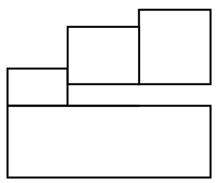
When reduced rectangles are used in this way, remember to multiply the line by the Reduction Index.

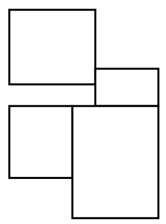
Also remember that the resultant rectangle may reduce as well.

Apart from these slight complications, the rule is simple and most important.

Where only three Elements border a line it is not possible to Add an Element from it.

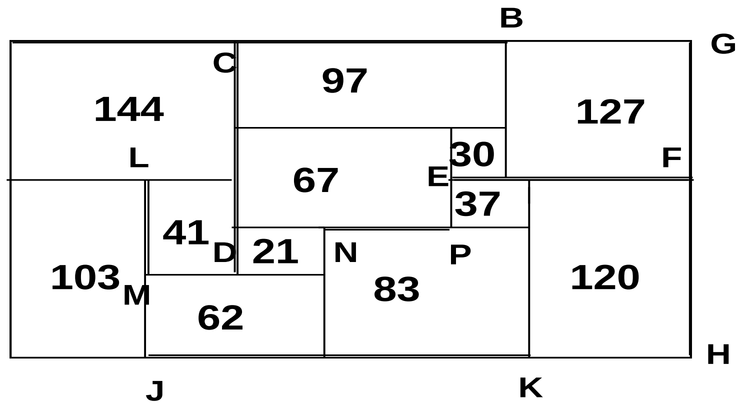
Where four or more Elements border a line then it is an Add-Line and it is found that TWO possible Elements can be added from it as indicated –





Note that there are two possibilities regardless of the composition of the Elements e.g. two each side of line or one against three.
Where 5 Elements border a Add-Line there are always FIVE possible positions where Elements can be Added – despite various compositions of Elements.
Where 6 Elements border a Add-Line there are always NINE possible positions.
Briefly the formula is as follows THREE None

FOUR	2	= 2
FIVE	2 + 3	= 5
SIX	2 + 3 + 4	= 9
SEVEN	2 + 3 + 4 +5	= 14 and so on … i.e. 20, 27, 35 etc



No matter where any line is drawn in a rectangle the ADD & DEDUCT RULE means that an extra element drawn at that line will have the same value! So an element drawn at AB has a value of 241, CD of 185, EF of 157, GH of 247, JK of 145 LM of 41, NP of 46.

[12] 368x247 Reduction 1. THE REDUCTION IN BOTH THE ORIGINAL & RESULTING RECTANGLES MUST BE CONSIDERED. They may well differ.
The DEDUCT RULE is similar:- if 41 is removed line LM is stil l 41 in length.
If 37 is removed, line EP rema ins at 37- as 37 is prime the resulting rectangle will be full size or Reduction 1.

D8.4. STATUS OF ADDED SOLUTIONS CAN VARY

Adding a square to a Perfect solution may result in a Perfect, Imperfect, Non-zero or Zero solution, and with others any of the four classes may result. New solutions should therefore be created using inferior types also.

Also the symmetry can change, and/or the Reduction Index. If the non-adjacent line from a full dimensioned solution is prime (1,3,5,7,11…) the resulting solution will always be full size with Reduction index of 1.

It will therefore always be Perfect. If a double prime the Reduction Index is either 1 or 2, and so on.

Below shows how the solutions are built up. Full dimensions are shown throughout.

Order 7	Order 8	Order 9	Order 10	

24 x 21	40 x 35	69 x 61	110 x 99	209
		“	111 x 98	209
		“	120 x 104	224
	45 x 30	75 x 55	130 x 94	224
		66 x 64	114 x 110	224
chart		“	105 x 104	209
showing		66 x 55	114 x 95	209
build-up of		“	115 x x94	209
solutions			+ 3 more	

D8.5. DEDUCTIBLE SOLUTIONS

Solutions are termed Deductible whenever at least one internal Element can be removed without the resulting rectangle becoming Compound, that is, with two adjacent Elements occurring. Using all Order 10 rectangles and adding various Elements at various internal points the following Order 11 rectangles emerged - 7 x 7, 8 x 6(1), 8 x 6(2), 10 x 9 , 11 x 8, 14 x 9, 14 x 10, 30 x 26, all Invalid and 97 x 96, 98 x 96, 98 x95, 112 x 81, 185 x 151, 185 x 183, 187 x 166, 191 x 162, 194 x 183, 195 x 191, 199 x 169, 199 x 178, 205 x 181, 209 x 127, 209 x 144, 209 x 159 ,209 x 168, and 209 x 177 (8 Invalid and 19 valid).

Obviously there is no change in the sides index whether internal Elements are added or deducted. This means that in attempting to discover all rectangles for a given Order, looking at each sides index in turn is very helpful.

D8.6. NON-DEDUCTIBLE SOLUTIONS

This leaves the following solutions to be discovered by some other means -
Order 11, 4 x 2 22 x 18 and 24 x 22 all Invalid and 177 x 176, 185 x 168 and 191 x 177 Valid. Six altogether.

There is a tendency for Non-deductible solutions to have comparatively few inner Elements and therefore many outer Elements. There seems to be a disproportionate amount of Invalid solutions also.

D8.7. SIDE INDEXES AND NON-DEDUCTIBLE SOLUTIONS

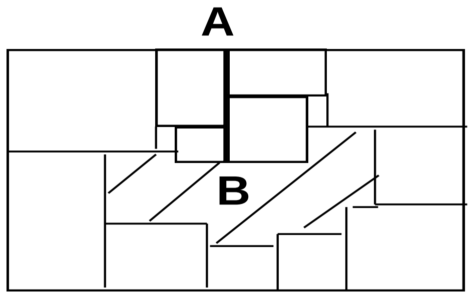
It is possible to give a table where all Rectangles for the sides and Orders shown is Non-deductible.
Non-deductible Rectangles to Order 13 are -
Order 7, 1 Order 8, 1 Order 9, 2 Order 10, 4 Order 11, 6 Order 12, 9 and
Order 13,15. The amount is roughly the total of the two Orders Below.
By Order 13 the proportion of total solutions has already dropped from 100% to 5% , and it is clear that most Rectangles are Deductible.

D8.8. FINDING THE NON-DEDUCTIBLE SOLUTIONS FOR AN ORDER

Suppose all Order 13 rectangles are known and the Deductible solutions for Order 14 found.

From the Above table, any solutions found with sides S2326 S2335 S2344 S2353 S2525 and S3334 is Non-deductible.

Unfortunately some will have sides listed against Order 13 too. The trick is to look at Order 13 rectangles where an Element can be inserted along the side to produce a 2326 solution, and so on..



INSERT AN ELEMENT AT AB TO CREATE A SOLUTION OF A WANTED SIDES INDEX
eg. HAVE S2435 WANT SIDES S2535

D8.9. REMOVING FROM OR ADDING ELEMENTS ONE AT A TIME TO SIDES OR CORNERS OF SOLUTIONS

So far only *INTERNAL ELEMENTS* have been considered. But it is often possible to remove a Side or even Corner Element and the result to be a valid - by valid we mean 1. Not Single Ended, and 2. Simple (no smaller rectangles within) and 3 with Sides S2223 or greater. There are therefore some Solutions which are Completely Non-Deductible - that is ones where not one Element anywhere (Internal or Side or Corner) can be removed without causing a useless Solution.

Ignoring [5] 2 x 2 for which no Elements can be "Added" to it, the smallest such Solution is [7] 8 x 7 = [7] 24 x 21 full size.

I have only found the following "CND" Solutions to Order 11:

[7] 8 x 7 (24 x 21) [9] 6 x 5 (66 x 55) [11] 14 x 9 (224 x 144) [11] 22 x 18 (176 x 144) all Invalid in practice. This implies that all Solutions to and including Order 11 - 53 of them - can be drawn from adding appropriate Elements ONE AT A TIME to
[7] 24 x 21 or [9] 66 x 55!

On testing all Order 12 Solutions not one CND Solution was found by me.

It is possible that all Order 12 and Order 13 Solutions may be drawn from one of the above four CND Solutions!

D9. REPEATERS

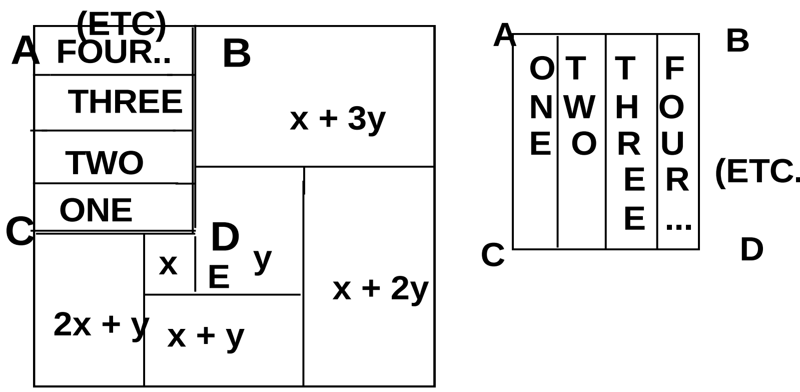
D9.1. CALCULATION OF REPEATERS

If a corner Element in a given rectangle is replaced by 2,3,4,5… repeated corner Elements as below, what effect does it have on the Orders, Sides and Semi-perimeters?

The series following shows a relationship, and other solutions are found to have similar progression series:-

REPEATER CORNER SOLUTIONS

THE REPEAT MAY BE ADDED HORIZ OR VERTICALLY AS SHOWN WITHIN THE SELECTED CORNER.



VERTICAL REPEATS

AS BD = 4y 3x + y (AT CD) HAS TO EQUATE MULTIPLES OF 4y (AT CD)

[7] 40x35 WITH 1 ELEMENT AT ABCD

HORIZONTAL REPEATS

x+4y AT BE HAVE TO EQUATE WITH MULTIPLES OF 3x+y PLUS x AT BE

The Above when calculated gives the following results-

1H at corner 7 Elements 24 x 21 SP 45 1V at corner 7 Elements 24 x 21 SP 45
2H at corner 8 Elements 40 x 29 SP 69 2V at corner 8 Elements 32 x 34 SP 66
3H at corner 9 Elements 56 x 37 SP 93 3V at corner 9 Elements 40 x 47 SP 87
4H at corner 10 Elements 72 x 45 SP 117 4V at corner 10 Elements 48 x 60 SP 108.
Increments of 1 16's 8's 24's 1 8's 13's 21's

Note increments of 16 8 24 and 8 13 21. Where 16 + 8 = 24 and 8 + 13 = 21 and 24 and 21 are significant.

All the repeated corners have the value of 12 as full dimension.

By the Add and Deduct rule all Repeaters must have the same value.

D9.2. RELEVANCE OF REPEATER SOLUTIONS

As a group, Repeaters are not acceptable solutions in this Book. But their theory is useful and relevant. In particular bits of them can be used effectively with additional elements or adjustments to obtain further solutions which includes Squared-Squares.

Where two different double cornered solutions can be found, they can sometimes be placed together and the four squares made into one Element to produce highly reduced rectangles. This subject is continued in Section G3.

D9.3. ANALYSING DATA IN REPEATER SOLUTIONS

I took the solution [9] 69 x 61 and calculated the eight Repeater Solutions caused by doubling horizontally and vertical each corner element in turn. Some resulted in a rectangle where the height is greater than the width as indicated in the values below. Each 'solution' is of course Order 10 in effect. The set of solutions are with double corners are -

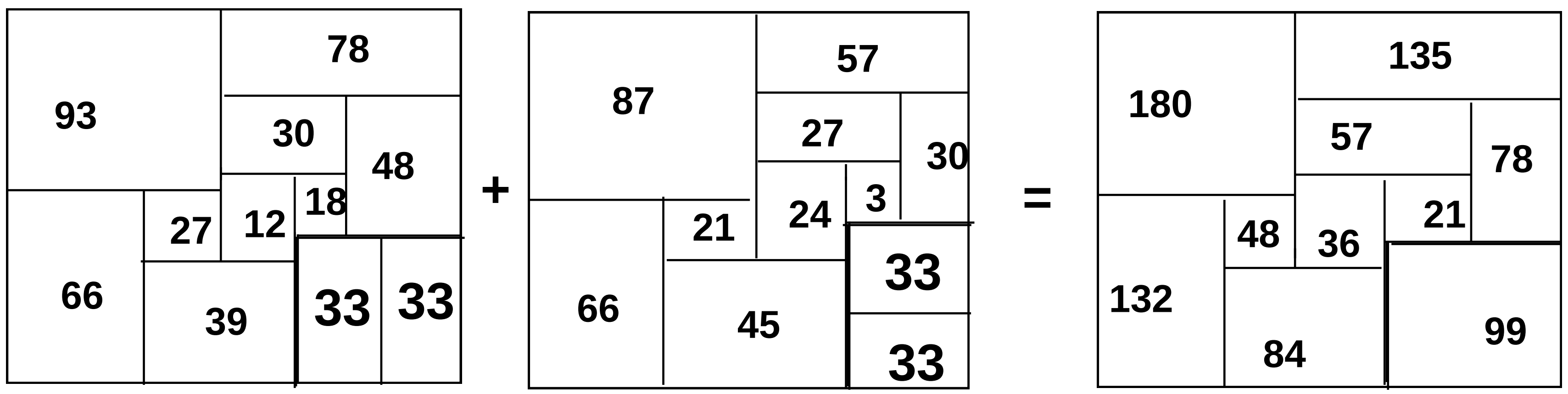
114 x 85 114 x 82 103 x 82 109 x 85

93 x 98 93 x 101 104 x 101 98 x 98 from which may be observed by adding each pair - 207 183 207 183 207 183 207 183 1. Four sets of 207 (x) 183. This is 69 x 61 multiplied by 3!

This feature is always found true for any given solution i.e. the vertical and horizontal Repeater dimensions when added together give amounts three times the dimensions of the original.
2. The second dimensions come in pairs i.e. two of 82 and 85 and two of 98 and 101.
The difference between pairs is found to be constant, 3 in this example but the difference varies solution to solution.
It is a pity there appears no way of calculating the dimensions of individual Repeaters by simply knowing the original is
[9] 69 x 61 with its corners 33 36 25 and 28. So, the value of these discoveries may be minimal.
However if dimensions only are sought, upon calculating one Repeater its counterpart can be easily found without algebra.

D9.4. ADDING HORIZONTAL & VERTICAL REPEATERS PAIRS

Below is a pair of Repeaters for a solution selected at random. The patterns are identical but for the vertical and horizontal Repeaters. If the corresponding elements are added in turn the result is below 3. The result is the Source Rectangle! However apart from any possible reduction the element values are Three times the normal full Dimensions.
Note that the corner element replacing the Repeaters is three times 33. For clarity, only full dimension have been shown in the Order 11 Repeaters.
The property is true whatever example is taken providing of course that the same corner is selected for the Repeaters.
This means that having calculated one Repeater only, and knowing the original Rectangle, the elements of the second Repeater are easily found without more algebra.

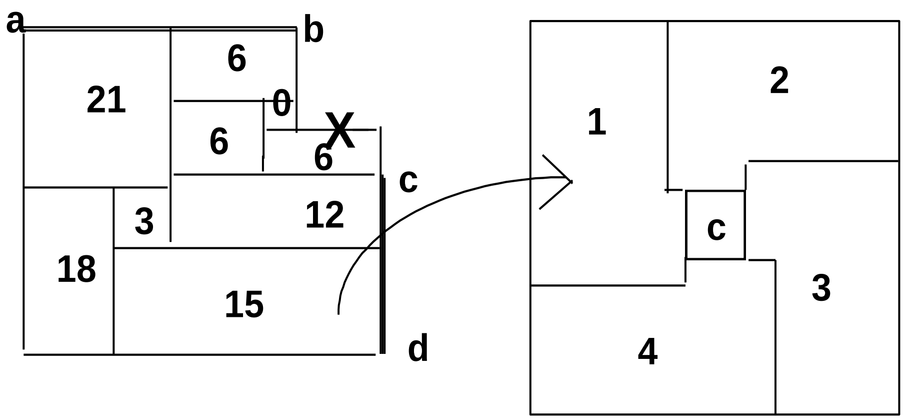


1st Pattern, 2nd Pattern, 3rd Pattern,
Is [10] 105 x 102 + Horizontal Repeater, [10] 105 x 102 + Vertical Repeater. Is [10] 105 x 102 up-rated 3 times,

D9.5. DEDUCTING PAIRS OF REPEATERS

The result in deducting instead of adding the Elements was totally unexpected!
Not surprisingly a mixture of negative as well as positive Elements arises, and although the construction appears really chaotic, adjustments made to the shape always suitable correct it. An L shaped construction arises and as the deduction of the repeater part gives zero the Order of this 'L' shape has dropped two. In below deducting the elements in the above Repeaters has been shown after adjusting the negatives. Zero elements do not always arise as here.

Owing to the negative adjustment, the ‘L’ pattern may appear somewhat changed from the original - no problem in this.



FOUR-FOLD SQUARED-SQUARE WITH CENTRE ELEMENT c

If the ‘last’ element - in this case 6 - is axed ab is always found to equal cd and this allows calculation of a fourfold Squared-square of the format above! See Section L regarding these Squares.

D9.6. COINCIDENTAL CONSTRUCTIONS

I stumbled across the following constructions which are coincidental and do not apply to most Repeater pairs.

1. Cases where a pair of Repeater solutions have a common side, the pair of solutions can be placed side by side with the four Repeater squares made a single element, 30. See Below 1.

2. Construction into a larger solution by putting two repeaters together then adding two side and two top elements. Four adjacent Elements of 9 make one of 18. See Below 2. NB. Further solutions are also possible (a) by adding a Pentad at AB and removing 34 and 37. (b) Or by adding an element at CD and increasing the outer elements. © By adding 3 at E and repeating construction up-side-down a Symmetric Solution arises. See Section E where all these Add-ons are fully described.

D10 to D21 EFFECTS OF ADDING OF ELEMENTS IN VARIOUS WAYS TO CONSTRUCT NEW SOLUTIONS

The Reason for Above 2 giving the Rectangle is by observing 37 x 29 and 35 x 34 the semi-perimeters of which are 66 and 69 a difference of -3 together with AC - BD being the same, namely -3. These SP’s (69 and 66) are always the values at the top even if the construction fails to be a Rectangle.

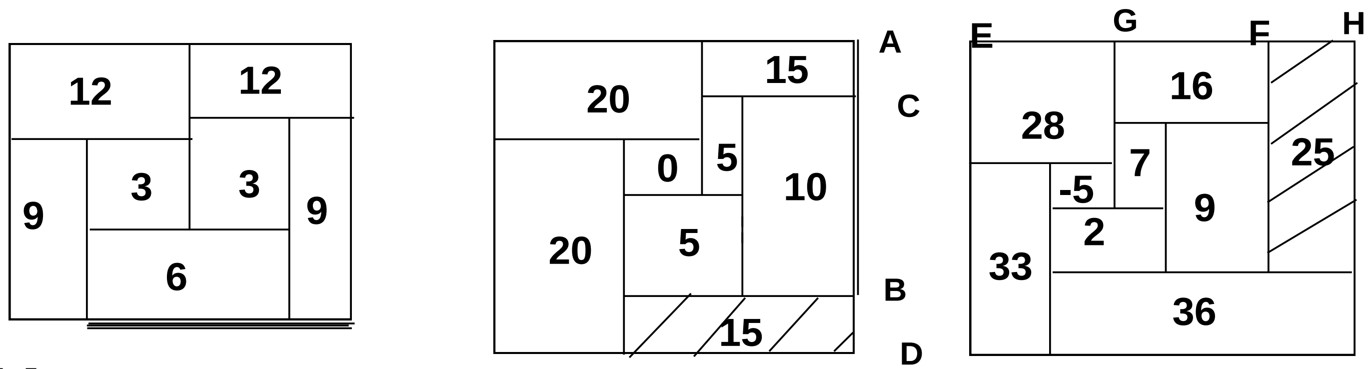
These differ from the add-ons in Section E and F. The Add-ons are not as convenient. In Sections E & F some or most of the Elements do not need to change, but in D10 to D21 the Solutions require total recalculation by Algebra or a set of known Solutions readily available. Put differently the Solutions contain Elements 100% different from each other.

Most of the following in D10 to D16 require either TWO or THREE known (or calculated) Solutions to start with, from which connections can be found enabling the dimensions of Solutions in ascending Orders to be calculated.

Such series in theory continue forever.

D10. ADDING SINGLE SIDE ELEMENTS IN S2223 SOLUTIONS

D10.1. EFFECT OF SUCCESSIVELY ADDING AN ELEMENT TO S2223 SOLUTIONS



[7] 24 x 21 (8 x 7)   [8] 40 x 35 (8 x 7)   [9] 69 x 61

Above a single Element as shown by the bars has been added. Note that adding at CD will also produce [9] 69 x 61 but normally there are two choices such as EF and GH. All rectangles produced are S2223.

They always contain distortions and negative values if the original pattern is strictly maintained. [10] 110 x 99 (10 x 9 zero) and [10] 111 x 98 come from [9] 69 x 61. But S2223 solution [10] 30 x 26 does not.

The full dimensions of larger rectangles can be determined from these figures. The differences in the dimensions is 3 5 and 8 all Fibonacci numbers and the differences continue 13 21 34 55 89... each being the sum of the previous two.

24 40 69 111 187 297 496 784 <-- larger dimensions

diffs 16 29 42 76 110 199 288

(16+29-3) (29+42+5) (42+76-8) (76+110+13) (110+199-21)

21 35 61 98 166 263 441 695 <--smaller dimensions

diffs 14 26 37 68 97 178 284

(14+26-3) (26+37+5) (37+68-8) (68+97+13) (97+178-21)

3 5 8 13 21 34 55 89 << differences in dimensions

The Above shows that the series continues [10] 111 x 98 [11] 187 x 166
[12] 297 x 263 [13] 496 x 441 [14] 784 x 695 [15] 1314 x 1170 [16] 2043 x 1810
[17] 3446 x 3069 [18] 5345 x 4735 [19] 9024 x 8037 [20] 13992 x 12395
[21] 23626 x 21042 [22] 36631 x 32450 [23] 61854 x 55089 [24] 95901 x 84955
[25] 161936 x 144225 [26] 251072 x 222415 and so on. Unexpectedly, note that the Fibonacci numbers are deducted and added alternately - Why is this!

<u>The Semi-Perimeters range 45 (Order 7) 75 130 209 353 560 937 1479 2484 3853 6515 10080 17061 26387 44668 69071 116943 180856 306161 473487 (Order 26) . . . which may prove to be a useful representative set of values.</u>

Note that such a series cannot be found unless three solutions are known initially.
It is also possible to calculate the corner Elements which arise from the series:-
Order 7 - (5 + 4 = 9) Order 8 - 9 + 6 = 15 Order 9 - 15 + 10 = 25 Order 10 - 25 + 16 = 41
Order 11 - 41 + 25 = 67 Order 12 - 67 + 42 = 109 Order 13 - 109 + 68 = 177 Order 14 - 177 + 110.

This may be a bit clearer by explaining
(1) That each corner number is used in the following Order (e.g. 5+4=9 9+6=15) and
(2) Numbers following "+" form a growth series 6 10 16 25 42 each the sum of the previous two. And,
(3) The table is gradually built up forwards but also backwards in the case of Order 7.
Logically the next group of similar type is [10] 120 x 104 (30 x 26 invalid)
[11] 199 x 178 and [12] 315 x 278 where the dimensions differ by 16 21 37 respectively. Although not Fibonacci numbers, the series is formed by the addition of the previous two, as before, namely 16 + 21 = 37 21 + 37 = 58 and so on. So far I have been unable to find the relationship for the series.

Another series found by me links up as follows:-
[11] 190 x 171 difference 19. [12] 324 x 284 difference 40 [13] 522 x 463 difference 59 [14] 873 x 774 difference 99
[15] 1382 x 1224 difference 158.

Now the differences between the first numbers (i.e. 190 324…) are 134 198 351 509 919 which are respectively for the later values 134 + 198 + 19 = 351 198 + 351 - 40 = 509 351 + 509 + 59 = 919.

Now the differences between the second numbers (i.e. 171 284…) are 113 179 311 450 820 which are respectively for the later values 113 + 179 + 19 = 311 179 + 311 - 40 = 450 311 + 450 + 59 = 820.

When these numbers are added to the earlier dimensions [13] 522 x 463 [14] 873 x 774 and [15] 1382 x 1224 are the result.

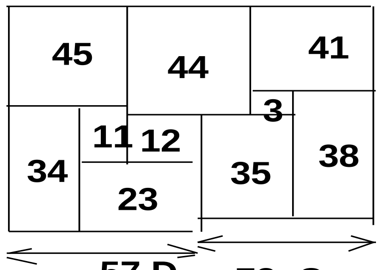
Above shows the basic procedure is probably easier to follow.
Again, the corner Elements can be determined bit by bit as follows Order 11 (34 + 52 = 86) Order 12 86 + 28 = 114 Order 13 114 + 80 = 194 Order 14 194 + 108 = 302 Order 15 302 + 188 = 490.

D11 ADDING DIADS TO 2224+ SOLUTIONS

D11.1. EFFECT OF ADDING TWO END ELEMENTS

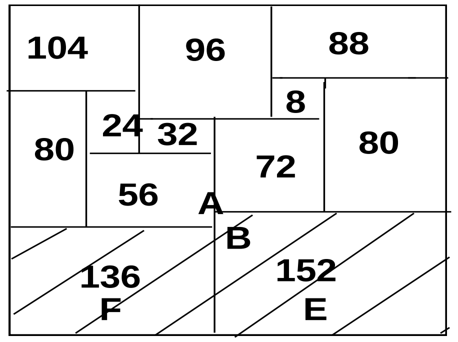
Whenever a SR has a side of 4 Elements or more another SR exists with two end Elements added -

[10] 130 x 79 SP209

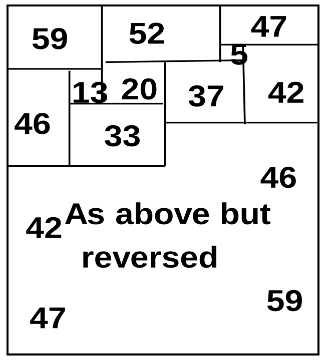


57 D 73 C
DIFFERENCE 16

[12] 320 x 288 (288 x 320) SP608



DIFFERENCE 16 ..=AB



[18] 194 x 158 ¬
Symmetric is obtained by taking the difference in the Elements from the two patterns to the left.

If similar comparisons are made it is always found that AB = C - D or E - F = C - D i.e. the difference between the ends and the two Elements are the same in the above case, namely 16.

D11.2. A CURIOUS RELATIONSHIP

In the Above example if the 57 and 73 are reversed and each deducted from 209 (the Semi-perimeter) 136 and 152 are obtained! Does this property hold good whenever two end Elements are added? [11] 191x177 is shown below.

The ends are 123 and 68 and 368-123 and 368-68 are 245 and 300 respectively. 245 + 300 = 545 and there is an Order 13 solution 120 x 109 (600 x 545) having ends of 245 and 300. See below 3 … but now for the surprise!

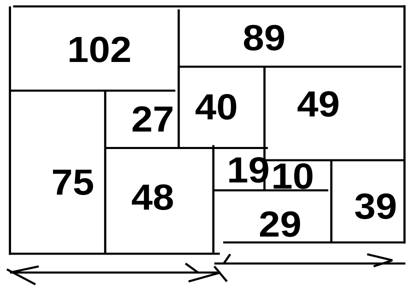
This is not below 1 with 2 added Elements, but below 2 is!

Note however that the full upper dimensions is the same in below 2 and below 3.

When the shaded Elements are removed a Compound solution is produced. Note that the side 191 is repeated.

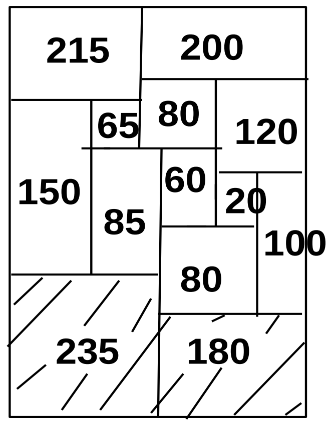
Another property also applies when considering the four solutions. [11] 185 x 151 to [13] 615 x 487 and [11] 191 x 112 to [13] 600 x 415 - 177 - 112 = 65 which times 2 is 545 - 415 = 130. Similar solutions prove the point.

[11] 191X177 SP368

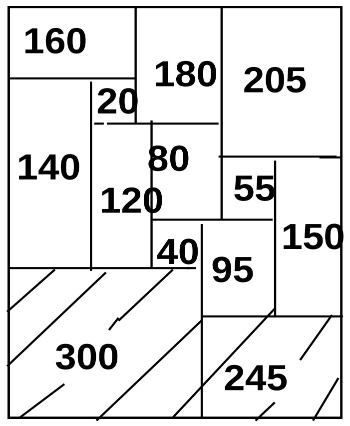


123
68
368 - 68 = 300
368 - 123 = 245
5 4 5

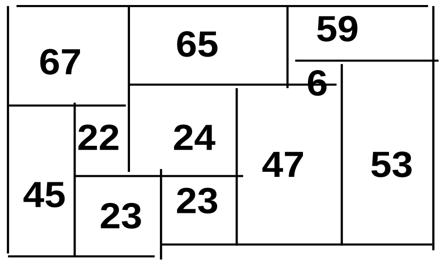
[13] 600X415 SP1015¬



[13] 600X545 ¬



[11] 191X112 NONSIMPLE

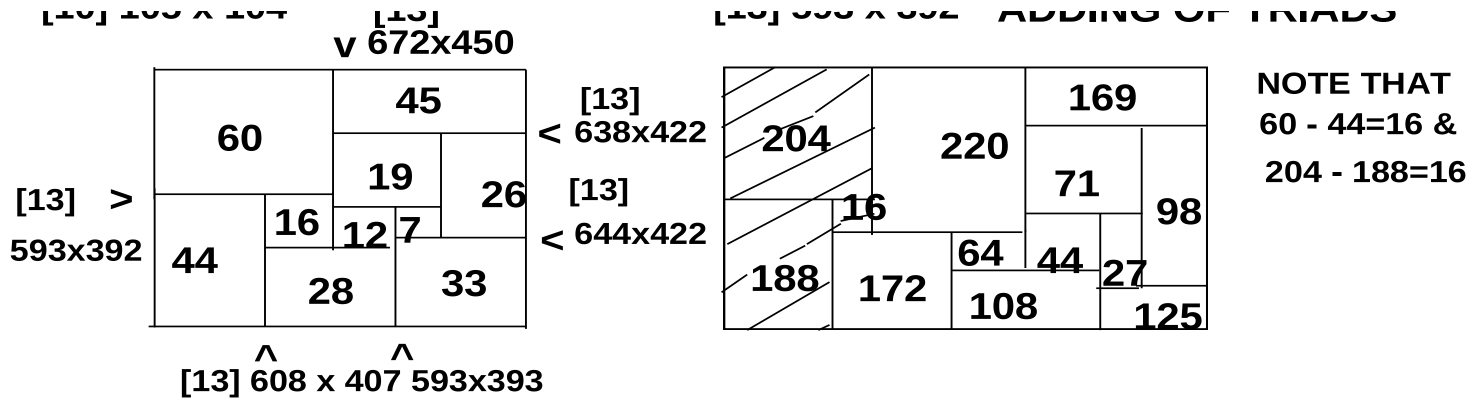


3303-68=235
303-123=180

D12. ADDING SERIES OF TRIADS

D12.1. EFFECT OF ADDING TRIADS (THREE END ELEMENTS)

Below shows the effect of adding an end Triad of 3 Elements. Note that the connecting Element in Below 2 is 16 which is both 60 - 44 and 204 - 188. This feature is always true when triads are added in this way.

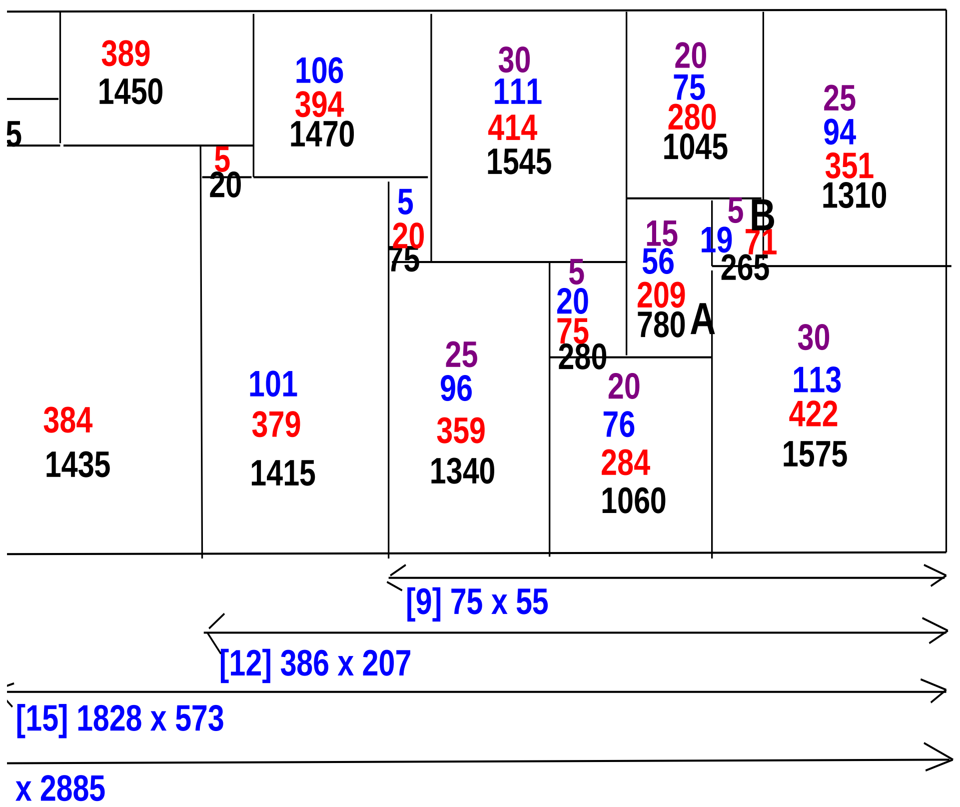


SOLUTIONS OBTAINED BY ADDING OF TRIADS AT VARIOUS POINTS.

D12.2. EFFECT OF ADDING REPEATED TRIADS

The following shows 4 solutions in one with three successive Triads added on the left to the original Solution of [9] 75 x 55 (shown in purple). What follows is a useful find.

For each Triad added a total recalculation is needed, and at first sight there seems no relationship from one solution to the next. However, notice the left hand linking Element of the Triad is the same each time (in this case 5). Notice also the repeats of 20 and 75.



But the numbers 30 113 422 1575 . . . in right hand bottom corner are related. For 113 x 4 -30 = 422 and 422 x 4 - 113 = 1575. The next value is therefore 1575 x 4 - 422 = 5878. This relationship holds true throughout e.g. 379 x 4 - 101 =1415, 56 x 4 - 15 = 209 and so on.

The only ‘snag’ in applying this is that we need to know (or calculate) a solution and a similar one with a Triad added, but once 2 solutions are known then an everlasting series can be found using only arithmetic.
If E1 and E2 are the known full Elements in a given position in the Rectangle then E3 = E2 x 4 - E1 and E4 = E3 x 4 - E2 and E5 = E4 x 4 - E3 and so on. It is possible to express all such formulas in terms of E3 E2 and E1 alone if required.
E(n2) = 4 x E(n1) - E(n) is the general case. In the above we only need to know the new values of A and B from this formula to complete the whole next Rectangle by arithmetic.

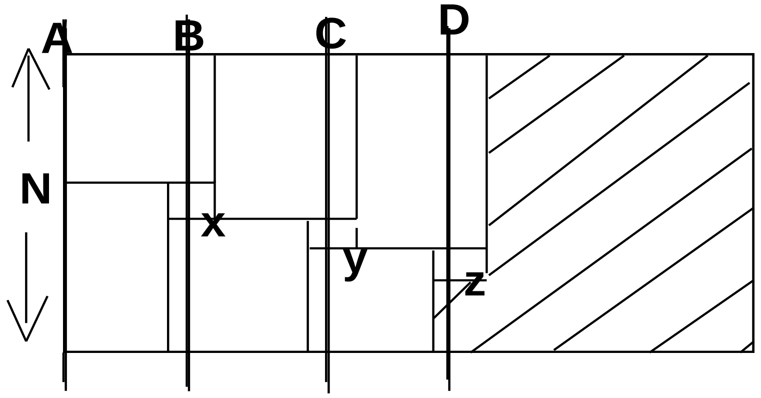
Note that the series concerns Full Dimensions and the sizes of individual Elements keeps increasing. It should be born in mind that occasional reductions are possible. e.g. [9] 75 x 55 reduces to [9] 15 x 11.

D12.3. CONNECTING DIMENSIONS FORMULA FOR REPEATED TRIADS

There is a relationship between the numbers 75 386 1828 8265 … and 55 207 573 2885.

As said before, two Solutions need to be known initially of the format [O] M x N and [O+3] M1 x N1 and we need to find the Full Dimensions for [O+6] M2 x N2. After some searching the Writer found N2 = N1 x 4 - N.

M2 is harder to find and is M2 = M1 x 4 + N1 x 2 - M - N: this can be rewritten using N2 but this form is more convenient. These two formulae can be repeated ad lib. Without explaining why, the second formula was found after realizing that



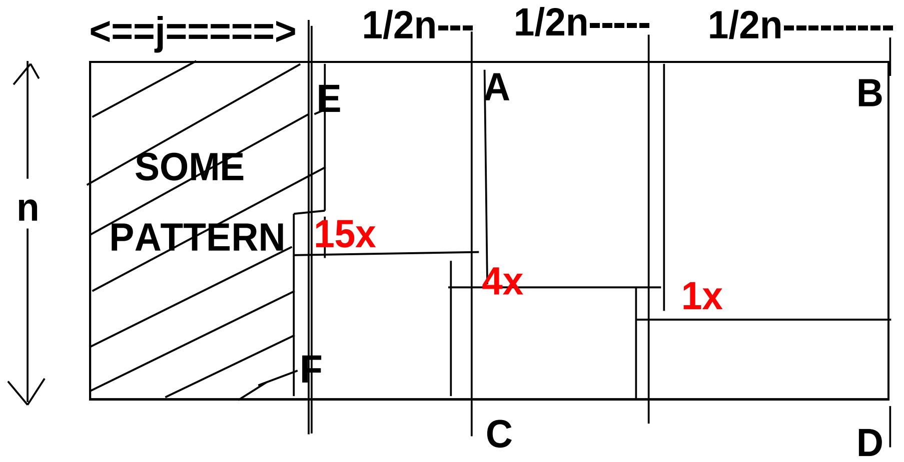
Lines B, C, D have been purposely drawn through the middle of Elements x y z.
Distances AB AC AD . . are then found to be half N, N, one & a half N . .
Also AB = BC = CD . .

The vertical centre point of each Triad is effectively one-half the value of n (the smaller Dimension). See Above.

These formulae are important and mean that sizes of Rectangles can be found without calculating any individual Elements.

D12.4. ADDING REPEATED TRIADS

So if we consider rectangles with at least one Triad at one end we can divide such a Rectangle into 1 or more sectors each of which is ½n by n in size (or area) providing we divide the linking Elements horizontally down the middle. Here 3



ABCD forms a Square of n x n

BDEF forms a Rectangle 1.5n x n

Triads are shown and if the value of n is known, so is the distance EAB as it is 1½n. If the remainder of the distance across is j, the whole distance
m = 1½n + j i.e. the dimensions are (1½ n + j) x n and the Elongation is n / (1½ n + j).

If we add a further Triad will the <u>Elongation</u> be n / (2n + j)? This would be true if j remained the same proportion of n, but no, it does not! In fact the proportion j / n increases each time a Triad is added. Having said this there will be an optimum size for j / m.

We saw earlier that if the extreme right linking Element is 1 then the linking Elements to the left will be 4, 15, 56, 209 etc. (if greater than 1 then 1, 4, 15, 56, 209 etc. have to be scaled up accordingly). This explains why the proportion of j / n increases as the Order increases three at a time.

D13

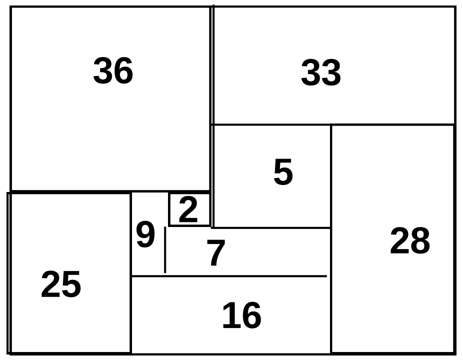
D13.1. A FEATURE OF S2223 SOLUTIONS

<u>WHERE ARE ILLUSTRATIONS FOR THESE???</u>

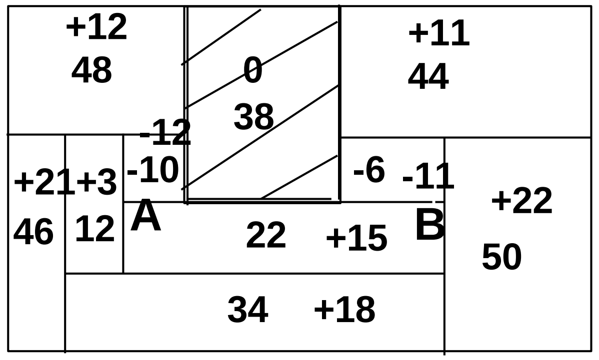
In Above 1, 377 - 368 = 9 = 1½ x 6 231 - 225 = 6 33 - 28 = 5 and 36 - 25 = 11 difference = 6.
In Above 2, 1103 - 1088 = 15 = 1½ x 10 675 - 665 = 10 100 - 81 = 19 105 - 76 = 29 difference = 10.

D14 ADDING A MIDDLE ELEMENT

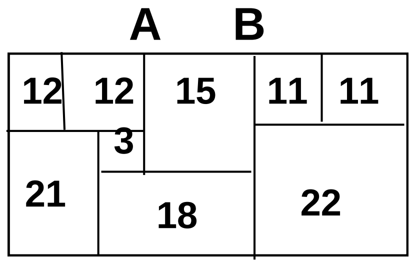
D14.1 EFFECT OF ADDING AN ELEMENT TOUCHING THE EDGE



[9] 69 x 61



[10] 130 x 94 (65 x 47 R2)



The Differences form a Rectangle!

A curious feature arises when an Element is added bordering the edge of any given SR. Take any SR say [9] 69 x 61 and add an Element at a line touching the border as above.

[10] 130 x 94 is the result and the new Element 38 (= 36 + 2 or 33 + 5).

As usual every Element changes value, but by how much? Element 36 becomes 48 an increase of +12, 5 becomes -6 down 11, and so on. In Above 2 the same format has been deliberately shown causing some negative Elements to appear.

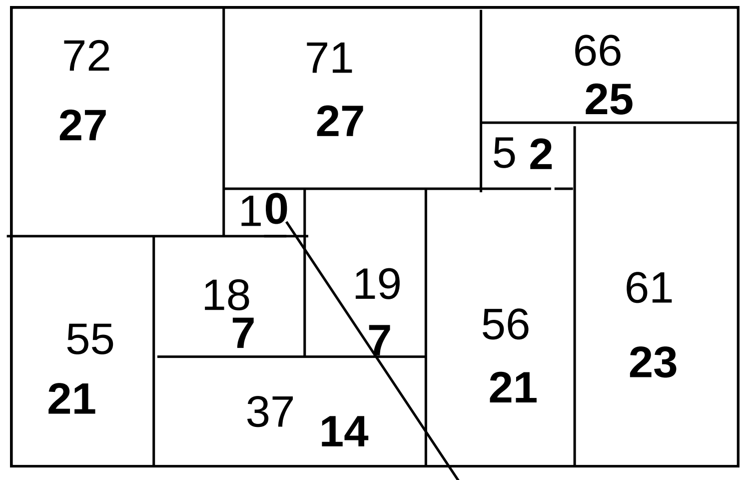
Looking at the black numbers in above 2 , we see values which fit as a Squared-Rectangle!

At first sight it looks wrong as at AB -12 and -11 does not total +15! But on drawing a rectangle as above 3 AB is in fact becomes external and the rectangle is true! Also to fit, some 'sliding' is necessary.

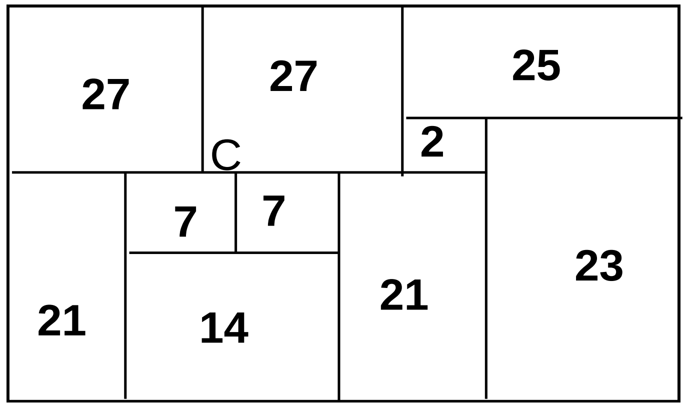
Without further examples being shown, this feature is always found to be true when adding a border Element to any rectangle. It is also true when an internal Element is added as seen below.

It is true to say that the rectangles above 1 and 3 when added give above 2 (apart from the added Element 38).

Rectangles of type above 3 will be termed <u>hidden rectangles</u> and are always Compound to a lesser or greater degree. Although interesting the Above feature does not offer much practical help in constructing new rectangles from old and there is no quick way of calculating new rectangles.



[11] 209 x 127 (FROM [10] 130x79 WITH
ELEMENT ADDED HERE

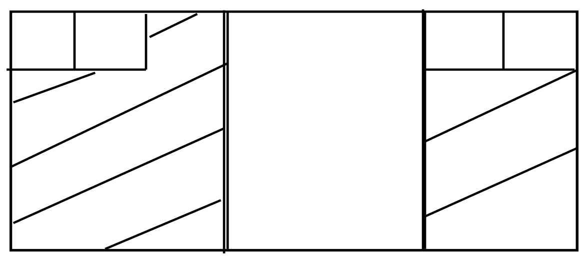


HIDDEN RECTANGLE
[10] 79 x 48 NONSIMPLE

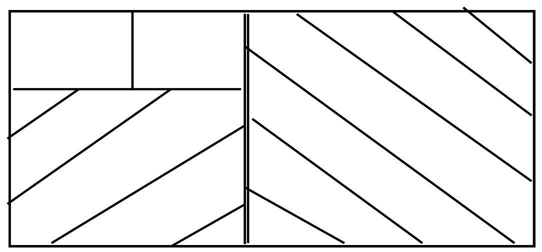
In above 1, the bold numbers are the increase from the respective Elements of [10] 130 x 79 and these are repeated in above 2. Again, a Compound rectangle arises. (Note that 0 cannot be inserted at C making [11] 79 x 48)

D14.2. RECTANGLE + RECTANGLE = ANOTHER RECTANGLE

In testing a number of cases, I found the Hidden-rectangle had Repeater Elements and is always Compound and Invalid. Also the rectangles were two rectangles side by side, some with a single rectangle adjoining them and sometimes without - See Below.



WITH AN ELEMENT BETWEEN ..



AND WITHOUT

TYPICAL FORMS OF
HIDDEN RECTANGLES

NOTE THE REPEATED
ELEMENTS WHICH
OFTEN BUT NOT
ALWAYS OCCUR,

D14.3. RULES CONNECTING DIMENSIONS OF HIDDEN-RECTANGLES

The table below shows a selection of rectangles where an Element touching a side has been added to form the rectangle in penultimate column by adding the Hidden-rectangle.

1. Often the upper dimension of the Hidden-rectangle is obligingly the same as the lower dimension of the original rectangle, but in other cases note the switch round of pairs e.g. 79 to 94 and 94 to 79 in the first two shown.

2. Notice that where this switch applies the added Element is the same (e.g. 56). Also the upper dimension of the both originals is the same (e.g. 130). The upper Element of the 2nd rectangle is the Semi-perimeter of the other solution in the pair.

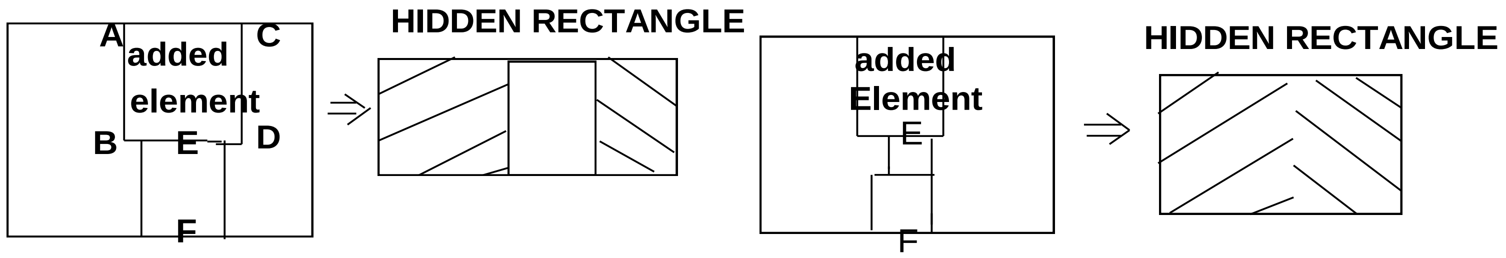
Orders	1st Rectangle		Hidden-Rectangle	2nd Rectangle	Added Element
10 - 11	130 x 79	down	94 x 89 Invalid	224 x 168	56* pair

10 - 11	130 x 94	up	79 x 33 Invalid	209 x 127	56* pair
10 - 11	130 x 94	<>	94 x 68 Invalid	224 x 162	58
11 - 12	194 x 192	<>	192 x 55 Invalid	386 x 247	116
11 - 12	205 x 181	<>	181 x 105 Invalid	386 x 286	106
11 - 12	209 x 177	<>	177 x 30 Invalid	386 x 207	111
11 - 12	209 x 159	down	168 x 7 2 Invalid	377 x 231	108* pair
11 - 12	209 x 168	up	159 x 72 Invalid	368 x 240	108* pair
11 - 12	224 x 162	<>	162 x 45 Invalid	386 x 207	96
11 - 12	224 x 162	<>	162 x 96 Invalid	386 x 258	84
11 - 12	185 x 151	<>	151 x 121 Invalid	336 x 272	100
11 - 12	185 x 183	<>	183 x 42 Invalid	368 x 225	61
11 - 12	195 x 191	<>	191 x 61 Invalid	386 x 252	70
11 - 12	209 x 127	<>	127 x 130 Invalid	336 x 257	73

Unfortunately there is no obviously way of linking the added Element to the dimensions. Nor does it seem possible to calculate the resultant square simply by inspecting the original.

D15.1. HIDDEN-RECTANGLE TYPES

Where only one Element applies between E and F Below then the Hidden-rectangle is divided by a single Element as shown in below 2. Where there is a minimum 2,3,4 … Elements between E and F however the hidden-rectangle is two rectangles side by side as in below 4. There must be at least two Elements bordering A - B, ditto C - D.



ONE ELEMENT BETWEEN A & B

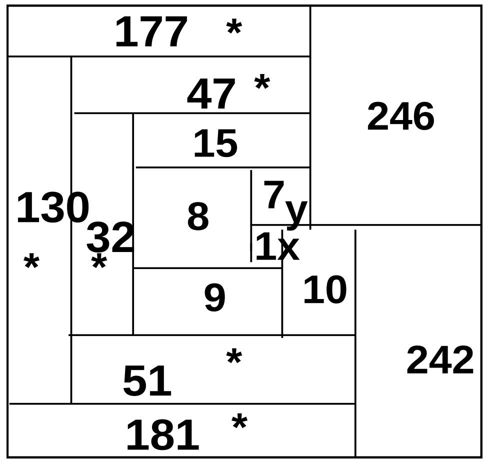
TWO OR MORE ELEMENTS BETWEEN E & F

D16 ADDING OF CLAWS

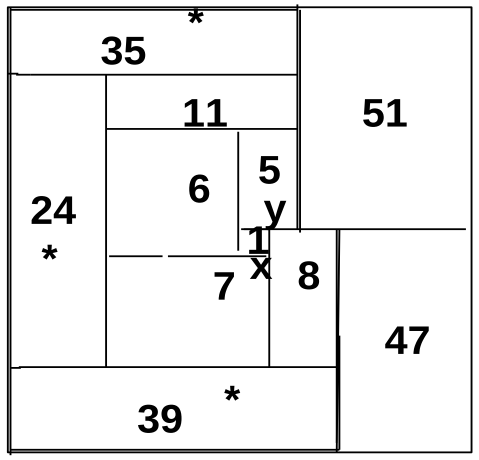
D16.1. ADDING CLAWS TO S2223 SOLUTIONS

Consider the 3rd solution below. The 2nd solution is this pattern with three Elements added (i.e. 35 24 39) and the 1st is the 2nd with three Elements added (i.e. 177 130 181).

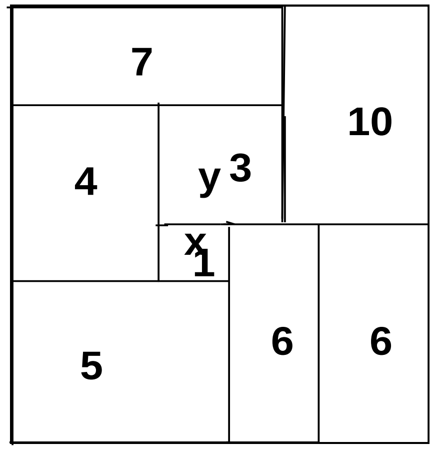
Although the algebra is not shown it can be seen that each may be calculated using x and y in similar positions as shown. Observe that x happens to be 1 in each case and y is 3 5 and 7. Can an arithmetic progression be assumed here an Order 17 solution calculated using x = 1 y = 9. Yes! In fact an infinite series is possible x =1 1 1 1 1 1 … with increment of nil and y = 3 5 7 9 11 13 … with an increment of two for orders 8,11,14,17,20 ….



[14] 488 x 423 ¬



[11] 98 x 86 ¬



[8] 17 x 16 repeater

This principle is found to apply in every case but note that
1. The arithmetic progression may have a negative, positive or nil increment.
2. The first two solutions chosen both have to be 2-2-2-3 sided, and both have to be calculated out (if not already known).
3. In substituting a Pentad for a Diad the property does no longer work.
4. The Reduction Index can vary throughout a series.
5. The status of solutions can vary throughout a series (see the example following with includes a Zero invalid solution for Order 14).
6. Looking at Repeaters can help to find the relationship easier. They are not useful solutions, e.g. A Repeater solution [8] 29 x 37 with x = 2 y = 5 and solution [11] 187 x 166 with x = 1 y = 8 can be extended with x = 0 y = 11; x = -1 y = 14; x =-2 y =17... for orders 14, 17, 20… The Order increases by three each time.

HIDDEN RECTANGLES

D16.2. HIDDEN RECTANGLE TRIAD SOLUTIONS

If any solution is considered and the same solution with a Triad added compared, an interesting feature applies.
Take [12] 368 x 265 which happens to be [9] 69 x 61 with a Triad added at the join of Elements 16 and 28.

In Below 1 the Elements before the minus signs relate to [12] 268 x 265 and those after to [9] 69 x 61.

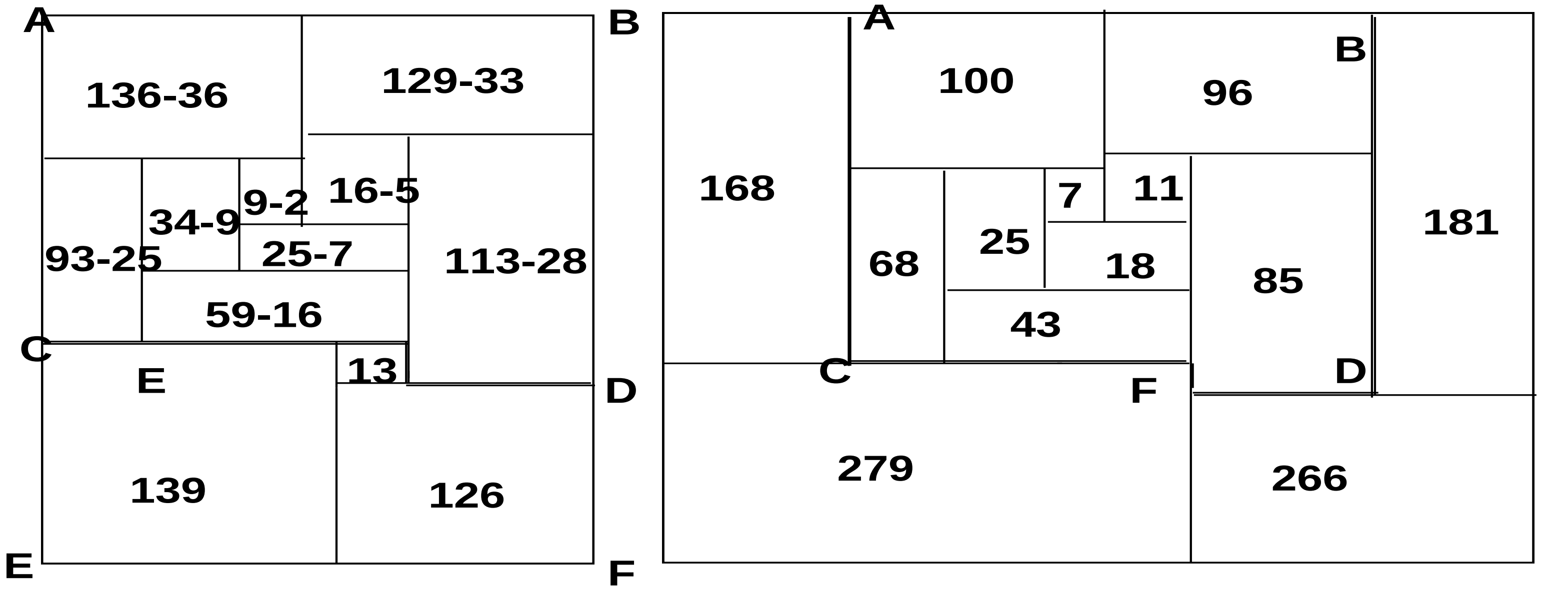
Although shown negatively as -69 x -61 in effect, the rectangle remains true.
Now in below 2 the results of taking the smaller from the larger Elements is shown in ABCD. If the linking Element (13) is dropped and 4 Elements added is shown a true rectangle of one Order higher (here 12 to 13) is created!

Notice that if suitable choices for x and y are made in the below 1 such as 9 - 2 and 16 - 5 or 7 and 11 then the new solution is calculated quicker.

Although a useful property, it does require two suitable solutions already known to use it, and unfortunately there is seems no quick way of calculating the solution with the added Triad. This is not the only link possible between two such solutions. There are at least two symmetric solutions which can be found - see later.

Note that AC = 229 and BD = 242 and that the 545 (of [13] 545 x 447) can be determined as follows- 265 - 69 + 229 - 61 + 242 - 61 = 545. This holds true with other values and solutions.

For any given solution, a Triad can be added in at least five different places, e.g. at E in below 1 where [12] 386 x 277 happens to result, and [13] 581 x 277 is found by literally deducting the solution 69 x 61 from solution 386 x 277.



[9] 69 x 61 and [12] 368 x 265 [13] 545 x 447

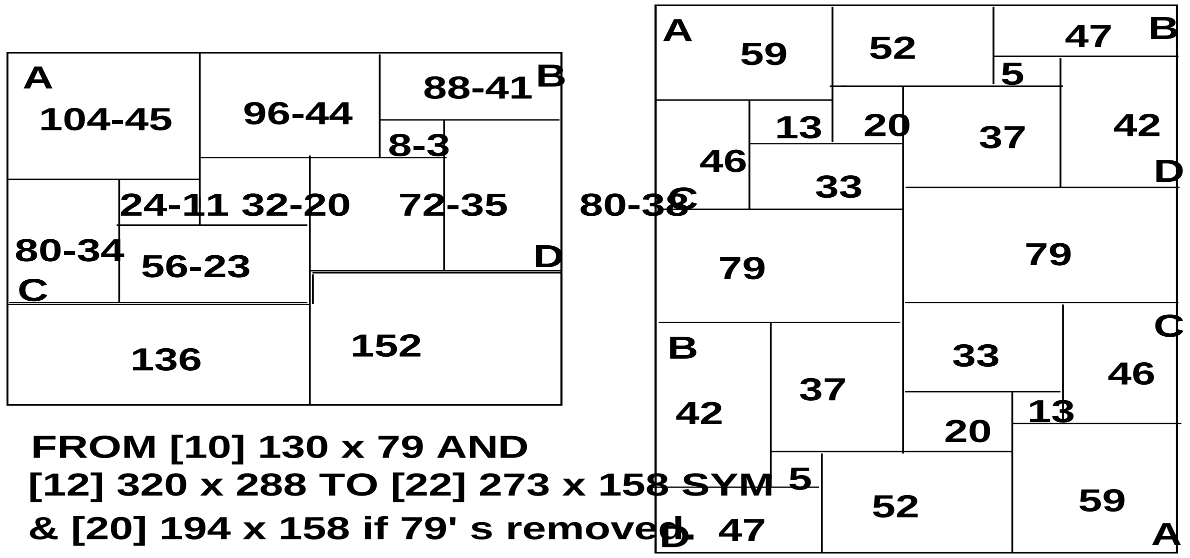
D16.3. SYMMETRIC HIDDEN-SOLUTIONS

In Above 2 if an Element 13 is shown at F then two Elements of 98 can be added. The Elements in ABCD may then be added in reverse Order to form a symmetric Mid-plus solution. The reader can check this out.

D16.4. HIDDEN-RECTANGLE DIAD SOLUTIONS

The numbers before the minuses Below in ABCD are the Elements of [12] 320 x 288 and those after the minuses are [10] 130 x 179. By deducting one set from the other, adding two component Elements (optional) and repeating the Elements in reverse the symmetric solution Order 22 in Below 2 is obtained, and also
[20] 194 x 158 with 79 and 79 taken out.

I have not observed any asymmetric solution possible.



FROM [10] 130 x 79 AND
[12] 320 x 288 TO [22] 273 x 158 SYM
& [20] 194 x 158 if 79' s removed

D16.5. ANOTHER SYMMETRIC SOLUTION

If in D11.1 we deduct instead the Elements of [9] 69 x 61 from the solution of [13] 581 x 477 and make various adjustments another symmetric solution is found. Again both Mid and Mid-plus solutions are possible.

D16.6. LISTING THE SOLUTIONS FOUND FROM TRIAD ADDITIONS

To recap, starting with a solution A and adding a Triad somewhere on an edge to give a solution B, further solutions may be found as follows -

Solution C - deduct A from B remove 3 Elements and add 4 - asymmetric solution.

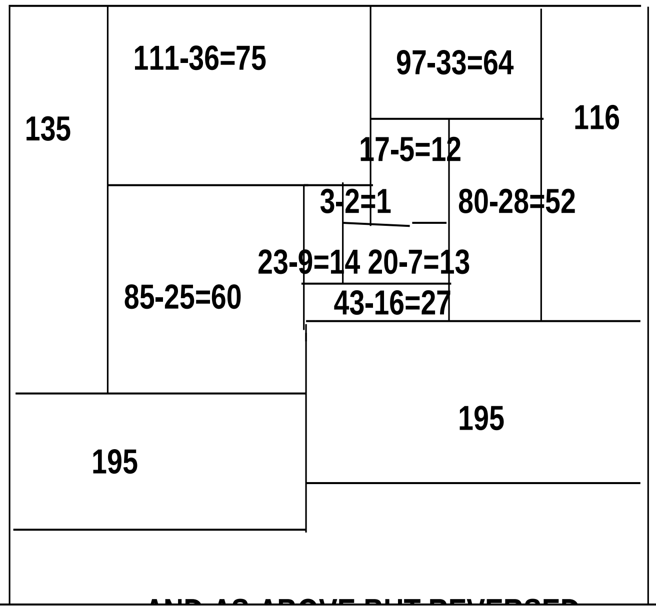
Solution D - deduct A from B adding various Elements to form Mid-plus symmetric solution.

Solution E - deduct A from C adding various Elements to form Mid symmetric solution.

Solution F - deduct A from C adding various Elements to form Mid-plus symmetric solution.

D16.7. LISTING THE SOLUTION FOUND FROM DIAD ADDITION

From solutions A and B deduct A from B and add various Elements to form a symmetric solution (D11.3. refers)



SYMMETRIC SOLUTION
ANOTHER IS FOUND
WITH 195'S REMOVED.

D17 ADDING OF SUCCESSIVE CLAWS

D17.1. ADDING OF CLAWS AND HIDDEN RECTANGLES

For this relationship we simply require any solution with a Triad End, for example Fig 1 - [9] 75 x 55.

If a CLAW is added as shown in Fig 2 the rectangle is calculated by algebra to be [12] 321 x 287. Note that the Elements within the bold line have all changed with the exception of 5. Now if all the differences are taken, Fig 3 is obtained. Although not a rectangle Fig 3 can easily be made into one by ignoring the zero! Thus it is a hidden rectangle.

So what happens if another CLAW is added as Fig 4? This is easy to calculate by adding the Fig 3 numbers to each respective Element. So Element 15 in Fig 1 becomes 15 + 4 in Fig 2 and 15 + 4 + 4 in Figure 4 which is [15] 1447 x 1256! This idea can be repeated throughout so the top 5 becomes 5 + 0, 5 + 0 + 0, etc. i.e. always 5 whilst the bottom 5 increments by 1 i.e. 5 6 7 etc. It is easy to complete the Rectangle by arithmetic.

Obviously an eternal series can be found by using Fig 3 as an Arithmetic Progression with Orders 9, 12, 15, 18, 21 . . . . . without the need for any algebra. This idea is true for any solution with a triad and the Element linking the Dad always stays the original amount.

This idea of adding one solution to another to obtain a third was seen earlier.

There is no need to work out the second Solution by algebra although using algebra to calculate the Hidden Rectangle is. Often this will be a Repeater solution - no problem.

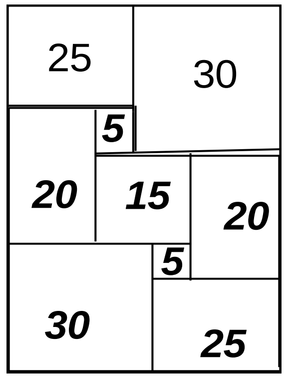


Fig 1

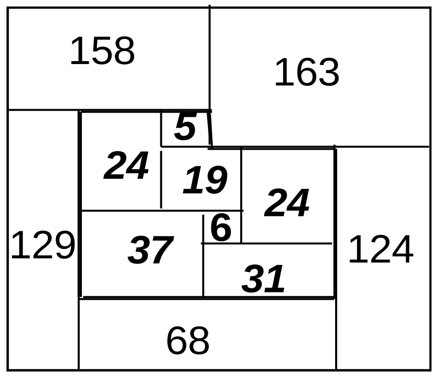


Fig 2

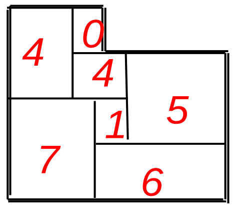


Fig 3

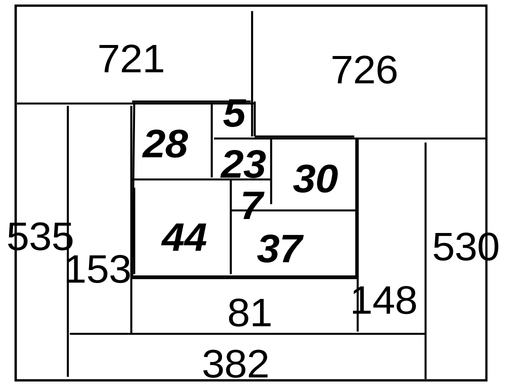
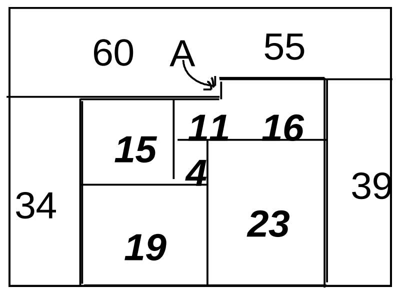


Fig 4

and so on . . .

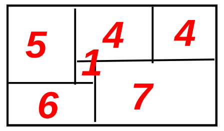
D17.2. ADDING OF CLAWS WITH DIAD SOLUTIONS

The same principles for Triads in the last section also apply to Diads. As before the series is infinite.



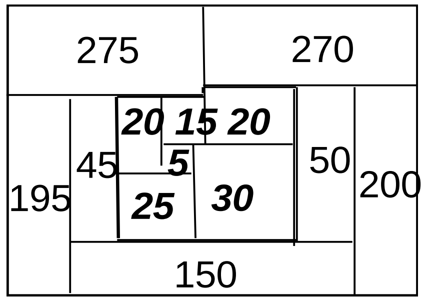
[10] 115 x 94

Plus this



Hidden Rectangle

=



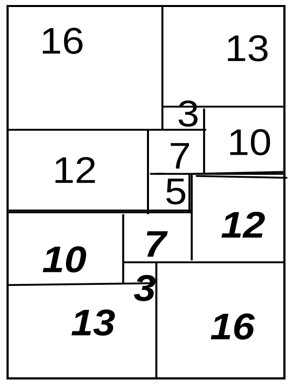
[13] 545 x 470 (109 x 94)

and so on . . .

The next in the series is interesting since the double claw can be replaced by an Add Element to obtain another Solution (this is mentioned in Section E) and a valid Rectangle is found - and this solution is like [10] 115 x 94 Above with an added Element at A. The next solution is [16] 2503 x 2167

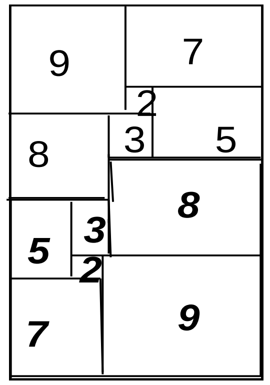
This means two infinite series are related!

It is interesting to see that if we Deduct the Hidden Rectangle instead from the first example in two stages, Symmetric solutions are also found! Each of these has a running series which goes on forever!



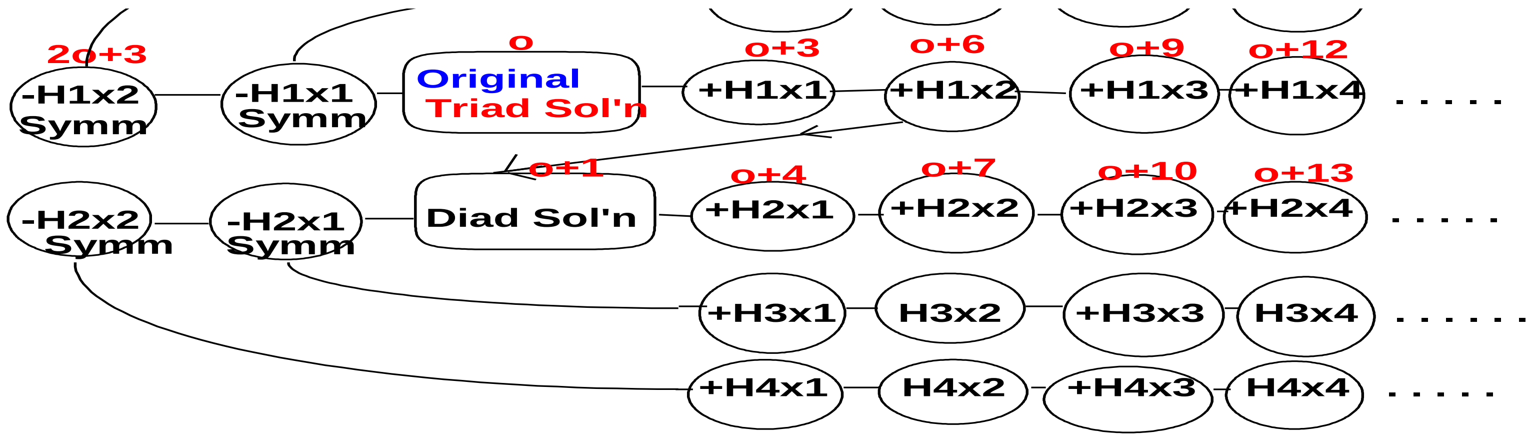
[13] 51 x 29

[12] 29 x 16



.

The next diagram shows how several Solutions are obtainable from an Original of Order O.



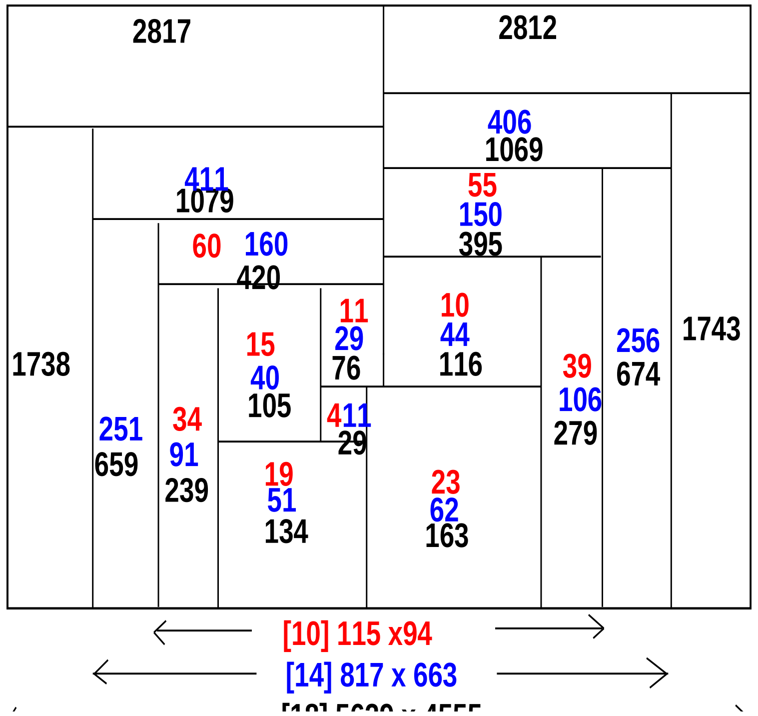
ADDING OF ARCHES

D17.3. ADDING OF ARCHES

If arches are added to a Solution in succession there is, again, a formula connecting the values of the Elements. [10] 115 x 94 has been chosen here and shown in red figures, but any solution can be used. By adding one Arch the Solution found is [14] 817 x 663. With a further Arch added [18] 5629 x 4555.

Look at the red values 11 and 4 (as values for x & y). Once new values have been calculated the whole Rectangle can be calculated readily. With an Arch added these Elements become 29 and 11. To obtain the next values they are found to be three times the current value minus the one before! Thus 29 x 3 - 11 gives 76 in the Order 18 solution. The next value is therefore 76 x 3 - 29 = 199 and the one after 199 x 3 - 76 = 521 and so on.

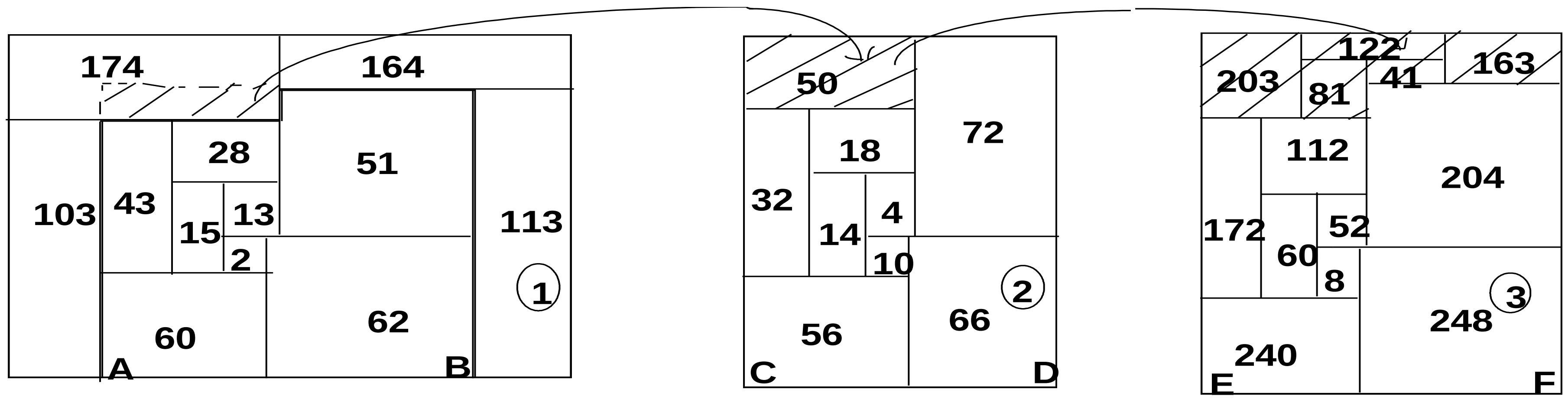
Two solutions, one Order 4 more than the other, are necessary for an eternal series to be calculated. So far I have found no formulae linking the Dimensions.



D18. ARCH TO SINGLE ELEMENT RELATIONSHIP

D18.1. ARCH TO SINGLE ELEMENT RELATIONSHIP

If a 222* solution is taken and the four corner Elements are replaced with one as shown below (which requires recalculation) the result is interesting.



[12] 338 x 277 [9] 69 x 61 redrawn as 122 x 138 [15] 488 x 615 ¬

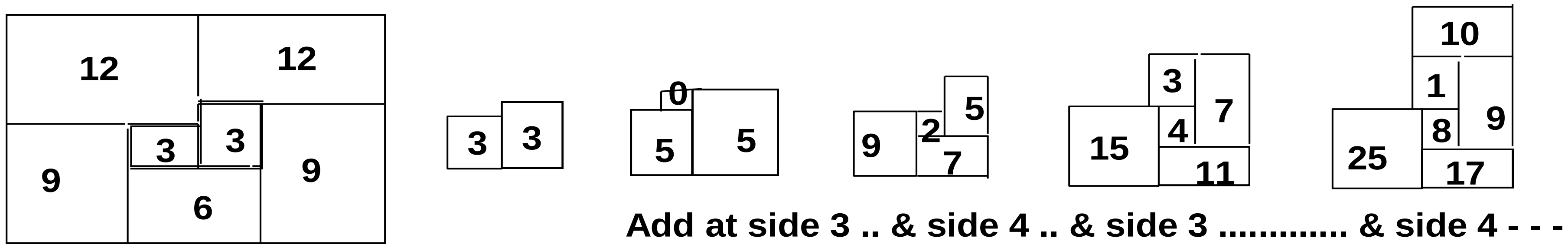
1. In a Solution m X n, AB is always 2m - 2n, in this case 122.
2. In above 2 CD is found to be 122 also - if all Elements are multiplied up by two, and the solution shown sideways.
(The pattern may modify slightly - in this case 4 is smaller than 10).
3. In Above 3 the single Element has been replaced with a Pentad.

Section E explains how above 1 and above 3 are related, but for present purposes EF is found to be 488 or 4 x 122.
4. So all three patterns are linked. In 3, most Elements are the same as those in 1, multiplied by 4.
But in 2 the Elements are different both in value and proportions.
5. No matter what 222* Solution is selected, it is always found that AB CD and EF are identical once suitable up-ratings (often by 2 4 or 8) are made.
6. Section E shows that the Dimensions of 3 (615 x 488) are readily found from the Dimensions of 1 by applying a formula.
7. But can the Dimensions of 2 (as 122 x 138) be determined by a formula from 1 and/or 3?
Taking m x n as 338 x 277 AB (122) is always found to be 2m - 2n.
In 3 taking m X n as 615 x 488 a quarter of 488 = 122 so clearly this Dimension is related!
What about the other value 69 (x8) - does this relate to 338 x 277 and/or 615 x 488??

D19. A SERIES OF S2223 SOLUTIONS AND ITS RELATIONSHIP

D19.1. S2223 SOLUTION SERIES

Below is an interesting series starting with [7] 24 x 21 which has sides 2223. By adding Elements at sides 3 and 4 alternately, further S2223 Solutions for Orders 8 9 10 11 … are created.
By the Adding Rule the added Element sizes are seen to be 0 (i.e. 3 - 3), 5 (i.e. 7 - 2) , 3 (i.e. 7 - 4) and 10 (i.e. 1 + 9).
However the values of other Elements alters each time (e.g. 3 and 3 to 5 and 5 to 8 and 7 etc.).



[7] 24 x 21 (8 x 7)

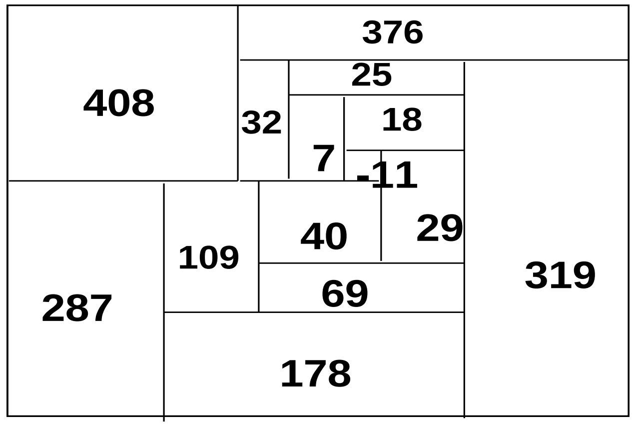
Surprisingly a definite series connects all the values:-
[7] 24 x 21 3 3 <==== Elements
[8] 40 x 35 5 5 0
[9] 69 x 61 9 7 2 5
[10] 111 x 98 15 11 4 7 3
[11] 187 x 166 25 17 8 9 1 10
[12] 297 x 263 41 27 14 13 -1 12 11
[13] 496 x 441 67 43 24 19 -5 14 9 23
Looking down the first column of Elements we observe increments of 2 4 6 10 16 26 ...

The second column has increments of 2 2 4 6 10 16 26 ... which ÷ 2 is the natural Fibonacci Series of 1 1 2 3 5 8 13 … where each value is the sum of the two before it. Note that the other columns have the same series except that some are negative Fibonacci series as in 3 1 -1 -5 ...

Alternately it is easily seen the values in the first Column (3 5 9 15 etc.) are the sum of the previous two PLUS 1.

Ditto second Column - sum of the previous two minus 1.

So the Order 14 Solution is easily - the starting inner Elements being 41 + 67 + 1 = 109, and 27 + 43 - 1 = 69.



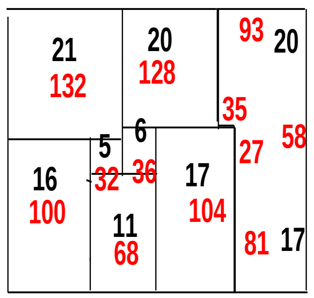
[14] 784 x 695

Semi-perimeters 45 75 130 209 353 560 937 1479 2484 3853 6515 10080 17061 26387 44668 69071 116943 180856 306161 473487 . . . .

So far I have found no formulae directly linking the Dimensions.

D20 DIAD TO PENTAD TO OCTAD

D20.1. DIAD TO PENTAD TO OCTAD ... RELATIONSHIP

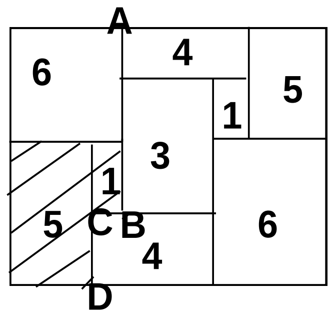


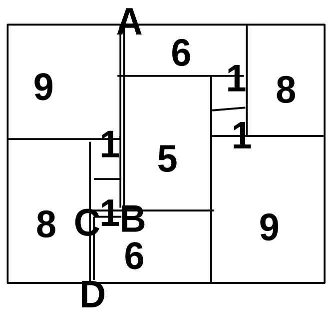
What happens when a Diad in a particular solution is replaced by a Pentad, then by an Octad, then by an 11-Add and so on? Above [12] 353 x 232 with a Pentad end has been selected (see red values) and the Invalid solution [9] 61 x 37 with a Diad calculated (see black values).

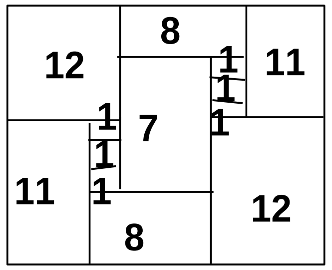
We notice the left hand red Elements are all divisible by 4.

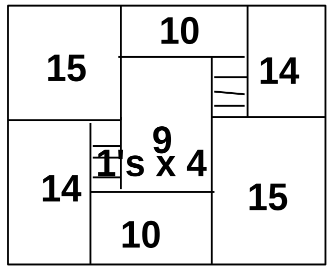
D20.2. CONNECTION BETWEEN DIAD & OCTAD ETC.

Below the Solution [9] 15 x 11 has been chosen as the simplest Solution containing a Triad and as the Solutions which follow it are so easy to find. But the general principles mentioned also apply equally well with other solutions.









[9] 15 x 11 Valid [11] 23 x 17 Invalid [13] 31 x 23 Invalid [15] 39 x 29 Invalid

In Above, the value 1 has been replaced 2, 3 and 4 times vertically and the Invalid Solutions calculated to be [11] 23 x 17 [13] 23 x 17 and [15] 39 x 29. Continuing this series is easily done with the corner Elements increasing each time by 3 and the Dimensions increasing by eight times six.

1. Now the left Triad in above 1 can be converted into an Octad without the Ratio of 7:1:4 changing (see ABCD).

2. Likewise the Triad Elements 9 8 and 1 in above 2 can be changed into an Octad. Actually an Octad Plus because of the 3 repeated Element 1's. Now an Octad Plus can be converted into a 14-Add. This gives two Valid solutions from an Invalid one - in fact there are two and only two Valid solutions ever possible at one time, one with an Order 5 less than the other.

3. From Above 3, the lhs can be converted by degrees into 22-Add Plus and 28-Add solutions.

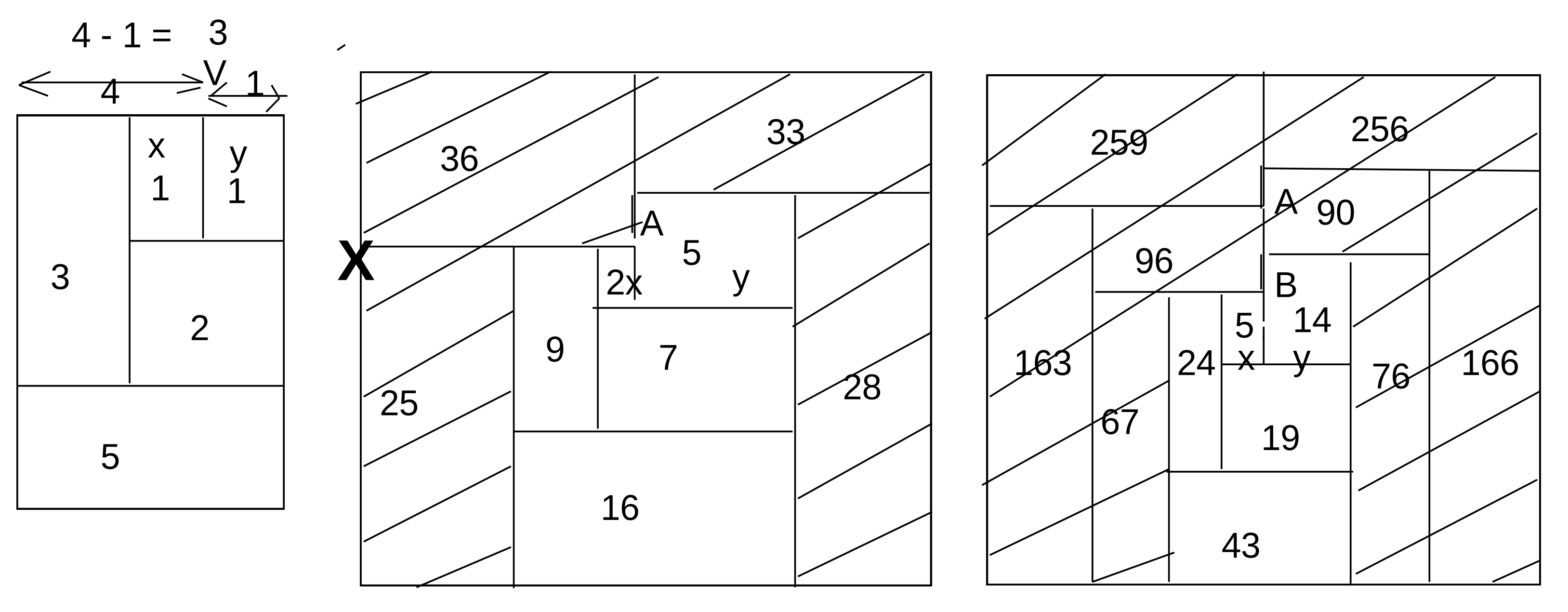
4. From Above 4, the lhs can be converted by degrees into a 28-Add Plus and 34-Add solutions.

Knowing the relationship between Diads and Octads etc. and equating the Algebraic coefficients it is possible to find a Formula linking the two shaded Elements (above 1) with an Element in the Octad from which the Octad values are easily found. In Above calling x = 1 and y = 5 then new value <u>v1 = (8x + y)/15</u>. Since 13 is not divisible by 15 we have to make nv 13 and up-rate all Elements to the right by 15 to get the new solution.

An equivalent formula can be found for above 2 converting the Elements 9 8 1 and 1 into a 14-Add Ending. This is found by algebra to be <u>v2 = (15x + 2y) div 28.</u> In fact there is a series of formulas which follows an easy sequence. *I have yet to find and check it however!*

<u>D20.3. ADDING OF SUCCESSIVE CLOAKS INSERT BETTER!</u>

Allied to the previous section there is another everlasting series of Solutions which may be created by the adding of Cloaks. A Cloak is 4 Elements added as shown in below 2. A further Cloak is added in Below 3.



[5] 5 x 8 INVALID

[9] 69 x 61

[13] 515 x 422

We start with any Solution best not Symmetric. Note that as above it may be sometimes an Invalid one!

We then need to calculate using algebra the same Solution plus a Cloak using x and y (in this case) as shown and [9] 69 x 61 is found (x = 2, y = 15). Adding another Cloak we find [13] 515 x 422 (x = 5, y = 14).

Without explaining why, the link between the values of x and y, and indeed the Elements generally is to triple the value of x in the 2nd solution and deduct the first value of x. (In the case of Triads it was x4 first - second, i.e. quite similar!).

Thus 3 x 2 - 1 = 5 and 3 x 5 - 1 = 14.

The next in the series is [17] 3573 x 2896 which has x as 13 and 37. Now 3 x 5 - 2 gives 13 and 3 x 14 - 5 = 37. So we can continue the series as long as we like. Sometimes the values of x and y will reduce giving a RI greater than 1.

D20.4. KINK VALUES

Note above that the lines at A have a distance of 3, and B a distance of 9. Note also in above 1 that the position of insertion of the Triad is 4 from the left and 1 from the right, and that the difference 4 - 1 = 3 the value of A. This property is always true. The equivalent value of C in [17] 3573 x 2896 is found to be 24. This series runs 3, 9, 24, 63, 165 ... divided by 3 gives 1,3,8,21,55..

Do you recognize the relationship? Yes?

The Fibonacci series runs 1,<u>1</u>,2,<u>3</u>,5,<u>8</u>,13,<u>21</u>,34,<u>55</u>,89,<u>144</u>... and we omit the alternate values! The Kink Values are actually a multiple of the alternate F series.

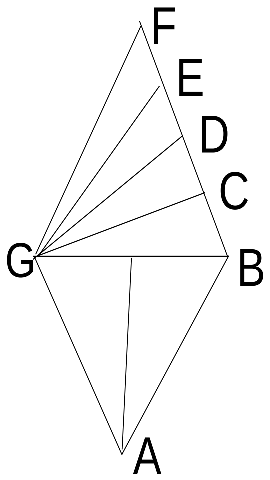
Look at X in Above 2. If we use [9] 61 x 69 as our base and were to add a Cloak at this point the kink value will be 11, which is 36 - 25. The Kink Values are 11, 33, 88, 231 etc.

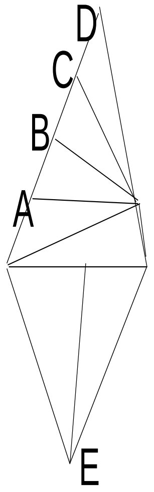
D21 HOOK ADD-ONS

D21.1. HOOK ADD-ON SERIES

(In this Section we are always dealing with *Full Dimensions* of Rectangles, and sometimes the Lower Dimension is quoted first. For instance [7] 24 x 21 and
[7] 21 x 24 are regarded separately and quoted thus).

This section looks at the effect of adding pairs of Elements at the Pole of a Squared-Rectangle. Below 1 is a Diagram where the SR generated from Pole B down to Pole A happens to be [5] 8 x 3 INVALID. Pole C down to Pole A happens to be [7] 24 x 21. Suppose we add two Elements BC & CG making the Pole at C. Now the Rectangle is found to be [9] 64 x 66 (i.e. [9] 33 x 32 multiplied by two and reversed). With the Pole at D [11] 168 x 185 is found and at E, [13] 440 x 497.
Are there relationships?





D21.2. HOOK NUMERICAL SERIES - 1ST NUMBER

Yes, the series 8, 24, 64, 168, 440 ... does have a relationship. 64 - 24 = 40 and the series, {8} 16 {24} 40 {64} 104 {168} 272 {440} 712 {1112} ... is easily seen each value being the sum of the previous two!
In order to avoid having to calculate in-between values, it is found that each number happens to be three times the previous number, less the one before that, i.e. 64 = 24 x 3 - 8, 168 = 64 x 3 - 24, 1112 = 440 x 3 - 168 and so on.
Other series can be found from the same original solution CGAB [7] 24 x 21.
Look at above 2 where the Poles start from the left side with AE BE CE & DE …

The Complexities of these also follow a series with each value the addition of the previous two. We have 21, 66, 185, 497 ...

D21.3. HOOK NUMERICAL SERIES - 2ND NUMBER

We would hope that 66 x 3 - 21 would give 185 as in the first series but it gives 177 - 8 less! Likewise 185 x 3 - 66 gives 489, 8 less than the required 497! Could it be that the next number is 497 x 3 - 185 + this odd value of 8? Yes - it is found to be 1314!

In fact an everlasting series can be found from this particular series by adding 8 each time, but why 8? Is it 8 for other series too? The Writer soon found others use 8 but the rest not!

Before assigning mathematics to the dustbin, the series connects backwards to the Invalid Solution [5] 8 x 8 (which reduces to [5] 2 x 2). This is where the 8 arises.

The Writer has found a series of [7] 21 x <u>24</u> [9] 66 x 55 [11] 177 x 176 [13] 465 x 497 and [15] 1218 x 1339. By knowing the first two of these Solutions only, the 1st Series are easily found e.g. 177 = 66 x 3 -21, 465 = 177 x 3 - 66 and so on. But we need to know of the third in the series [11] 177 x 176 before it is possible to calculate the 2nd series.
176 x 3 - 55 = 473, 24 short of the required 497. Now this 24 comes from [7] 21 x <u>24</u> (with apparently has no link back to [5] 8 x 8).
So, 497 = 176 x 3 - 55 +<u>24</u>, and 1339 = 497 x 3 - 176 +<u>24</u> !

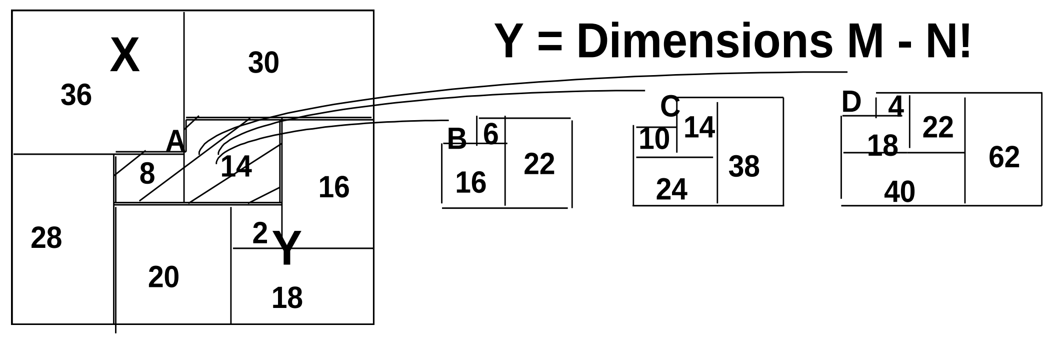
Annoyingly we cannot find 176 from the two previous Solutions - 55 x 3 - 24 = 141 is 35 short. Isn't mathematics strange? We simply have to have to know three initial Solutions for starters!

Some solutions in a Series are found to be Invalid or highly reduced in size.

D22.1. A PARTICULAR SERIES OF 2233 SOLUTIONS

In this section we consider side 2233 Rectangles with four Elements arranged in such a fashion that they leave a _| shaped piece in the centre as shown in [9] 66 x 64. Below (which reduces to [9] 33 x 32).

By adding an Element at A [10] 114 x 110 is found. By adding at B [11] 196 x 190 and then at C [12] 324 x 314 ...
By examining each solution in turn, some fascinating things are discovered.
1. Look at Element 2 below. This is the difference in the Dimensions of 66 - 64. In fact the Element in this position is always the difference in the Dimensions in [10] 114 x 110 this Element becomes 4, in [11] 196 x 190 the Element 6. The Element numbers are 2 4 6 10 16 26 42 ... each the sum of the previous two, and in this series twice the Fibonacci series 1 1 2 3 5 8 13 etc.



[9] 66 x 64 full size [10] 114 x 110 [11] 196 x 190 [12] 324 x 314

2. Can the Dimensions in the series be linked? At first apparently not, but since the Elements are alternately added vertically then horizontally it becomes evident that we must treat the series as *two* series, one for odd Orders, and one for even Orders! To establish a formula we need to calculate the first three Solutions which for the odd Orders are [9] 66 x 64 [11] 196 x 190 [13] 538 x 522 and continue [15] 1434 x 1392 [17] 3780 x 3670 [19] 9722 x 9634 ...
Similar to the add-on in Section D21 we observe that 3 x 196 - 66 = 522 and that 538 is 16 more! Likewise 190 x 3 - 64 is 506 again 16 less than 522!

So the formula is <u>3 times the dimension less the previous dimension plus a TOP-UP number</u> which has to be established. Thus e.g. 3780 x 3 - 1434 + 8 gives 9722 and so on.

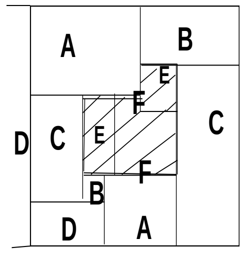
3. By the Adding & Diminishing Rule we know the Element added at A will be 6 i.e. 14 - 8, at B will be 10 i.e. 16 - 6 and so on.

4. In all these types of series, the Reduction Index is always even and therefore at least 2.

5. The Writer was fascinated that where the reduced sizes are concerned [9] 33 x 32 [11] 98 x 95 and [13] 269 x 261 all contain Reciprocal Pairs for the semi-perimeters throughout. But the Order 10, 12, 14 ... solutions do not! Look at the corner Element 36 above (X) & Element 2 (Y) which reduce to 18 and 1. These two Elements are Reciprocal. 18^2 is 324 and 1^2 is 1. 324 + 1 = 325 which is 65 x 5 (the reduced semi-perimeter 33 + 32). In the Order 15 solution X & Y reduced by factor 2 are 354 and 21 and the Semi-perimeter 1413. Now 354^2 = 125316 + 21^2 = 441. 125316 + 441 = 125757 = 1413 x 89.

6. In [10] 57 x 55 the X & Y Elements are 30 and 2 for SP of 112. But $30^2 + 2^2$ is NOT divisible by 112! But 30 squared <u>minus</u> 2 squared is 896 which is 112 x 8! Similarly with [12] 162 x 157 SP 319 where X & Y are 82 & 5. $82^2 - 5^2$ = 6699 = 319 x 21! Note these coefficients 5, 8,89 & 21 are all Fibonacci numbers! NB Pairs do not always work using full dimensions for this series.

7. The Dimensions for the Even Order solutions in this series are also calculated thus - <u>3 times the number less the previous number plus the TOP-UP In this case 16</u>. 8. For the set of Solutions considered the pairs of Reciprocals are in the same relative positions as shown below. A with A, B with B etc. Note however these pairings differ with other sets of solutions with sides 2233!



THE SOLUTION HAS RECIPROCAL PAIRS THEN THE ELEMENTS WILL PAIR UP AS SHOWN (All remaining pairs will be within the shaded area)

E. CALCULATING BY LINKS

E1. EXPLANATION OF SECTION & CREATING SOLUTIONS USING ADD-ONS

E1.1. INTRODUCTION TO LINKING SOLUTIONS

Creation of large numbers of new solutions by algebra is tiresome and time consuming, but thankfully there are easier ways of doing this and avoid much algebra.

Once *certain types* of solutions are known, it is possible to create new ones from them.

The solution which is suitable to be manipulated in some way will be referred to as the Origin and the solution found from the Origin will be termed the Result.

The Origin is always deemed to be Order [o] with dimensions of m X n and is indicated by [o] m X n.

In the majority of cases the Result can be calculated in terms of m and n only, once the necessary LINKING FORMULA has been established.

For example the Result might be [o + 1] 5m - 4n X 4m - 3n indicating that the Order has risen by one.

Sections E1 to E15 all deal with the creation of new solutions from ones already known.

There are so many different devices used it is so confusing! It is like solving a jigsaw in which all the pieces first have to be gradually found and then grouped into sets of series, before the full picture can be seen!

E1.2. STUDY OF LINKS IS COMPLICATED

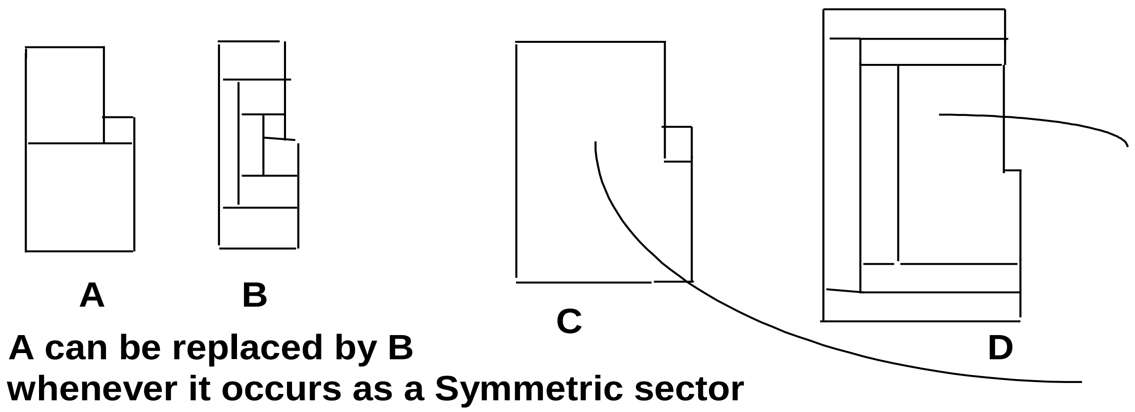
I have spent hours studying the complicated and bewildering masses of patterns, and have begun to understand the theory as a whole. There are infinite possible Links, and the quantity far more than expected.

In fact it is not possible to provide more than a proportion of them in this Book.

I have decided it is probably easier to mention my ultimate findings first and then deal with specific Links after. This does not follow the pattern of perfect mathematics textbooks which gradually builds up the picture from first principles, but the study is confusing however it is presented. The important paragraphs which follow will not make much sense when first read, but hopeful will make more sense later on in the text.

E1.3. STUDY OF LINKS HAS SOME EASY AND BASIC FEATURES

Despite the above paragraph, I have has found basic features which always apply which are not difficult, and once known, these help the subject to appear a little easier! Two examples are shown below and these are of vital importance in the understanding of this complex subject. The patterns are parts of a Rectangle.
Simply A can be replaced by B, and C by D (and vice versa) whenever they occur.
In each, one element termed a PLUS (+) is removed and replaced with six as shown. The Elements increase by 5.



Provided these Areas contain exactly the same Elements, and form a strictly Symmetric pattern they are always interchangeable. i.e. a solution containing C can always be changed to D & vice versa, without the Rectangle failing.

A can be replaced by B whenever it occurs as a Symmetric sector of a Symmetric pattern within a Rectangle.

E1.4. LINKS - BASIC CONSIDERATIONS

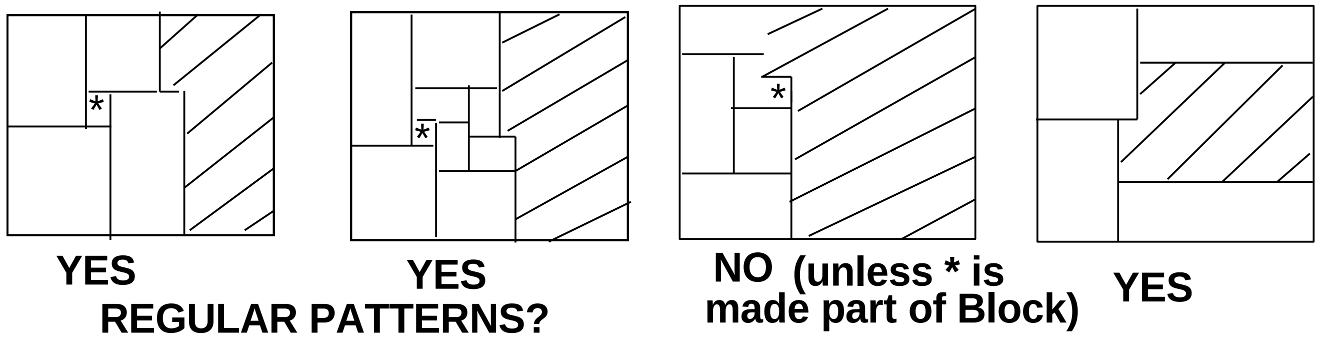
Any Squared-Rectangle consist of two parts although some Rectangles are entirely IRREGULAR in type.
1. A REGULAR pattern *, with the rest of the Rectangle deemed to be
2. IRREGULAR, even if some or all of it by chance is Regular. This Irregular pattern is a single BLOCK (sometimes two Blocks) and is always "L" shaped in varying proportions. The Block is shown shaded, and as such may represent any pattern which exists.
There may or may not be PLUS ELEMENTS as well, which must be regarded as being part of the BLOCK, not the Regular part. (NB - A Symmetric pattern with such an Element added, is always an Irregular pattern if viewed as a whole).
It can be observed that the alterations from one pattern to another (in order to create new solutions) always affects the Regular part only - i.e. the structure of the Regular part is changed, with the Result still Regular.

E1.5. WHICH PATTERNS ARE ACTUALLY "REGULAR"?

This is not that obvious! For example a Triad is not a Regular pattern but a Diad is!

Some patterns which seem Irregular are in fact Regular! Most patterns appear in several possible ways, which can be shown in one fixed format. The fact that some Elements may have to assume negative values to make the exact format is not a problem.

Link Elements (shown as * below) added inside the Regular Area may seem to destroy the Symmetry, but do not!



E1.6. GROUPS AND GROUPS WITHIN GROUPS

I have found that the Links form definite Groups although the Groups vary greatly in size from zero to hundreds!

Of these some of them can be joined together to form larger Groups.

How to present these Groups in the easiest and most logical fashion is problematic as there are two or more basic ways of doing this -

1. By Regular patterns with or without a Link Element, is a good division, but then so is -
2. By Patterns of sides S2223 type.

E1.7. LINKS COME IN PAIRS - AT LEAST IN THEORY!

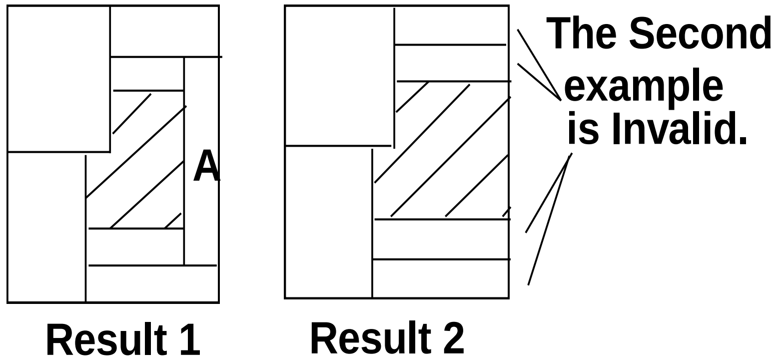
The Reader at this stage is simply informed that Links come in Pairs of Links, and this appears convenient.

Each of the Links are found to fit numerically - i.e. the Links themselves appear sound in theory, but there are many instances where one will give rise to an undesirable Invalid Solution.

E1.8. INVALID SOLUTIONS AND LINKS

Look at the two patterns below, shown as examples. Whatever Origin gives the Result 1 will also give Result 2 by deletion of A. But clearly Result 2 with its adjacent elements gives a useless Invalid Solution and is rejected.

The point being stressed is that this actual Rectangle <u>still fits exactly</u> with no gaps remaining, and in this sense is valid.



E1.9. VARIOUS LINKS CONSIDERED ONE AT A TIME

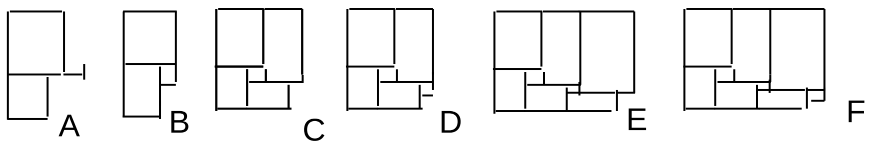
The reader should note the general formats particularly in Order to understand the mass of patterns shown.

Much effort has been made to present these as clearly as possible in logical groups as follows:

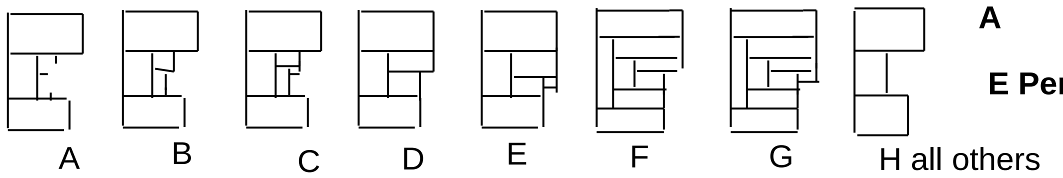
Adds E1.1. to E1.6. By adding one external Element.

“ E1.7. to E1.10. By adding one internal Element.

“ E2.1. to E2.4. By adding two Elements.
“ E2.5. to E2.6. In addition to these are Add-ons relating to Squared- squares only.
“ These are dealt with properly under the Squared-square section H2.
Ends E3. to E7. By using special endings (groups of 5, 8,… elements).
“ E4. 1st series E5. 2nd series E6. 3rd series
“ E7. 4th series. E8. 5th series E9. 6th series
“ E10. 7th series E11. 8th series E12. 9th series
Joins E13. to E15. By joining parts of two known Rectangles together with extras.
The two parts may be the same used twice, or both different.
Trial E16. By trial and error processes.
The ADD-ONS are better understood by drawings rather than by writing.
Throughout Section E, wherever an Element is added, it will be observed that the size of that Element is always the same as the line length it came from due to the Add and Deduct Rule earlier defined in D8.



A Diad B Diad+ C Double D Double+
E Triple F Triple +



A B C D Pentad
E Pentad + F Octad G Octad + H All others irregular

2223 - AAAA >> S2224 - AAAZ S2223 - BAAA S2223 - ABAA S2223 - AABA

E1.10. CREATING SOLUTIONS S2224+ TO S2223 CODE N1 - N3

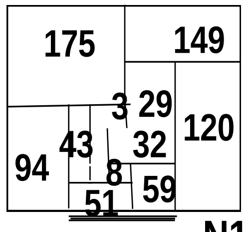
Consider any S2224 (or S2225, S2226, S2227...) solution - ABCD Below. If an Element S is added at EFGH and the four corner Elements expanded, a new Rectangle an Order one higher is produced.

Careful inspection shows the whole border increases by S all round and that the result is a true Rectangle.

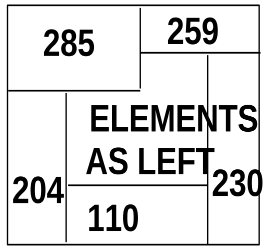
Below 1 is converted to below 2 by adding 110 (51 + 59).

Investigation confirms that the Reduction Index in the Result is always the same. Also it is found that solutions go from Invalid to Invalid; Imperfect to Imperfect and Perfect to Perfect.

In other words the status of the Result is the same as the Origin! If the Origin is called [o] m x n, m being normally deemed greater than n, then the new Element at EFGH is found to be 2 x (m - n) and the Result is [o + 1] 5m - 4n X 4m - 3n.

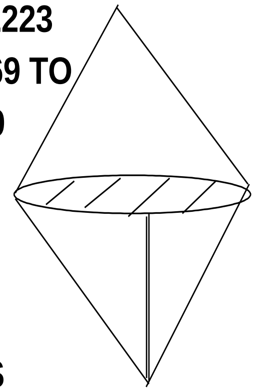


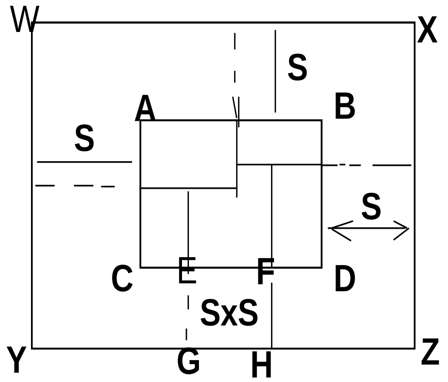
N1



N3

ADD-ON 222x TO 2223
FROM [12] 324 x 269 TO
13] 544 x 489





ALGEBRAIC PROOF THAT
THE PROCEDURE ALWAYS
WORKS & THE COMPARATIVE
ALGEBRAIC SIZES
[o] m X n
[o+1] 5m - 4n X 4m -3n

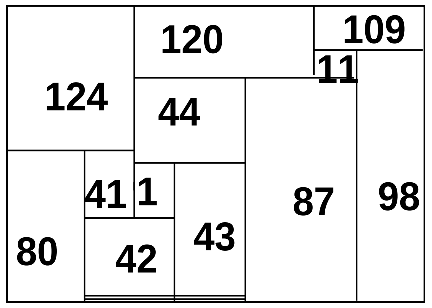
SMITH DIAGRAM
FOR 2ND
PATTERN

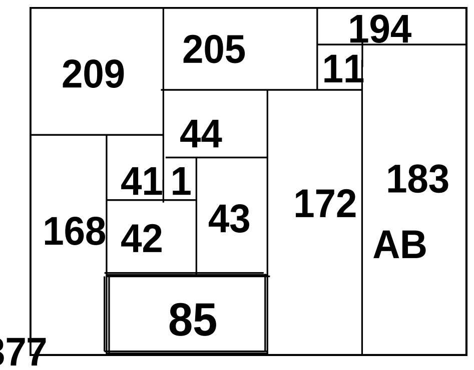
E1.11. CREATING NEW SOLUTIONS S2325 TO S2324 AND SIMILAR - CODE AB

This is similar to the Add-on in D3.1. with a Triad at one end. Below shows again that all the outer Elements increase by the amount of the Add-on, 85. In this example the new solution is 3 x 85 times 2 x 85 bigger, and one Order higher.

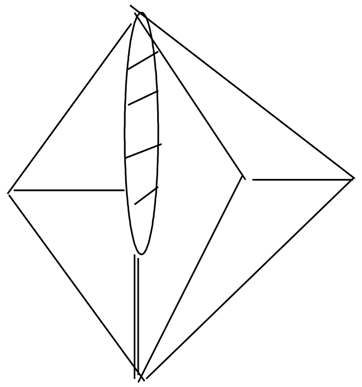
In similar solutions where there are any amount of Triads to the left and/or right, a new solution can be created also.

Using endings other than Triads fails. In Below the Element 85 is equivalent to 2m - 3n.

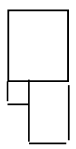


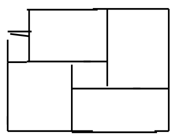


[12] 353 x 207 TO [13]608 x 377



SMITH DIAGRAM
FOR 2ND PATTERN





SIMILAR SOLUTIONS BUT WITH FURTHER
TRIADS ADDED TO LEFT AND/OR RIGHT IN
ANY AMOUNT ALSO PROVIDE A NEW
RECTANGLE.

[o] m X n to [o+1] 7m - 9n X 4m - 6n

E1.12. CREATING NEW SOLUTIONS S2343 TO S2333 AND SIMILAR

Yet another variation of D3.1. is apparent when the Rectangles shown are viewed sideways.

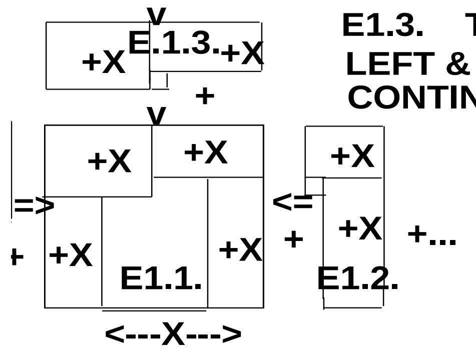
It is a D3.1. Add-on with a Triad added at the top (or left in below).

Similar Rectangles with further Triads added will also work.

In practice this Add-on does not provide many new solutions.
The Order increases by one. The Element 221 below is equivalent to -2m + 3n.

E1.13. BRINGING THE ABOVE ADD-ONS TOGETHER

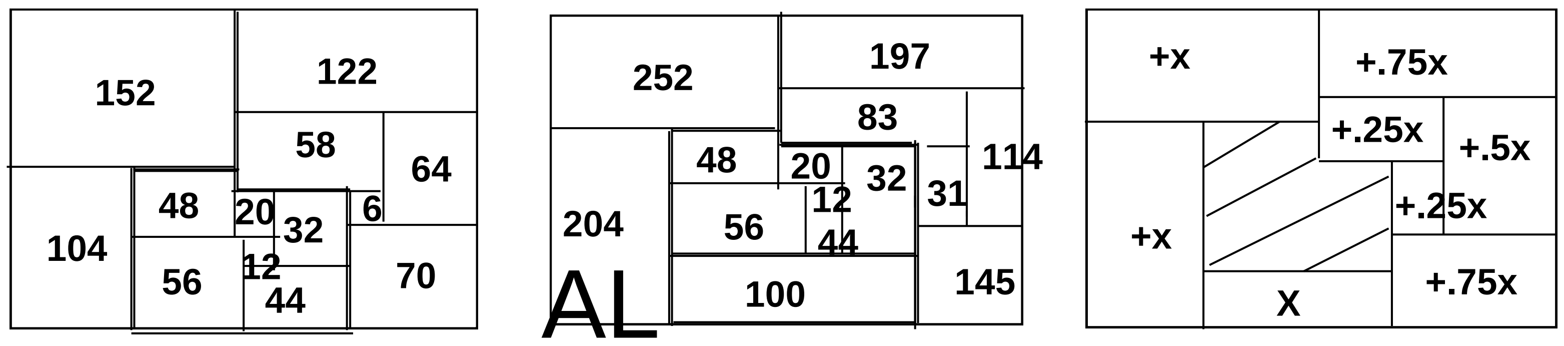
The three Add-ons and their series so far described are all related. Below7 shows how and also provides proof to the validity of each -



E1.3. THE SERIES RIGHT AND/OR LEFT & SERIES UP CAN BE CONTINUED AS FAR AS DESIRED.

E1.14. ADD-ON INVOLVING A DIAD AND A PENTAD CODE AL

Below is an Add-on of this type. Although the Pentad and Diad parts usually need recalculation, the body Elements always keep the same proportions and shape.



[13] 274 x 256 [14] 456 x 449 SHOWING AMOUNTS EACH EXTERNAL ELEMENT INCREASES BY

The relationship as usual can be proved by algebra. However it is evident by above 3 that the relationship is true as the new Rectangle becomes 2x greater one way and 1.75x the other.

This Add-on made me aware of many more relationships which are summarized in the next section.

The Formula for this is [o] m X n to [o+1] 4.5m - 49/ 16n X 4m - 2.5n (multiplied by 1 2 or 4).

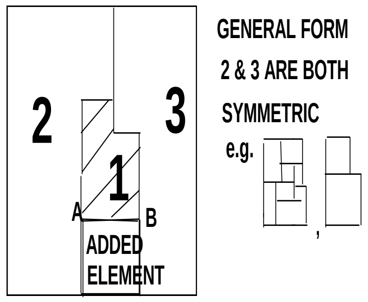
E1.15. GENERAL FORM

Below three areas are shown -

1. Body composed of a set of Elements with at least two Elements bordering AB 2. A symmetric format on the left, and
3. A symmetric format on the right.

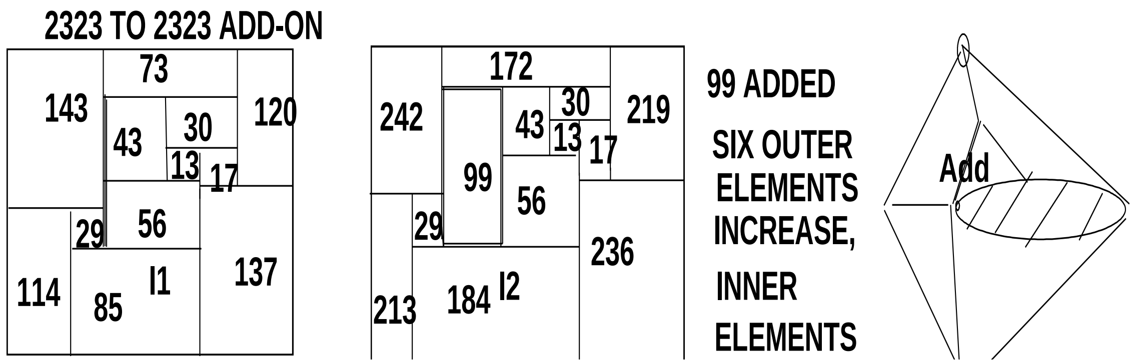
The Diad and Pentad Above are examples of symmetric formats - but note that a Triad is not! Whatever formats are applied, The body area always retains the same proportions (even if it is necessary to multiply all the Elements by a constant) when an additional Element is added at AB. Both the formats require recalculation and this can be tricky - but as these forms are not frequent if ones larger than pentads are used, this is not a major problem.

The principle is the important thing. Note this theory includes most of the Add-ons shown so far, e.g. using two Diads as in E1.1. The Order increases by one.



E1.16. CREATING NEW SOLUTIONS S2323 TO S2323 - CODE I2

The Formula for this construction is [o] m X n to [o + 1] -7m + 16n X -4m + 9n.



In some S2323 solutions it is possible to add an Element which straddles the two side Elements as Above, and create a new Rectangle, one Order higher. Whenever this is done all six outer Elements are increased by the value of the Add-on (e.g. 143 + 99 = 242 and so on, in above).

The resultant Rectangle is therefore increased by 3 times the Add-on x 2 times the Add-on.

Note it is not essential for the Add-on Element to touch a corner Element as it does above.

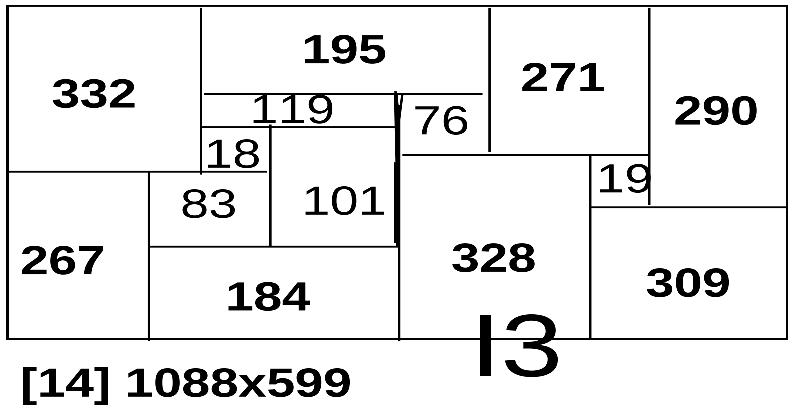
The amount of new solutions obtained by this method is poor as most S2323 solutions do not qualify.

E1.17. CREATING NEW SOLUTIONS S2424 TO S2424 INTERNAL- CODE I4

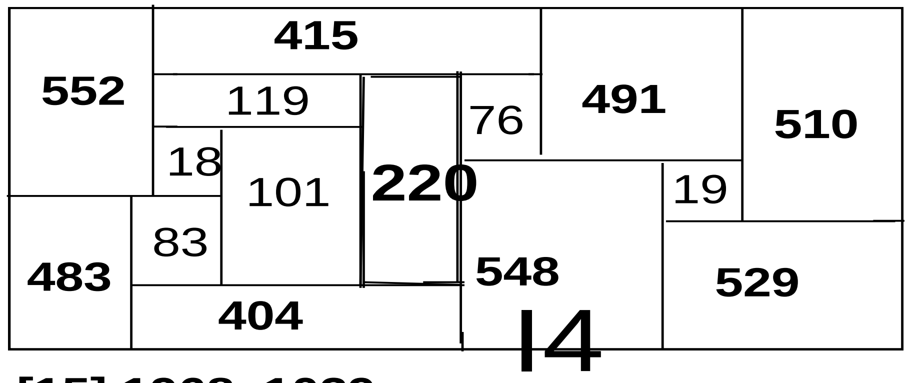
Really an extension of the last Add-on, the above is similar to the format in E1.5. except for a Triad ending. But if there are any amount of repeated Triads on one or both ends a new solution is possible.

The bold Elements show an increase of 220.

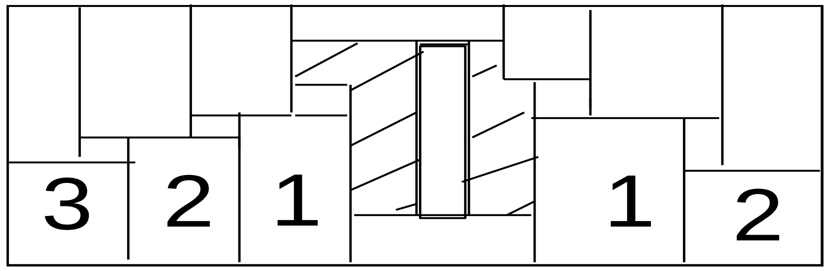
The Formula has not been worked.



[14] 1088x599



[15] 1968x1039

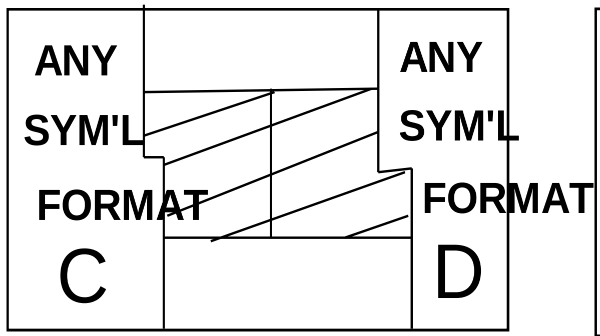


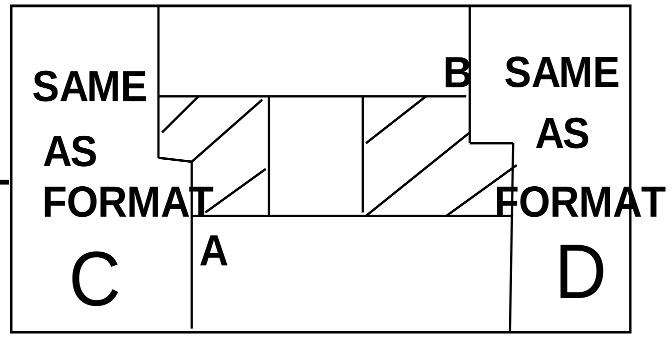
I0

THE PROPERY HOLDS REGARDLES
OF THE NUMBER OF TRIADS ON OI
OR BOTH SIDES

E1.18. GENERAL FORMAT FOR INTERNAL ADD-ONS - CODE I SERIES

The Diad endings or repeated Triad endings so far shown can be replaced by any Valid symmetric format e.g. a Pentad one side and an Octad the other. The adding of an Element as below alters all the values of Elements apart from the shaded areas, but the new solution one Order higher is always found to fit exactly. (E3.1 explains Pentad etc.).





THE ADDED
ELEMENT CAN BE
AS SHOWN, OR
IT MAY BE
TOUCHING LINE A
OR LINE B

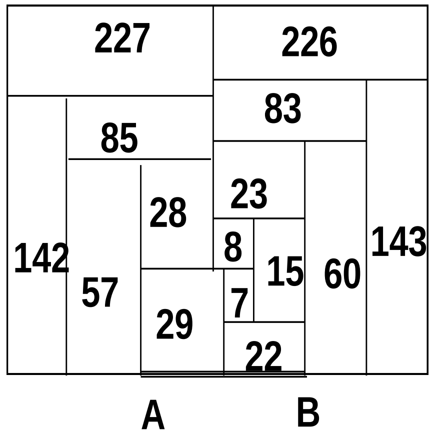
NOTE - C & D MAY BE EITHER THE SAME OR DIFFERENT PATTERNS

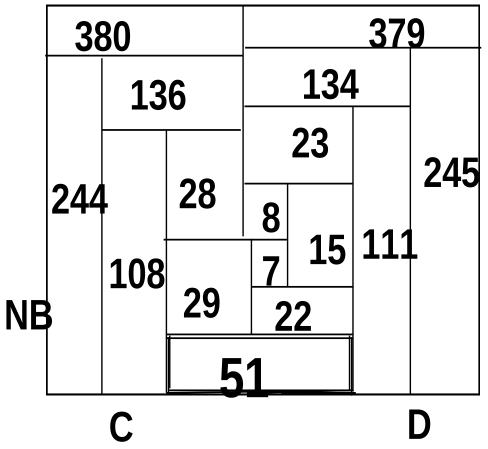
E1.19. CREATING NEW SOLUTIONS S2226+ to S222x - CODE NB

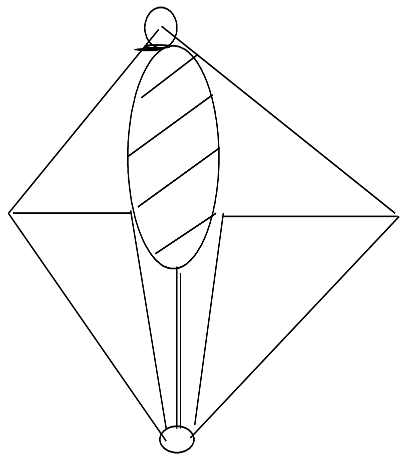
Below shows a pattern like E1.1 and in fact an Add-on of that type is possible here. This type is much less common and requires 6 or more Elements along one side and a ‘double arch’ construction as below 3. For S2228+ with a ‘triple arch’ and S222(10)+ with a ‘quadruple arch’, a further type of Add-on is possible.

There is no limit to this arch idea. In fact a chain series of Add-ons becomes possible - Below 2 gives another solution of D3.1. type with an Add-on at CD. From Below 1 can be found [16] 789 x 705, [16] 759 x 624 and [17] 1299 x 1164. The Formula is [o] m X n to [o + 1] 31m - 36n X 25m - 29n.

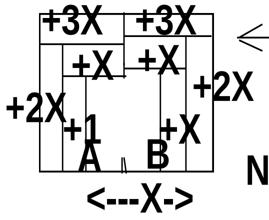
[15] 453 x 369 TO [16] 759 x 624







SMITH DIAGRAM FOR SECOND DIAGRAM



THIS ADD-ON ONLY WORKS WHERE THE RECTANGLE HAS THIS 'DOUBLE ARCH' PATTERN AND WHERE THERE ARE 2+ ELEMENTS IN A TO B.

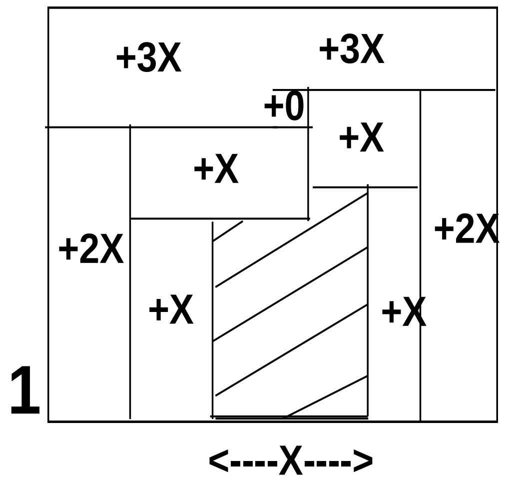
NOTE: TRIPLE ARCH PATTERNS++... GIVE ADD-ONS TOO.

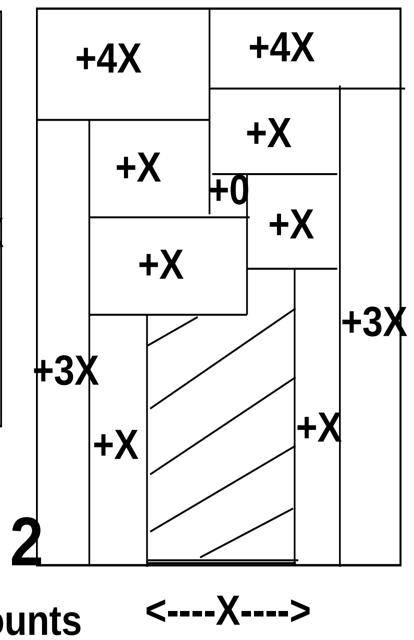
E1.20. OTHER RELATED ADD-ONS, NOT YET CODED

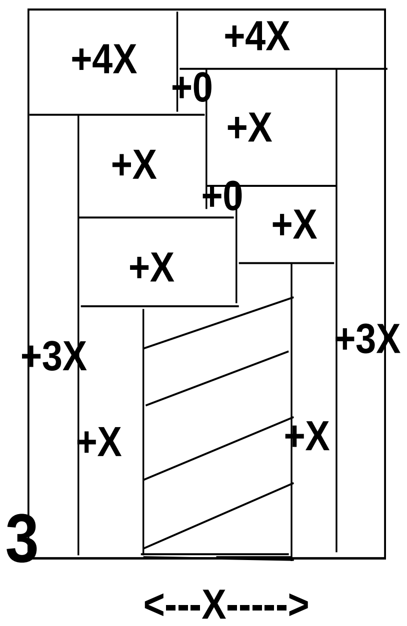
As well as the section E1.6. there are several other forms to be looked at, although they occur much less often.

Look below: if the added Element is size x then the effect on surrounding Elements is shown. e.g. If x was 5 then "+3x" means that the Element would increase by 15. Below 1 is as E1.6. with an additional Element.

All contain a double arch. All can be converted to another solution by the Add-on in E1.1. Below 2 and 3 only vary in the linking Elements (marked +0). Obviously these can be extended to 4, 5, 6 rows… each with several combinations of linking Elements



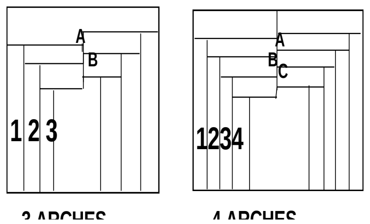




In Adding an element X at lines shown the Elements around increase by the amounts indicated. eg. +X means element increases by the amount of X. Each pattern can therefore be checked that the Add-On is valid eg in 3 the rectangle increases by 8X along & by 7X down. Note the +0 elements retain the same values

E1.21. WITH THREE OR MORE ARCHES

Another series exists in that the amount of arches can be increased to 3 and then 4 and so on. There is no need to give for all these and their validity can be proved easily as the pattern above .



ELEMENTS CAN BE ADDED HERE TO PRODUCE MORE SOLUTIONS. THERE ARE MANY MORE PATTERNS WITH EXTRA ELEMENTS AT A AND/OR B AND/OR C ALSO WITH 5 ARCHES & MORE

E2. ADD-ONS WITH 2 ELEMENTS

E2.1. CREATION OF NEW SOLUTIONS BY ADDING 2 ELEMENTS - MID AND MID-PLUS CODE o*

* Means the Code refers to a Symmetric Add-on, which is also true for Asymmetric Solutions.

It also means a separate Code has not been allocated for Asymmetric Solutions.

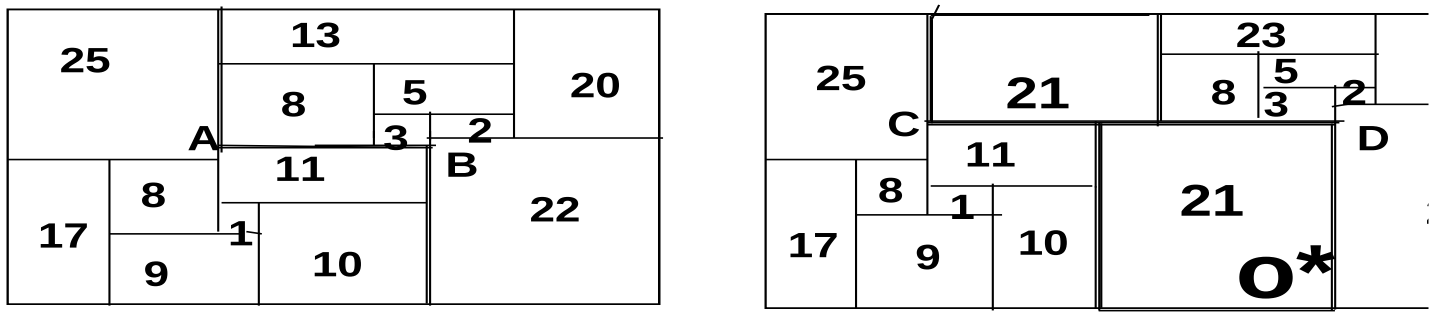
Whenever a Centre-line occurs in a solution 2 Add-ons can be added as below.

Such solutions are found by chance rather than design, and are obviously always imperfect.

It is the only Add-on where no Element increases in size! Many resultant solutions prove to be Compound.

Not surprisingly most examples found are symmetric. (See E2.3).

AB & CD ARE CENTRELINES - MIDWAY BETWEEN TOP & BOTTO



[14] 58 x 42 MID SOLUTION TO [16] 79 x 42 MIDPLUS

EACH ASYMMETRIC, BUT MOST EXAMPLES OF THIS SYMMETRIC.

FORMULA - [o] m X n to [o+2] m + n/2 X n

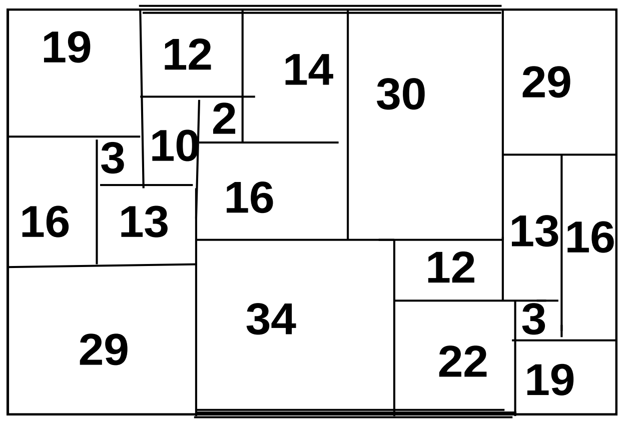
E2.2. ADD-ON BUT ASYMMETRIC - CODE q*

Below is an example of this unusual type.

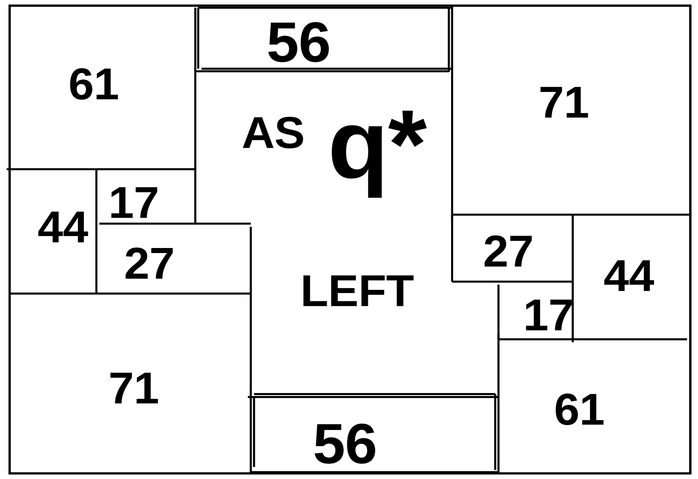
This happens to have a Pentad at each end (both 19, 16, 3, 13 ,29), but any Rectangle with a similar end type and containing the same Elements can be converted in this way by adding two Elements and increasing the Order by 2..

Note that with different Pentad endings the Add-on does not work. The Formula is [o] m X n to [o + 2] 3m - 2n X 2m - n.

ASYMMETRICAL



[19] 109 x 64 TO [21] 188 x 176

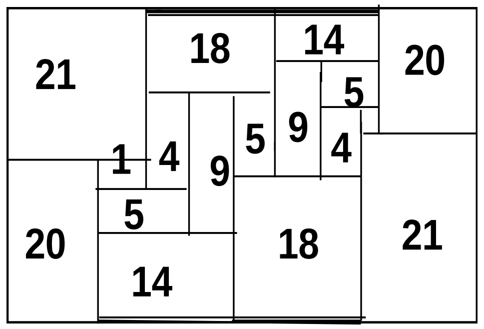


56 & 56 ADDED.

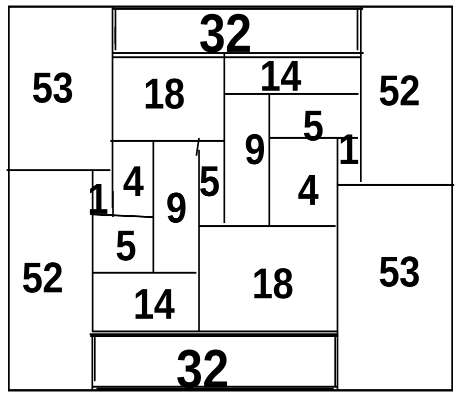
E2.3. CREATING NEW SYMMETRIC SOLUTIONS - ADDING 2 ELEMENTS - CODE q

Below is an example being a more common version of E2.2. Like many Add-ons the internal Elements re remain as before and all outer ones increase, in this case, by 32. The Order increases by two, and the Formula has been mentioned.

SYMMETRIC ADD-ON



[17] 73 x 41 SYM. TO [19] 137 x 105



32 & 32

ADDED.

q

E2.4. CREATING NEW INVALID SOLUTIONS BY ADDING 2 ELEMENTS

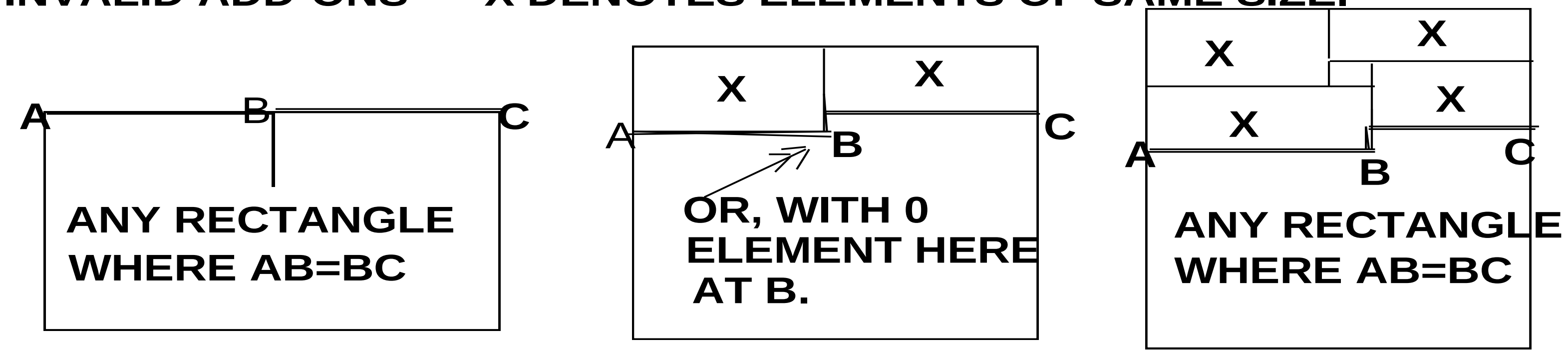
This type of Add-on is mentioned to complete the list but is of no practical use.

Whatever the status of the original Rectangle, the resultant one is either a useless Zero or Non-zero Rectangle.

A further zero Element is optional at B and Triads could be added ad-infinitum with zeros in between, but unexciting! All are Compound in reality.

The Formula is [o] m X n to [o + 2] or [o + 3] m X m/2 + n (reversed) followed by m/2 + n X m, m + n X m, 1.5m + n X m and so on.

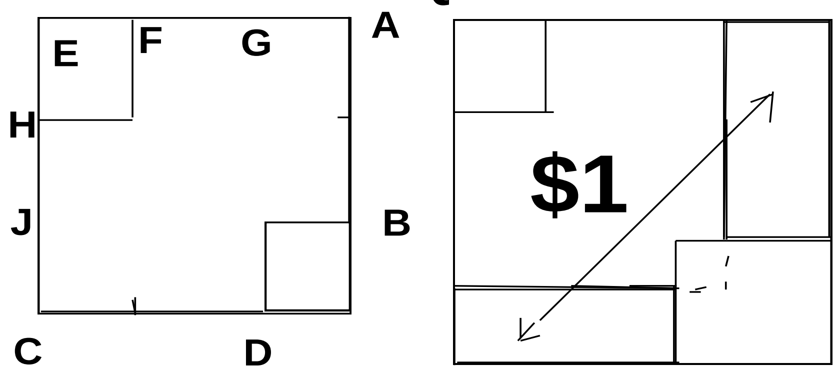
INVALID ADD-ONS   X DENOTES ELEMENTS OF SAME SIZE.



E2.5. ADD-ON FOR ALL SQUARES - CODE S1

See Below. The Order for the new solution increases by two. See also Squared-squares section for more.

ADD-ON FOR SQUARES



AN IMPERFECT SQUARE IS PRODUCED BY ADDING ELEMENTS AT AB & CD
THE ELEMENTS ARROWED HAVE THE SAME VALUED.
THE PROCESS CAN BE REPEATED WITH ELEMENT I AND AGAIN... FOR EVER & EVER!

The Formula for this is [o] m X n (m = n) to 2m - A X 2m - A where A is the value of the removed corner Element.

E2.6. ANOTHER SQUARES ADD-ON - CODE S2

Less useful than in E2.6. it can be done in squares of sides S3333 or greater by adding two Elements at lines FG and HJ Above and suitably extending the bordering Elements. See Squared-squares section.

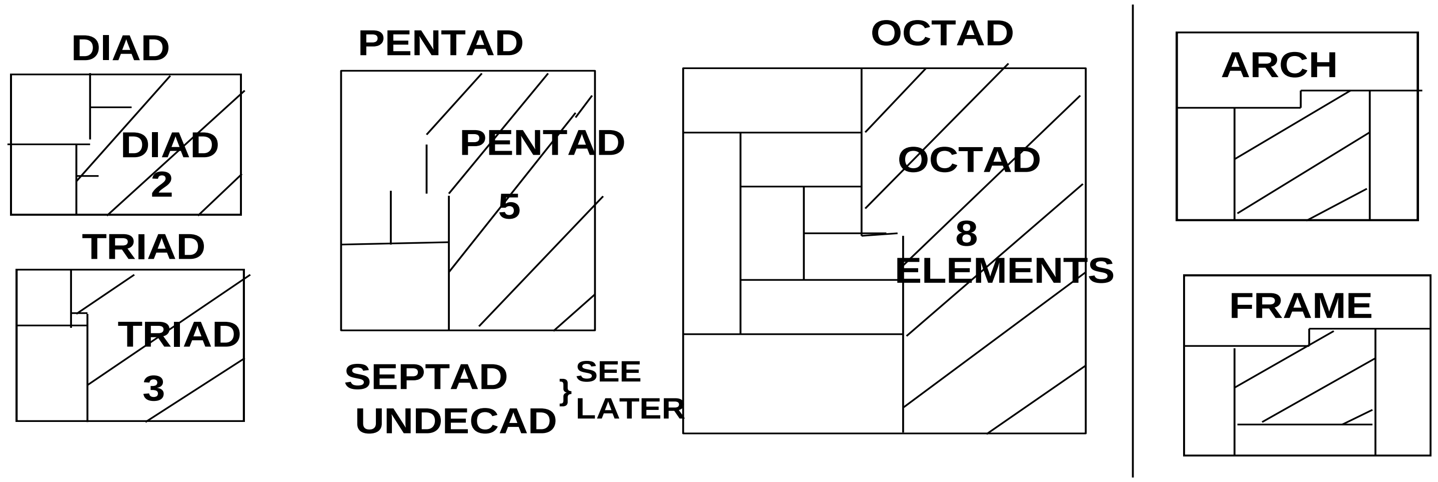
E3. USING ENDINGS - DEFINITIONS AND GENERAL

E3.1. DEFINING THE ENDINGS

In this and later section various endings are given names, some of which can be applied to two ends of a Rectangle or one, as required.

The term Arch is applied to Elements bordering three sides of a Rectangle, and Frame to Elements covering four sides as shown below. In addition to the single Elements is an “L” shaped piece called the body.

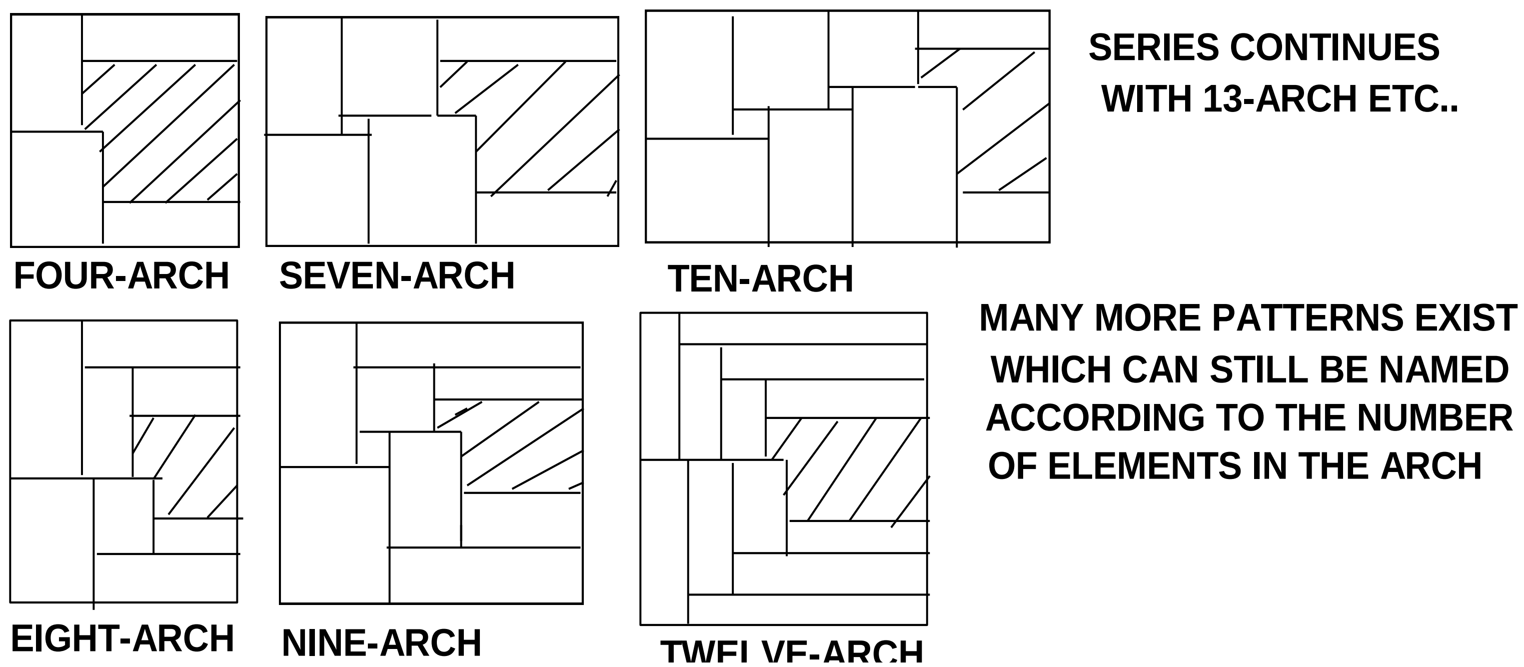
Thus a Pentad-arch consists of 7 Elements and a Diad-frame of 5 Elements, and so on.



Other endings called Septid, Undecad, Fourteenad, Octid, Octud, Nonid, Nonud and Nonod… are shown as and when required later.

E3.2. DEFINING THE ARCHES

Below some types of arches are shown.

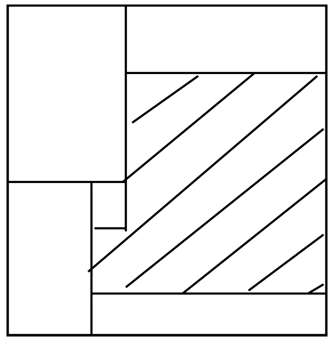


Note the linking Elements which are sometimes necessary (in above 2,3 and 5).

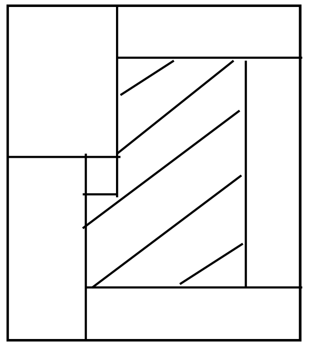
Although the patterns can be confusing, the numbers here are simply the amount of Elements.

E3.3. DEFINING THE FRAMES

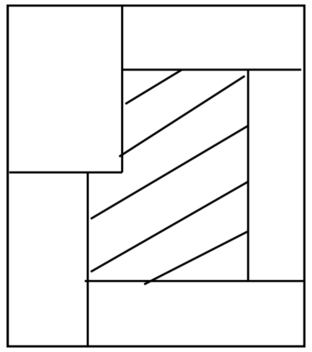
Variations of the above are now shown.



FOUR-ARCH+



FOUR-FRAME+



FOUR-FRAME

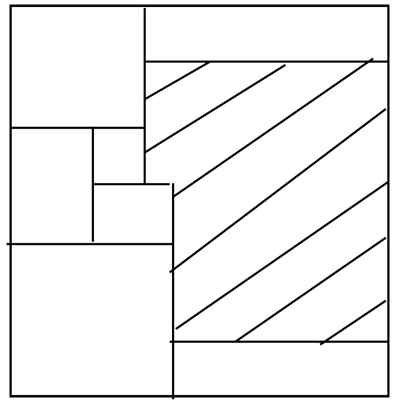
NOTE THAT THESE CAN BE APPLIED TO ALL OTHER ARCH & FRAME PATTERNS ALSO

"+" MEANS THAT AN ELEMENT IS INSERTED AT THE KINK OF THE SHADED BODY AREA.

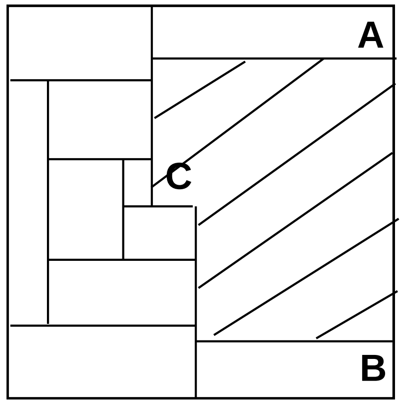
Note that the amount of Elements increases by one in “frame” cases, by one in “+” cases, and by two in “frame+” on the number shown.

E3.4. MORE DEFINITIONS

So far only Diad patterns have been shown. Here are some alternatives.



PENTAD-ARCH

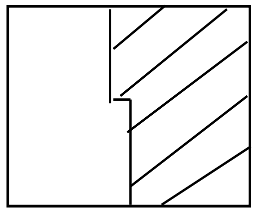


OCTAD-ARCH

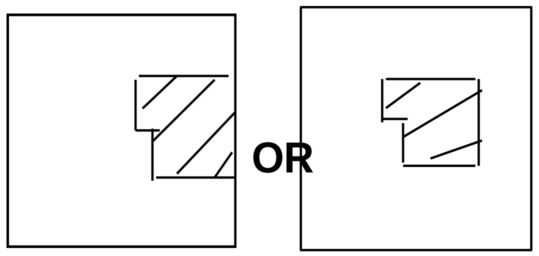
AND SO ON.. EACH HAS ONE ELEMENT . AT A & B. IF AN ELEMENT IS INSERTED AT C A "+" IS ADDED TO THE NAME. IF AN ELEMENT IS INSERTED BETWEEN A & B, THE NAME CHANGES TO "FRAME"

This section reveals important and useful means of creating new solutions from old.
Some are also helpful for the purpose of discovering very reduced solutions - see that section also.

E3.5. FEATURES GENERAL TO SECTIONS E4 TO E12



SYMMETRY PATTERN



ARCH PATTERN

1. Effect on Unknowns xy, xyz...

Each pattern has a symmetric pattern attached to a body, or an arch-type pattern enclosing the same body.

Whether the body is xy or xyz or xyza… as the arch-type pattern can always be constructed directly from it, the arch-type solution must have the same Unknowns as the original body. But with a symmetric pattern attached to a body there is no such fixed rule, although the symmetric part only is known to be xy, or xyz or whatever.

2. One-way and Two-way pairs

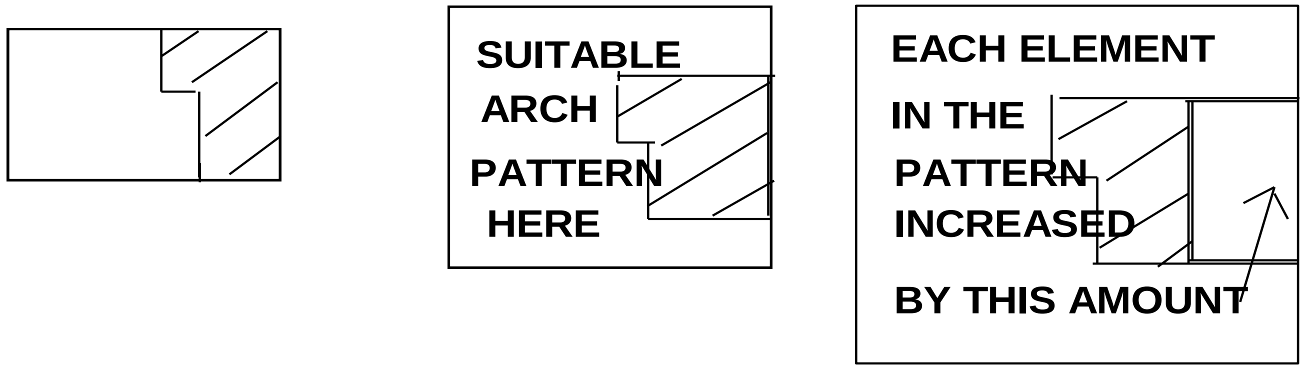
It will later be seen that any symmetry pattern is easily changed into an arch type pattern. But with some patterns (E10 onwards) it is impossible to identify the symmetry pattern solution from an arch-type solution.

As is seen later several symmetry patterns can share the same arch pattern connection. The pairs of such solutions are therefore one-way - from pattern to arch only. This is a pity since the arch solution often has a lower Order.

The patterns in E4 to E9 are two-way patterns which means that whatever solution is known its pairing solution can always be readily found.

3. Adding a further Element to the body in the arch-type pattern

If a matching pair is known, it is always possible to calculate a further solution by adding an Element to the body in the arch-type solution and recalculating the arch. Below makes this clearer -



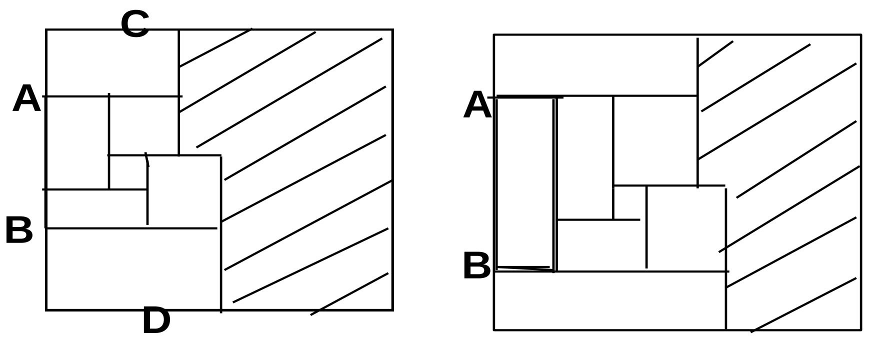
4. Adding a further Element to some symmetry patterns

Some patterns such as of the type chosen below can be substituted for another by adding an Element as shown.

<u>Whatever the body pattern, the same happens to fit in the new solution,</u> but often all body Elements have to be up-rated by a certain amount.

Since all Elements in below 2 change in size and have to be recalculated to fit, the calculation can be tricky.

(N.B. the connecting arch-type pattern is irrelevant in this substitution)



EXAMPLE OF A PATTERN WHICH CAN
BE SUBSTITUTED FOR ANOTHER
THERE MUST BE AT LEAST 4 ELEMENTS
BETWEEN C&D AND 2 BETWEEN A&B
THE TOP & BOTTOM ELEMENTS BECOME
BIGGER IN THE SECOND DIAGRAM.

5. Sides divisible by certain numbers

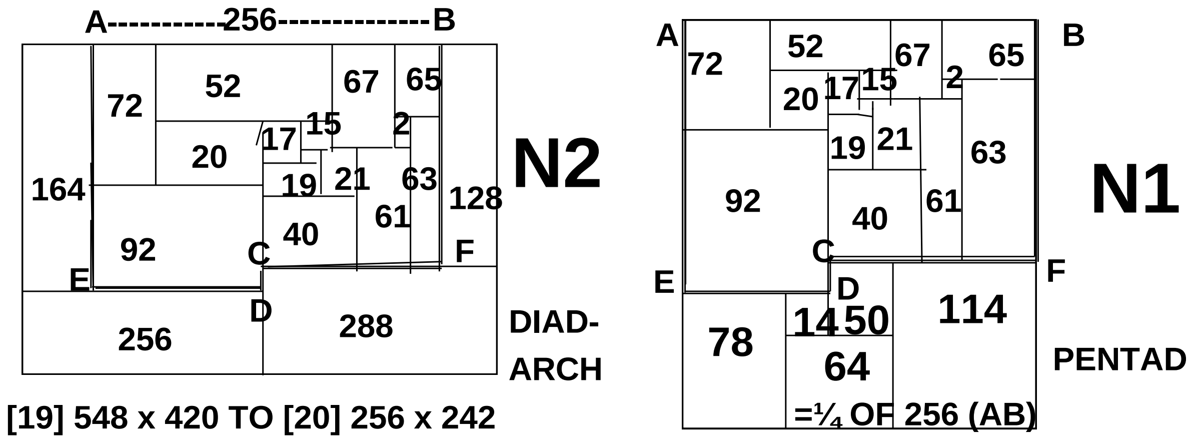
The vertical side above happens to always be divisible by 3 and this may be confirmed by algebra.

Every pattern has a particular divisor amount e.g. 4, 5, 6, 11, 19, 40 etc. and reference is made to it as the patterns are described. These numbers dictate as to whether multiplying up all Elements by a constant is necessary, and if so, by how much.
E.g. If the old Rectangle has a side 45 and the new has to be divisible by 15 then no multiplying up is necessary.
If 48 then multiplying by 5 is necessary. If 50 then multiplying by 3 is necessary for the new solution to be divisible by 15.

E4. ENDINGS - 1ST SERIES

E4.1. DIAD-ARCH TO PENTAD ENDING - CODE N2 TO N1

A curious relationship between 222# and Pentad solutions was discovered by me. If the four-arch of 4 Elements is removed and replaced by a Pentad another solution arises with similar Elements. Below a solution has been chosen where
AB is divisible by 4, and when this is so a higher Order solution of smaller size is obtained:-

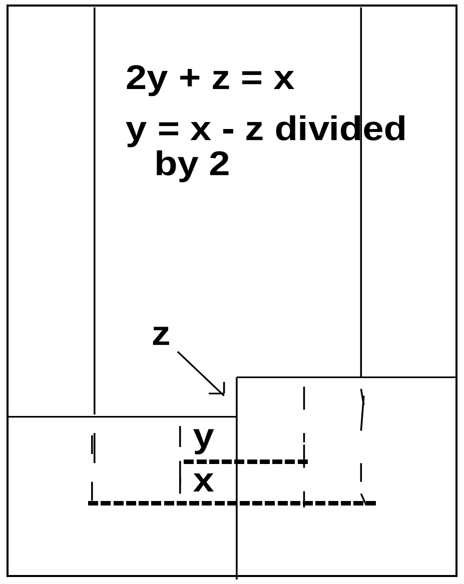


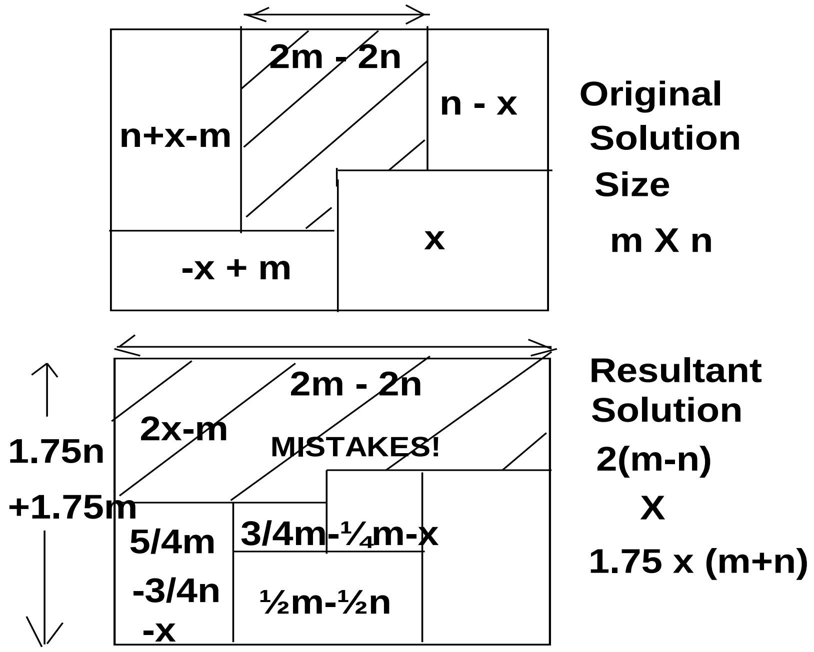
CD is 36 in each case. AB = 256 a ¼ of which is 64. To obtain the Element 14 in above 2 take half of 64 - 36.

The other parts of the Pentad are easily calculated, and found to fit. Now where AB is divisible by 2 but not by 4 it is necessary for all values in the original to be doubled. If AB is not divisible by 2 then quadruple everything.
This principle also applies to S2223 solutions.

E4.1.1. PROOF OF DIAD-ARCH TO PENTAD LINK

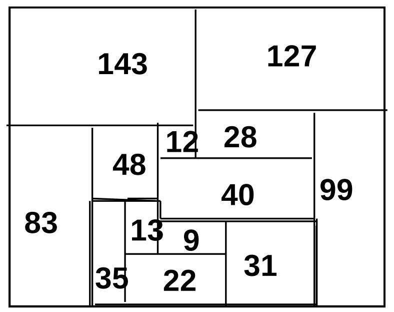
For the proof follow the algebra in below 1 2 and 3 where the patterns agree. From the initial dimensions of m x n to 2(m - n) x ¼(m + n). As ¼(m + n) is now the larger size and bearing in mind that some solutions will need multiplying up by four and others by two, the Formula is found to be [o] m X n to [o + 1] 8m - 8n X 9n - 7m requiring division by 1 or 2 or 4 according to the Solution.



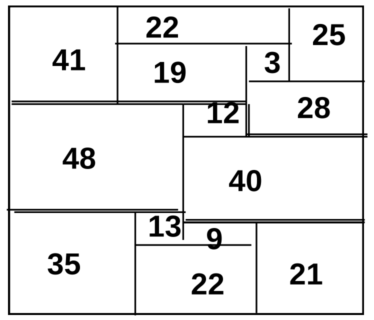


E4.1.2. TWIN SOLUTIONS SOMETIMES POSSIBLE FROM PENTAD LINK

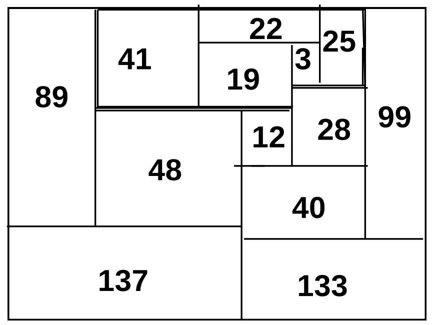
In S2225 solutions where there happens to be a Pentad pattern between the Arch an interesting Trio of Solutions arise, e.g.



[13] 270 x 226 (2)



[14] 124 x 88 imperfect



[13] 270 x 226 (1)

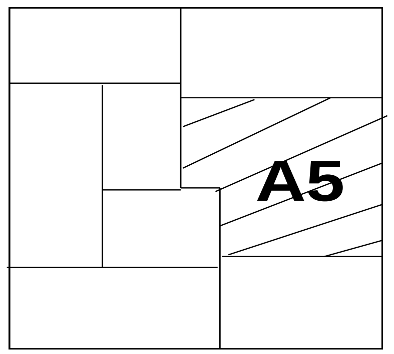
In above 1 & 2 the normal Diad Arch to Pentad relationship is shown. In Above 2 the lower Pentad is considered and changed back into another and twin Diad Arch!

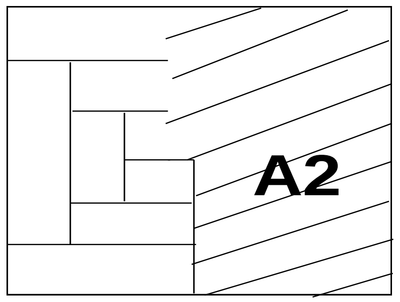
E4.2. PENTAD-ARCH TO OCTAD - A5 TO A2

See Below. The 7 Elements in the first become 8 in the second, and the Order increases by one.

The depth of the first is divisible by 4 and the second divisible by 15.

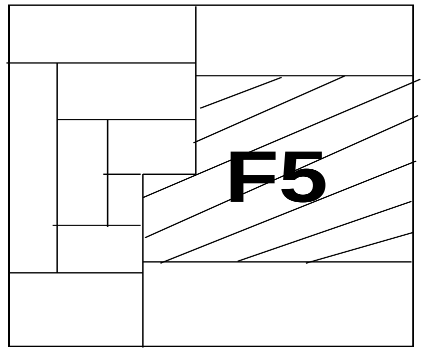
The Formula subject to check is [o] m X n to [o + 1] 16 / 15m +46 / 15 n X -8m + 7n ÷ by 1, 2 or 4.

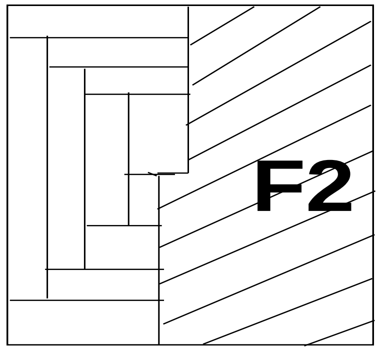




E4.3. OCTAD-ARCH TO UNDECAD - CODE F5 TO F2

Shown below, the 10 Elements in the first become 11 in the second, an increase of one in the Order.





E4.4. CONTINUING THE SERIES

The series continues

E5. ENDINGS - 2ND SERIES

E5.1. TRIAD TO PENTAD-ARCH RELATIONSHIP - CODE A1 TO A5

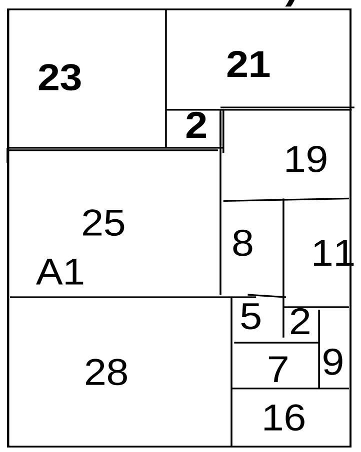
An interesting relationship is shown below. Note the following -

1. For the Pentad to fit, the Elements often need to be multiplied by 2 or 4 within the black area in Below 2 and 3.
2. In Below 2 3 Elements have gone and replaced with 7 so Order up by 4.
3. In Below 3 three Elements go and are replaced by eight so Order up by 5.
4. The status between below 2 and 3 is the same.
5. As so many solutions contain Triads it seems the amount of solutions of the type Below must be great, but as the Order rises by 4 or 5 the amount of solutions below Order 21 are quite small. Also as the dimensions of the resulting solutions are between 2 and 9 times larger, this restricts the amount of very reduced solutions found.

Obviously the bottom corner Elements increase by 44 to 97 and 99. This makes the horizontal side 240, ¼ of which is 60. Add the “kink” at AB of 2 to make 62, and divide by 2 to get 31. From this the Pentad is calculated.

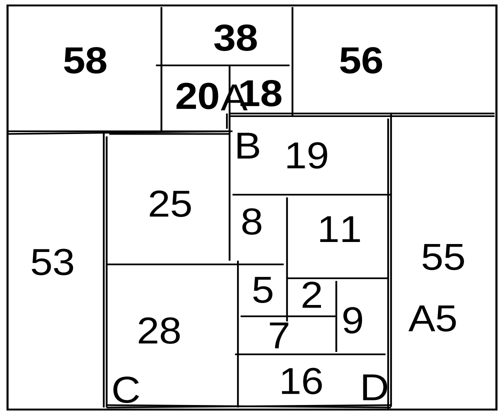
Where halves or quarters arise suitably up-rate the Elements by 2 or 4. This happens where the Element at CD is not divisible by four. The resultant solution has the same status, i.e. Perfect to Perfect, Imperfect to Imperfect etc.

(See Section D4)

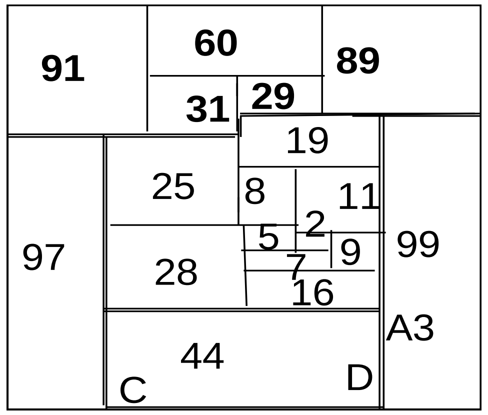


[13] 76x44 ¬

PENTAD-ARCH

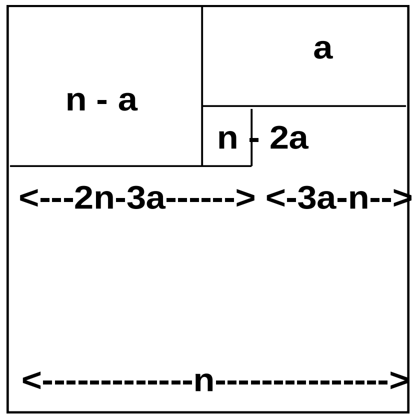


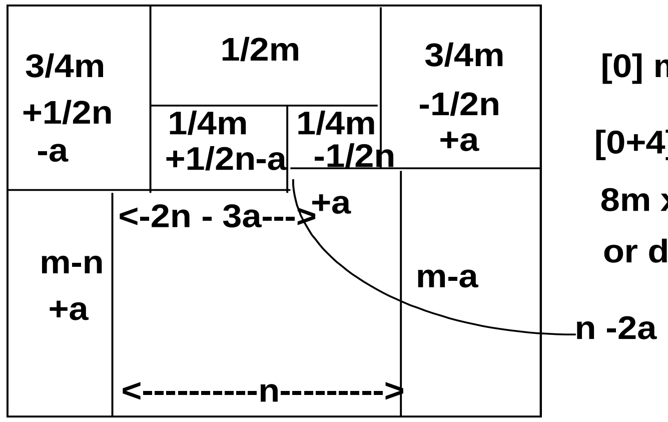
[17] 152X111 PERFECT



[18] 240X188 With Add-on

The proof of Pentad-arch relationship is below. Below 2 assumes m is divisible which may be untrue, 4 so the dimensions have to be multiplied by 4 so that the result is [0 + 4] 8m x 7m - 2n which sometimes reduces to half or one-quarter of this.





[0] m x n TO

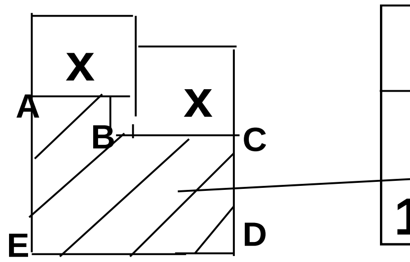
[0+4]

8m x 7m-2n

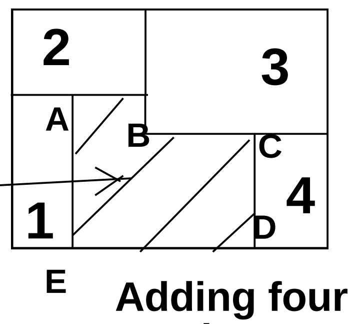
or divided by 2 or 4

E5.1.1. DIAD+ OR TRIAD TO PENTAD-ARCH 2ND GROUP

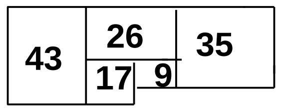
I have discovered that if a random series by trial and error is constructed and continued until two identical Elements are found, a Rectangle is found by removing the linking Element as below-

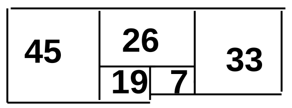


x Denotes two identical Elements



Adding four Elements as shown.



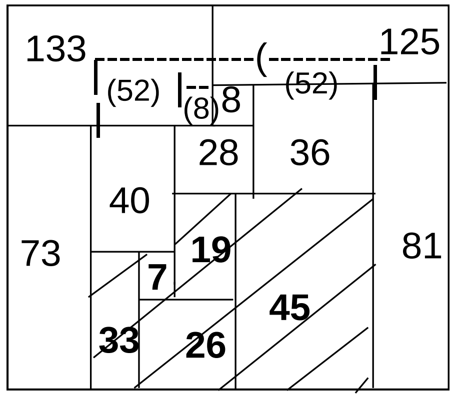


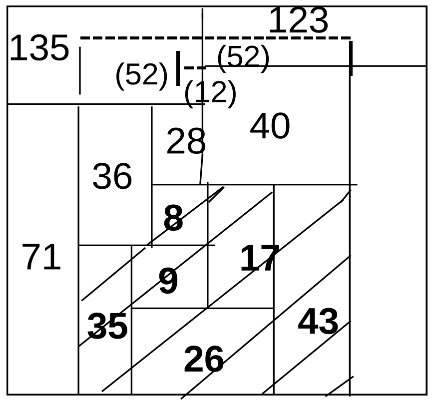
SEE BELOW 1 & 2

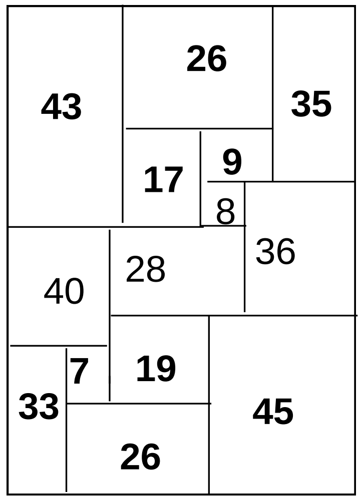
In running a special computer program to construct solutions of this type but containing Pentads, I found pairs turned up for every solution - but some pairs were duplications of the first. The reason is below. If Pentads above 3 & 4 are added to below 1 and 2 (with 4 Elements omitted) [14] 116 x 104 as Below 3 is obtained twice - once upside down.

Note the Elements 26 in below 3 being ¼ of 104. Double Pentad ended solutions must always be imperfect.

PENTAD PROPERTIES







[13] 258 x 206 (1)

[13] 258 x 206 (2)

BOLD PORTIONS ARE PENTADS

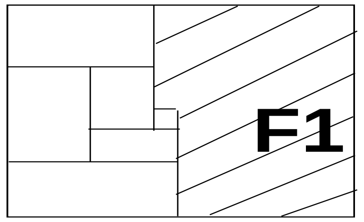
BRACKETED NUMBERS DO NOT SHOW

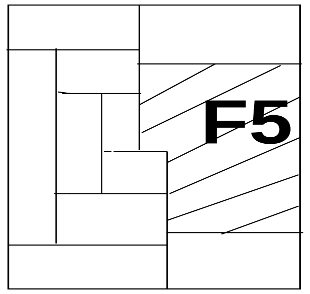
[14] 116 x 104

The Formulae are: 1st to 2nd [o] m X n to [o+4] 8m X 7m - 2n ÷ by 1, 2 or 4.
1st - 3rd [o] m X n to [o + 5] 8m + 8n X 5m - 7n ÷ by 1, 2 or 4.

E5.2. PENTAD+ TO OCTAD-ARCH - CODE F1 TO F5

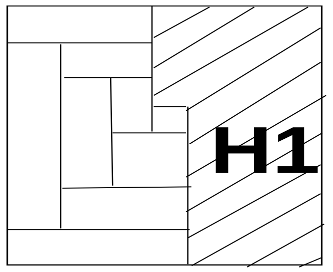
This relationship is shown below-
The depth of the first solution must be divisible by 4 and the second by 15.
The Formula is [o] m X n to 26m - 9.75n X 30m - 11.25n.

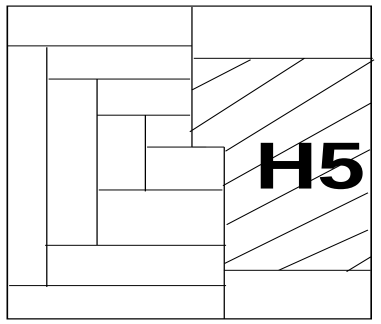




E5.3. OCTAD+ to UNDECAD-ARCH - CODE H1 TO H5

Shown below. Formula [o] m X n to [o + 4] ...
The depth of the first solution must be divisible by 15 and the second by 56.





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E5.4. CONTINUATION OF THE SERIES

It may be shown by algebra that the series Diad+ to Pentad-arch, Pentad+ to Octad-arch and Octad+ to Undecad-arch continues for ever. The series continues with the relationships Undecad+ to 14-arch, 14+ to 17-arch, 17+ to 20- arch, and so on..

E6. ENDINGS - 3RD SERIES

E6.1. RESULTS OF LAST TWO SECTIONS

The following were found to be linked -

E4.1. Diad-arch to Pentad
E4.2. Pentad-arch to Octad
E4.3. Octad-arch to Undecad
E4.4. And so on...
E5.1. Diad+ to pentad-arch
E5.2. Pentad+ to Octad-arch
E5.3. Octad+ to Undecad-arch
E5.4. And so on...

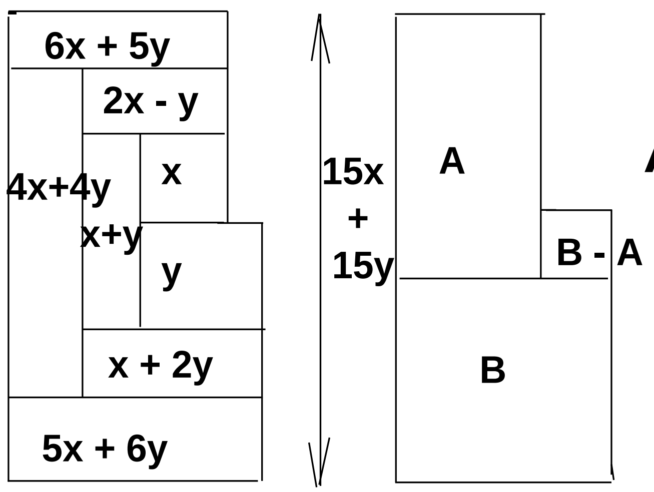
Now as Pentad-arch is linked to Octad (E4,2) and Diad+ (E5.1) this means that Diad+ must be linked to Octad.
Also we observe that Pentad+ and Undecad must also be linked.
In fact there is another whole series which forms Section E6!

E6.2. DIAD+ OR TRIAD TO OCTAD RELATIONSHIP - CODE A1 TO A2

We shall now see that other Triad solutions, not of ENZ type, can be converted into Octad endings.

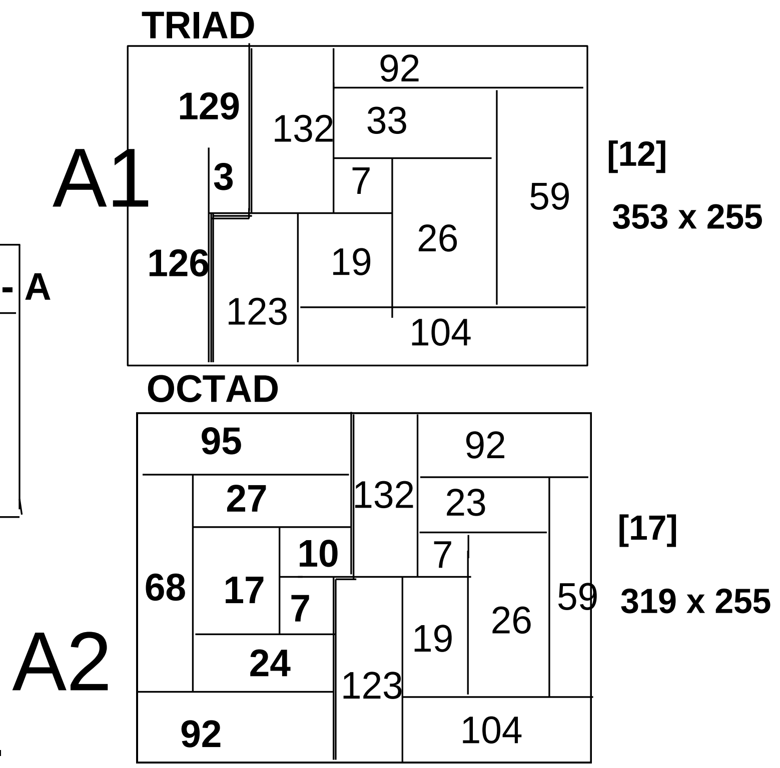
Consider Below. What conditions are necessary with A and B for an Octad to work? Obviously y - x will equate with B - A. Also, A + B will equate with 15x + 15y. A + B / 15 = x + y. -A + B =-x + y. adding these it is found 8B - 7A = 15y.
It is clear that A + B has to be divisible by 15. So I took known examples where the Triad side is divisible by 15 and sure enough Octads could be added successfully every time, and the proof is shown below.

So an Octad can be substituted for any Triad once A + B is made divisible by 15. Calculating the Octad solution is done using 8B - 7A = 15y. Taking Below chosen as 225 is conveniently divisible by 15. B = 104 A = 121 8B - 7A =-15 i.e. 15 x -1 225 / 15 = 15. Now 16+ - 1 is 15, and the Octad can be calculated.



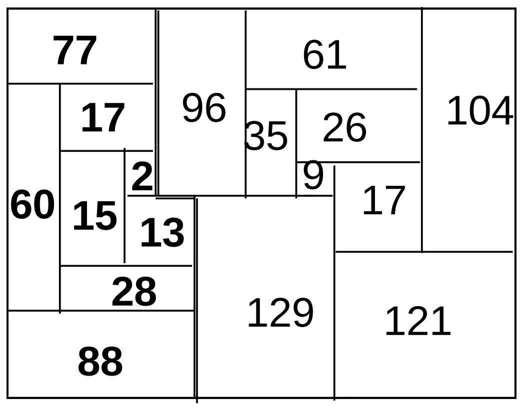
OCTAD END TRIAD END

Formula [o] m X n to
[o + 5] 15m - 2n X 15n
divided by 1 3 5 or 15 as needed.



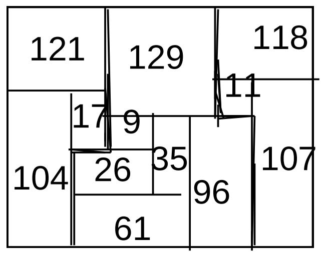
E6.2.1. TWINS FROM TRIAD TO OCTAD RELATIONSHIP

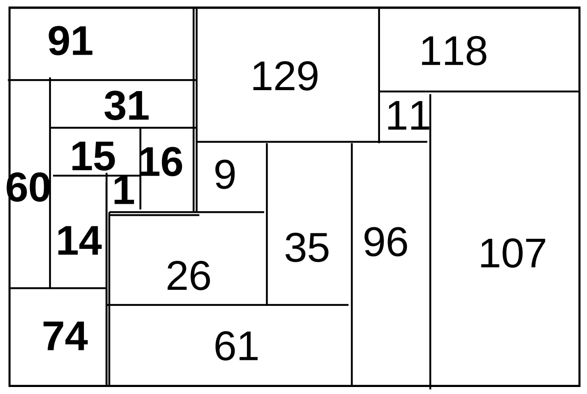
See Below 2. An Octad has been added to the right hand and the left.
The results are twins [17] 338 x 225 in below 1 & 3! The Octad addition is a means of creating reduced size Rectangles where the Triad end is divisible by 15. The Octad solution Order is up 5, but the resulting solution actually smaller.
The status of the original and resulting solutions can vary, Perfect to Imperfect, Imperfect to Invalid etc.
In the case Below Octads can be added to both ends at once but the result always imperfect for reasons already given.



[17] 338 x 225 (1)

[12] 368 x 223





[17] 338 x 225 (2)

E6.2.2. PENTAD+ AND UNDECAD RELATIONSHIP - CODE F1 TO F2

Above is an interesting substitution proved by algebra since each contains the lengths y - x, 56x + 56y, 30x + 26y and 26x + 30y. Note the different though small change from AB to CD which means that twins are not produced.
Since AY must be divisible by 56 and CZ by 15 as it is a Pentad end, it means the resulting solution must be a multiple of 840 or 56 x 15.

Thus the amount of solutions with highly reduced size is limited. Though interesting, it is not a huge source of solutions.
The Order increases by 5 being 11 minus 6 Elements.

E6.2.3. EXAMPLE OF PENTAD+ TO UNDECAD SUBSTITUTION

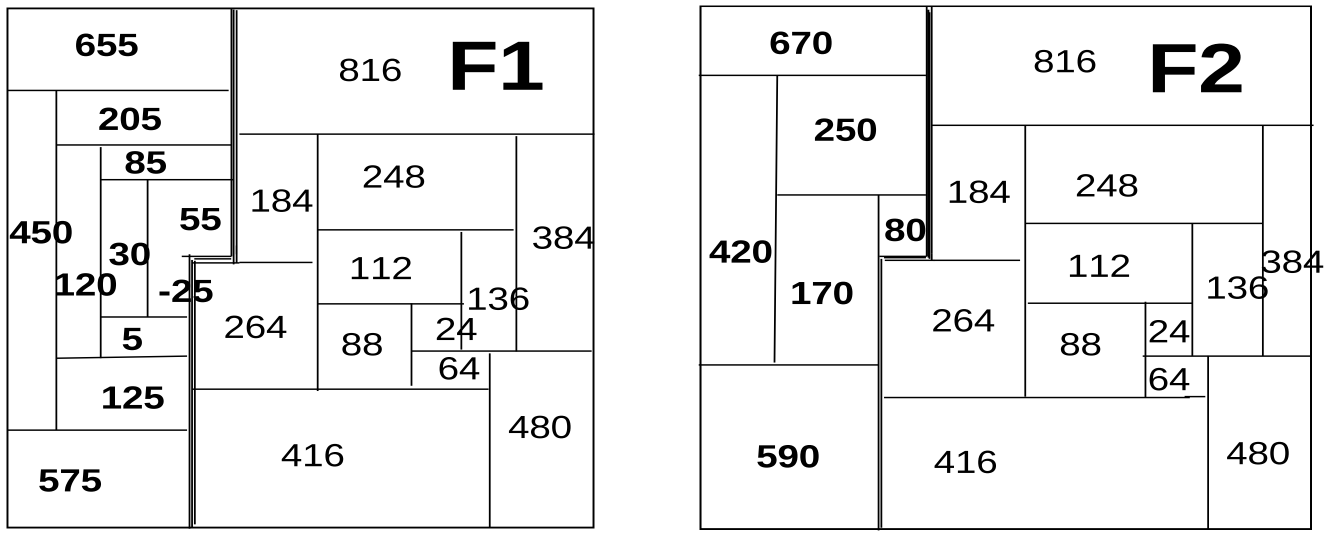
Below 2 (already known) was converted to below 1 by changing to an Undecad end.

The resulting solution was calculated from values provided by the fact that y - x had to equal 80 (264 - 184) and that 56x + 56y had to be 1680. This provided the values of x and y.

In this case y proved to be negative (-25) but for convenience the format has not been adjusted purposely.

NB the ¬ sign in this book indicates that the solutions need to be upended to read the dimensions shown.

[23] 1680 x 1471 ¬ AND THE RELATED [18] 1680 x 1486 ¬



NOTE - THE SECOND RECTANGLE REDUCES TO [18] 840 x 743

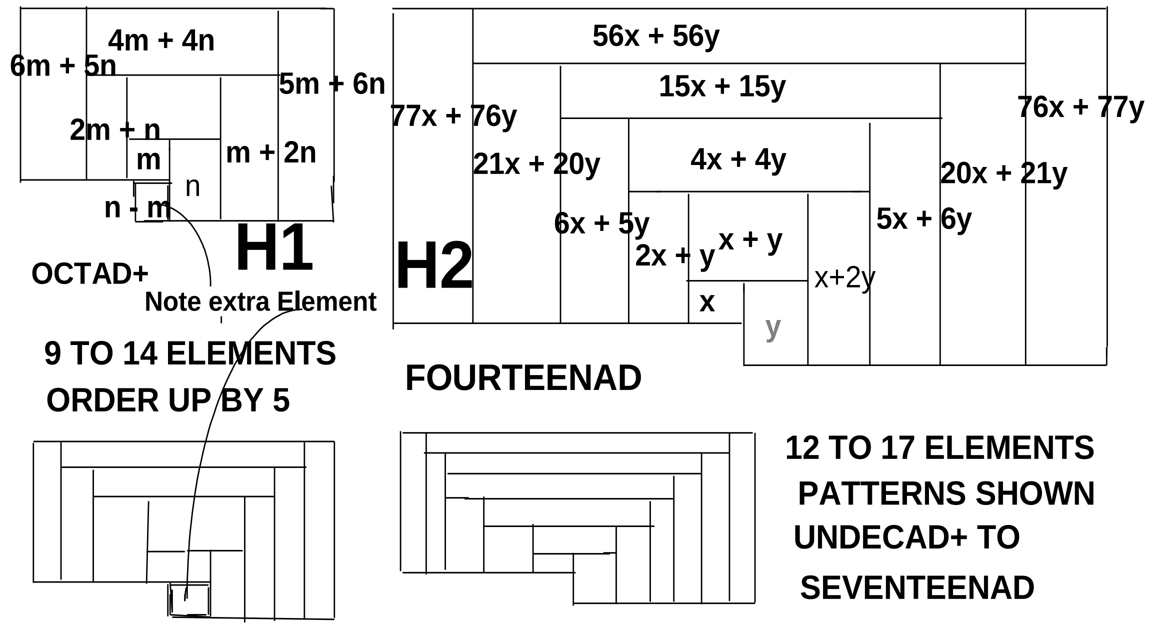
7

Formula: [o] m X n to [o+5] 56m x 56n - o.5m ÷ by 56, 28, 14, 8, 7, 4, 2 or 1

E6.3. OCTAD+ TO FOURTEENAD SUBSTITUTION - CODE H1 TO H2

So far there has been a Triad to Octad relationship and a Pentad+ to Undecad one where the substitution is 3 to 8 and 6 to 11 Elements. So what about 9 to 14? Formula not found. Yes, it does work!

The algebra is shown in the diagram. It is seen that the resultant depth of the solution is 209(x + y) and as this is divisible by 209 there are no solutions with a depth of 1000 or less.



E6.4. CONTINUING THE SUBSTITUTION SERIES

Above 3 and 4 shows patterns for 12 to 17 Elements.
This too is a Valid substitution, which may be made.

Note that an extra Element is essential in the original otherwise the substitution does not work.
The series is thus -
3 to 8 6 to 11 9 to 14 12 to 17 15 to 20 18 to 23.... The connecting Element in the original is essential and the Order increases by 5 in every instance.

To jump to the next in the series merely add “|__|” to each pattern.

E7. ENDINGS - 4TH SERIES

E7.1. SEPTID TO SEVEN-ARCH RELATIONSHIP - CODE P3 TO P1

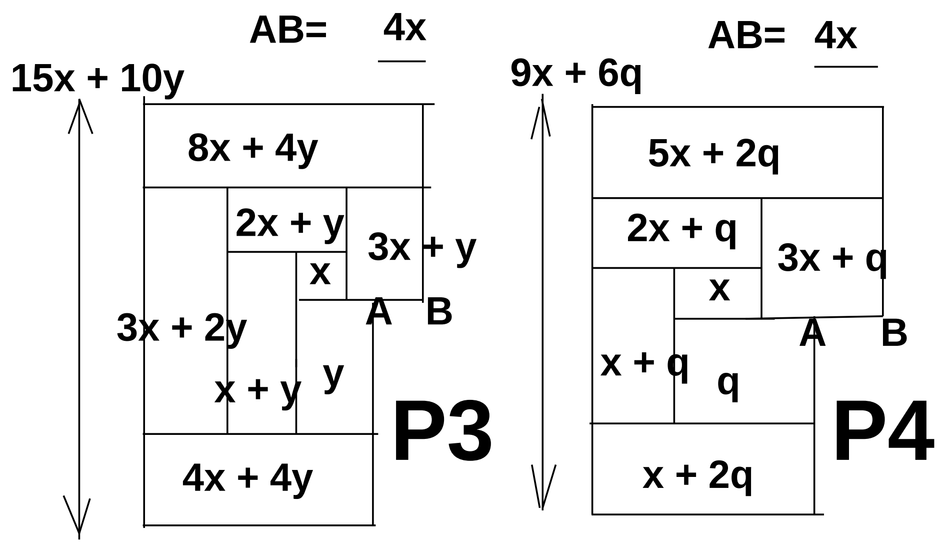
Below is a substitution where the Order is unchanged at 16.
This is a pattern where if an Element is added at AB (to become an Octid, see below) then a further solution can be drawn. Septids and Octids are interchangeable, although re- calculation is necessary.

E7.1.1. SEPTID AND OCTID RELATIONSHIP - CODE P3 TO P4

Below 2 is interesting as the right hand sides are identical, although the left hand side required recalculation.
Examining the algebra this relationship is found to be always true, but scaling up the numbers 5 times is sometimes necessary.

(NB. If 51 is multiplied by 5 it becomes 85 x 3 - the Add rule does apply here!). Above 2 is calculated as follows:-

The added Element (51) is always one-fifth of the depth (255) as 3x + 2y is one-fifth of 15x + 10y in below.



NEW RECTANGLE OLD RECTANGLE

The resultant Rectangle is one-third of the added Element longer i.e. 17 + 274 = 291. Also 95 + 17 = 112.

The Reduction index in the new solution is doubled (9 to 18). Note the same Element x and that AB = 4x in Above 1 and 2.

The new values need to be calculated for the new solution

E7.2. SEPTID TO OCTID RELATIONSHIP

Both the Add-ons Below are symmetric by pattern. The right hand sides are identical.

By examining the algebra of the left hand side in x and y it is found that a replacement is always possible, but multiplying by 5 times is sometimes necessary.

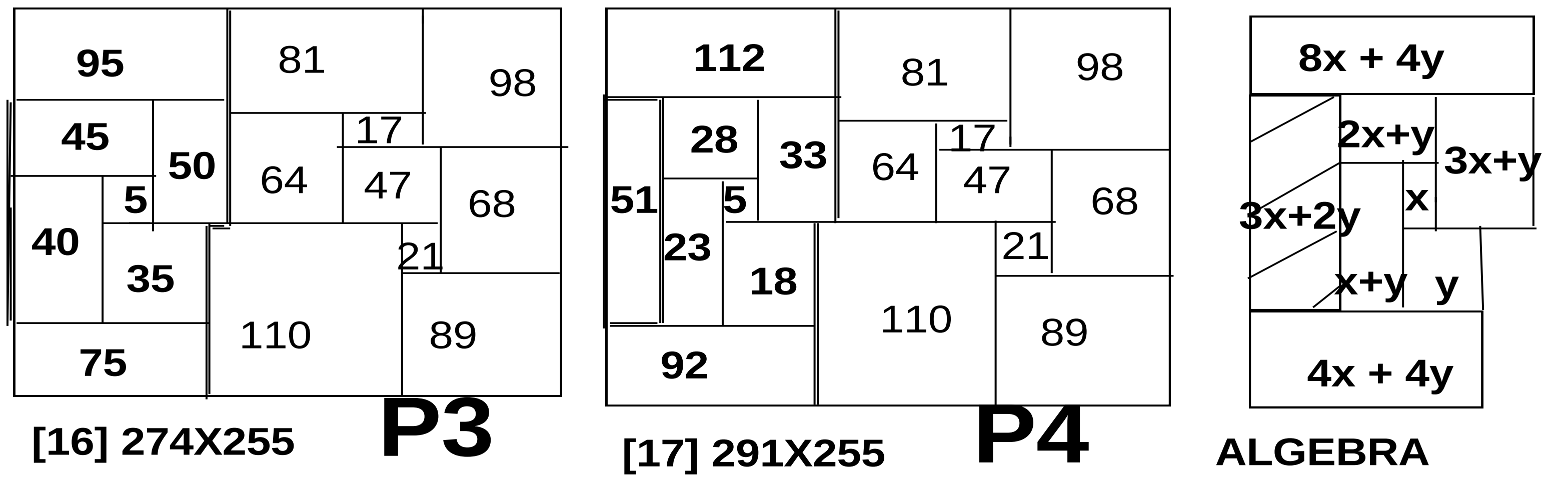
The size of the larger Order can be determined as follows.

The added Element is 51 is always one-fifth of the depth of 255, as 3x + 2y is exactly one-fifth 15x + 10y (Below 3). The resultant solution is one-third of this amount larger i.e. 95 becomes 95 + 17 = 112 or 274 x 255 becomes 295 x 255.

The Reduction Index in the second solution is always double - actually 9 in Below 1 and 18 in Below 2.

As the length DE is identical at C and D represent x and therefore must always be identical in this relationship.

SEPTID & OCTID ENDINGS RELATIONSHIP



[16] 274X255 [17] 291X255 ALGEBRA

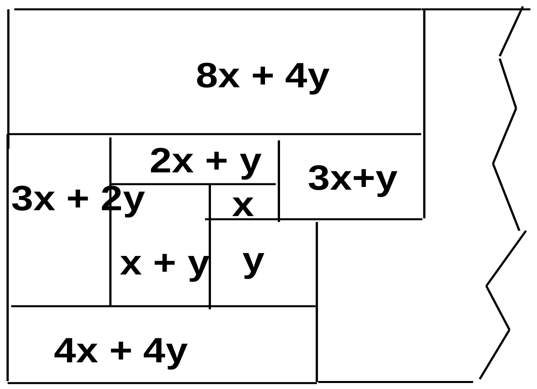
The Formula is [o] m X n to [o + 1] 15m + n X 15n.

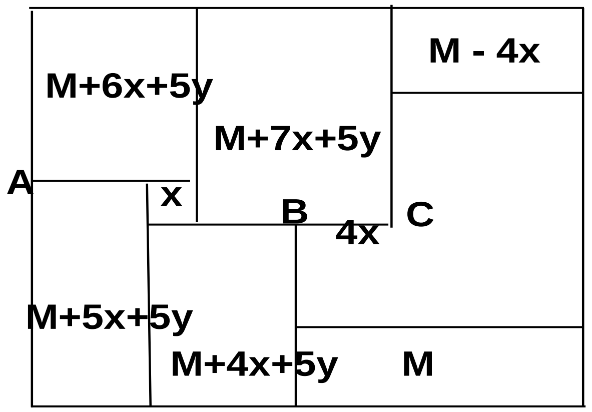
E7.3. OCTID TO SEVEN-ARCH RELATIONSHIP - CODE P4 TO P1

Compare the solutions below, noting the similarity in part with above 3.
The frame pattern in above 1 of 7 Elements (in black) and the left hand of above 2 are always found to be interchangeable! Note that 112 + 85 in above 1 is equal to 197 in above 2.
In Below 1 the depth of the Rectangle is 15x + 10y and thus clearly always divisible by 5. Thus AB in full dimensions is also divisible by 5.

E7.3.1. PROOF OF OCTID TO SEVEN-ARCH RELATIONSHIP

It is easy to show that this relationship always applies by using algebra -

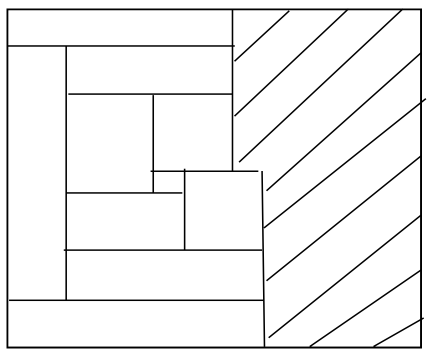


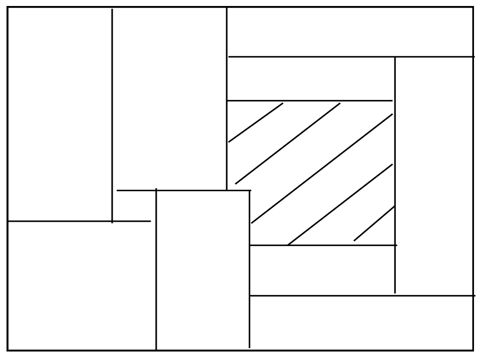


ALGEBRAIC PROOF
BC is x +3x+y - y ie 4x as at left. Calling the unknown distance M the remaining Elements are ca lculated. At A, M + 5x + 5y + x does agree with M+6x+5y and thus the design always fits.

E7.4. NEXT IN SERIES

Compared to the pattern in E7.1, Elements have been added at left, top and bottom in below 1.
Observe the three similar additions at right, top and bottom in below 2.
A solution of the first type can be transformed into one of the second.
With 10 Element shown in each there is no change in the Order.



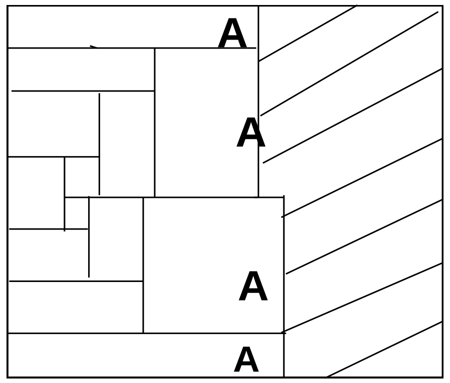


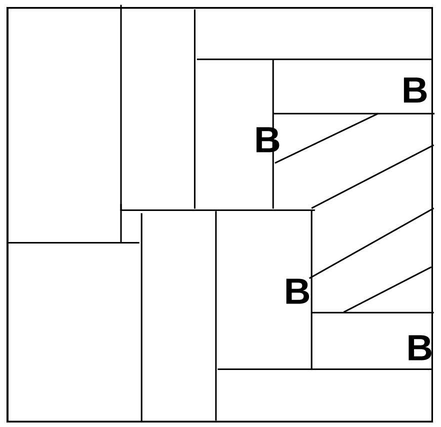
E7.5. CONTINUATION OF THIS SERIES

Without bothering to illustrate, Elements at left top bottom and right top bottom can be added respectively to each diagrams forever. These give 13 to 13 Elements, 16 to 16, 19 to 19 and so on. As before there is no change in the Order.

E7.6. ANOTHER SERIES

An alternative series which increases by 4's instead of 3's is below.
The Septid pattern has 4 Elements A added. The matching pattern is obtained by adding 4 Elements - shown by B.

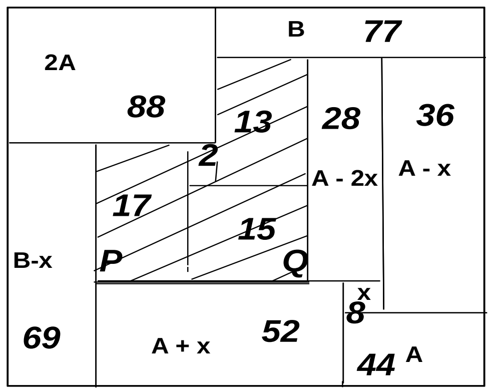


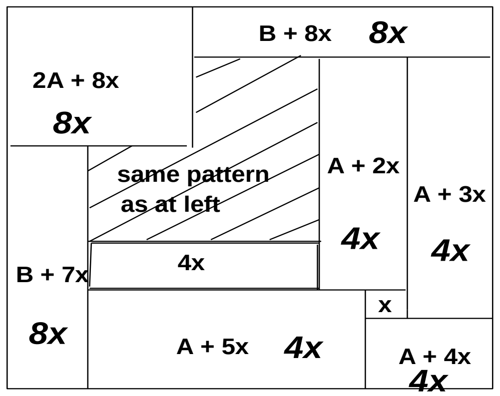


Now these increments of four can be applied to any pattern.
The Elements are 13 to 11 a drop in the Order of two.
The series continues 17 and 15 Elements, 21 and 19, 25 and 23 ...

E.7.7. A NEW ADD-ON LINK

On observing the rectangles for [12] 165 x 157 and [13] 293 x 285 I was made aware of a new Link which applies to some rectangles of Sides 2233.
These rectangles both contain the same internal Elements 13 17 2 and 15 indicated by shading in the diagram below.
The Elements for [12] 165 x 157 have been shown in below 1.
By adding an Element at *PQ* the eight External Elements need recalculation. However the new rectangle fits OK.
By looking at the general Algebraic format below, it becomes clear that this Add -On always holds good - provided there is 2 or more Elements bordering PQ initially. In 165 x 157 and 293 x 285 Rectangles, the x Element has the value 8.
Note the added Element is 4x (32) and all four Elements encircling x all conveniently increase by 4x (see Italic figures).
The top and left Elements are increased by 8 times and thus the resultant Solution is bigger by 16x in each dimension, i.e. by 128 in our example 165 + 128 = 293 and 157 + 128 = 285.





In our example it so happens that the top left corner Element is twice that of the bottom right corner in both rectangles. Also the Top right Element is x more than the bottom left one. This is always found to be the case.

E8. ENDINGS - 5TH SERIES

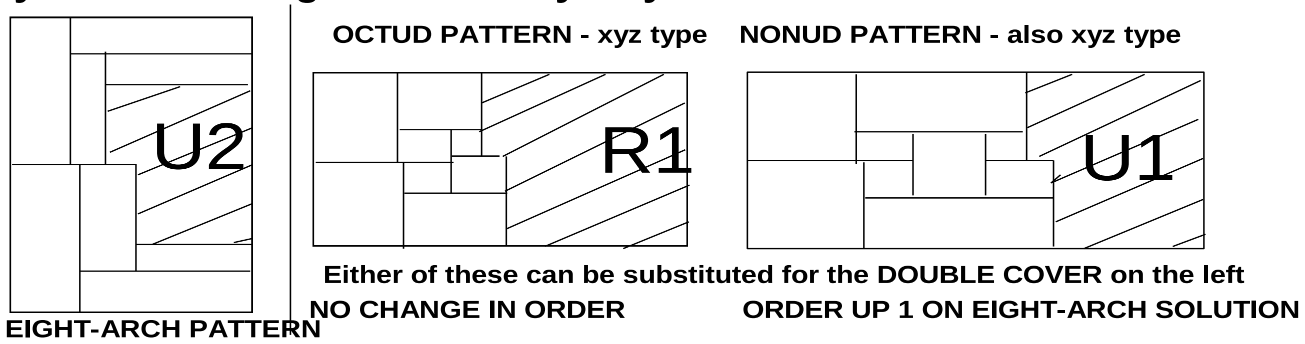
E8. PATTERNS LINKED TO EIGHT-ARCH PATTERN

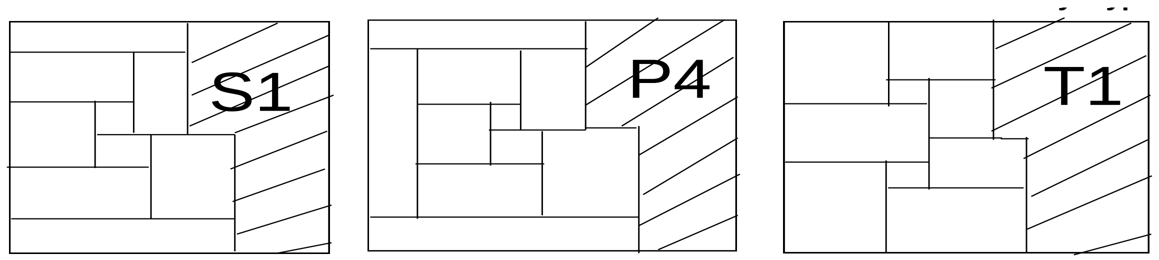
The next group can be conveniently put together as they share the eight-arch pattern - Below 1.

After this section each is mentioned separately.

Each of the patterns which follow can be converted into this eight-arch pattern, leaving the Order unchanged in pattern 1, down 1 in patterns 2,3 and 5 and down 2 in pattern 4. Patterns 2, 3 & 5 all require three Unknowns xyz and the other two, xy.

Note that with any known Rectangle containing an Eight-arch as shown below, it is possible to convert it into any of these patterns. Thus any new Rectangles found by any of these will have the same or lower Orders.

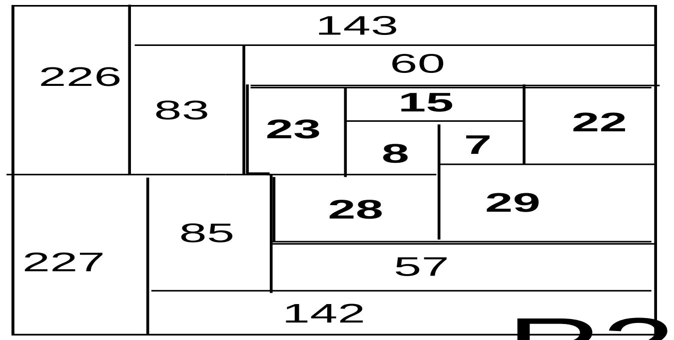




E8.1. OCTUD TO EIGHT-ARCH CODE R1 TO R2

Below an eight-arch solution of Order 15 is shown, with a corresponding Octud solution.

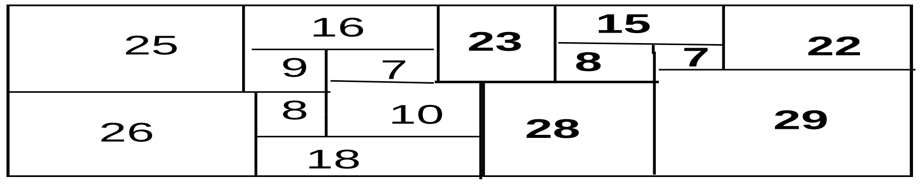
I happened to know that the pattern above 2,2 could be substituted to obtain [15] 101 x 51 but above 12,3 and above 1,2 will not work. But there is no obvious means of knowing by inspection which (if any) relationship applies! Eight Elements are substituted for eight without change in the Order.



[15] 453 x 369

R2

SUBSTITUTION TO SAME ORDER.



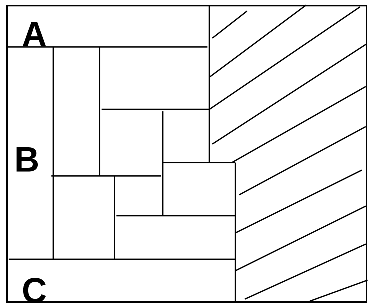
[15] 101 x 51 imperfect

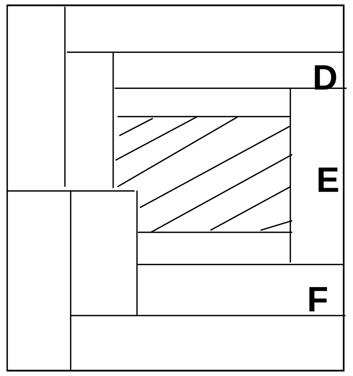
R1

E8.2. NEXT IN SERIES

The next matching pair is discovered by adding A, B and C and D, e, and F to the previous pattern as below.

11 Elements to 11 Elements and no change in Order.





E8.3. CONTINUATION OF SERIES

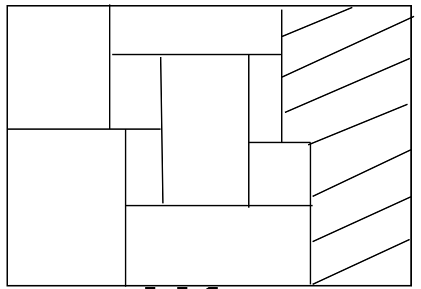
By successively adding 3 Elements on the left and 3 on the right patterns of 14 and 14 Elements, 17 and 17, 20 and 20....are obtained.

E8.4. ANOTHER SERIES

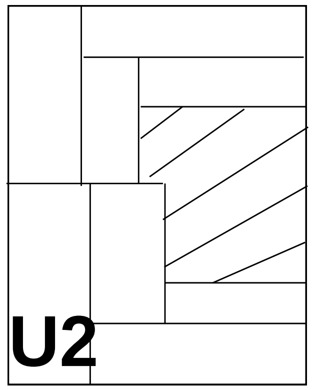
Refer to E7.6. Increments of 4 can be applied giving the series 12 to 12 Elements, 16 to 16, 20 to 20, 24 to 24...

E9. ENDINGS - 6TH SERIES

NONUD TO EIGHT-ARCH



U1



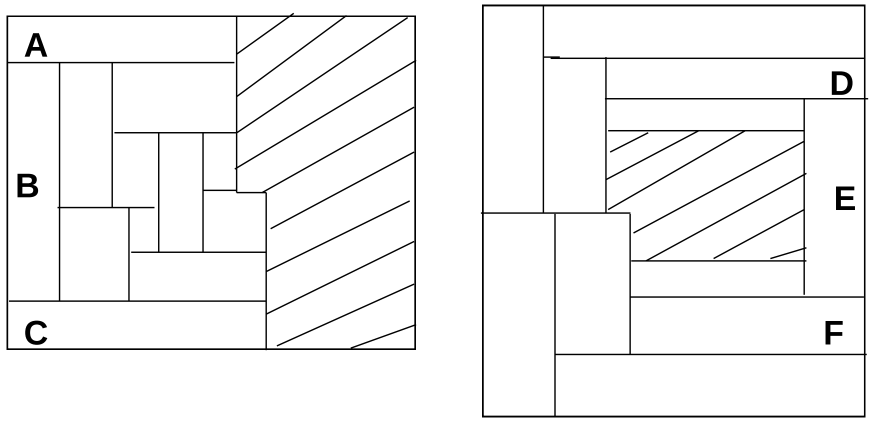
U2

E9.1. NONUD TO EIGHT-ARCH - CODE U1 TO U2

Above is a Nonud solution, which is the Octud pattern with an internal Element added.

E9.2. NEXT IN SERIES

Very similar to E8.2 but an internal Element added to below 1. Below 2 is unchanged.



E9.3. CONTINUATION OF SERIES

15 to 14 Elements, 18 to 17, 21 to 20... is obtained by adding successive Elements at abc and def.

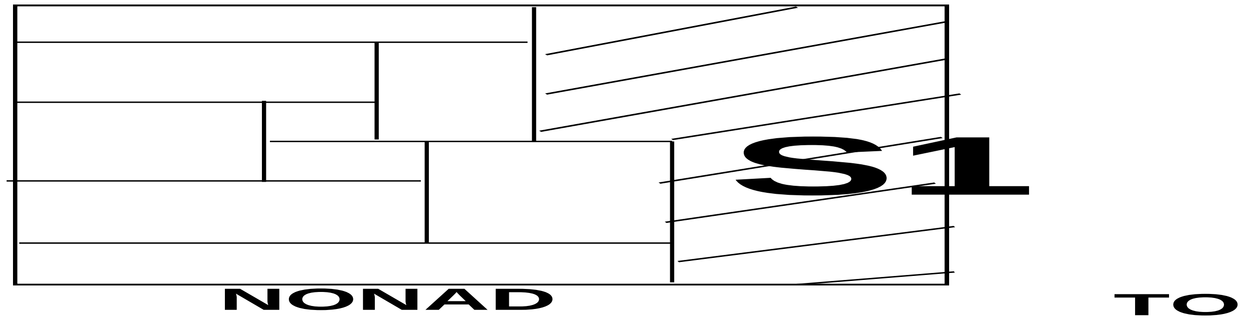
E9.4. ANOTHER SERIES

Refer E7.6. A further series can be obtained with increments of 4

E10. ENDINGS - 7TH SERIES

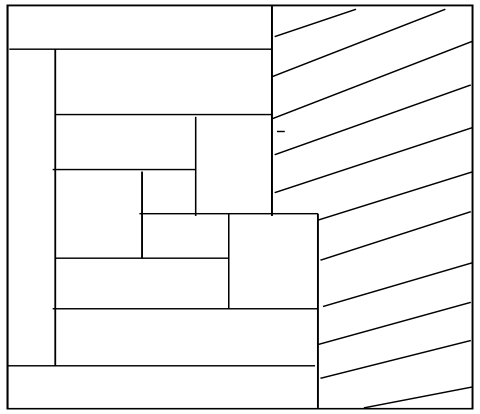
E10.1. NONAD TO EIGHT-ARCH

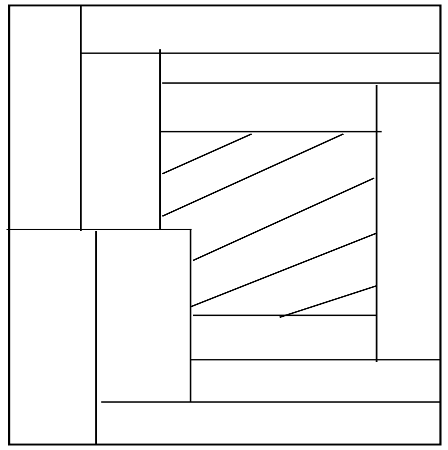
An xy series. A Rectangle of the first type below 1 can easily be converted into the second without any Elements needing to be up-rated. The Order drops one. Note that a solution of the second type does not always fit the first.
This is because the Eight-arch pattern links to several different pattern.



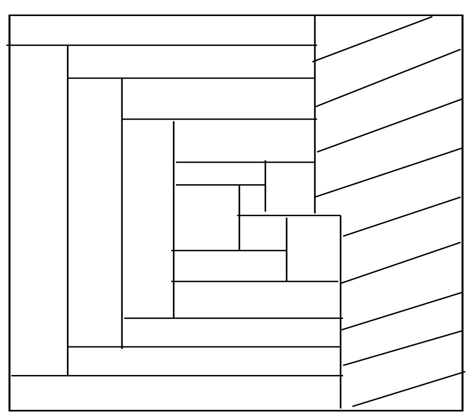
E10.2. NEXT IN SERIES

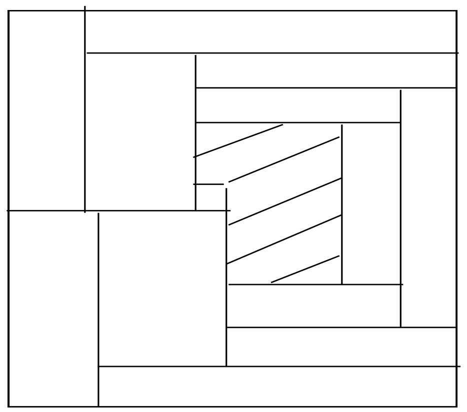
The vertical side of below 1 is always divisible by 40.
It starts with 12 Elements and finishes with 12 Elements, so Below 2 has the same Order.





This is the last pattern with 3 added Elements at left top and bottom.
The vertical side below 1 is always divisible by 556.
This pattern can be translated into below 2 by reducing the amount of Elements from 18 to 13, a Reduction of 5.

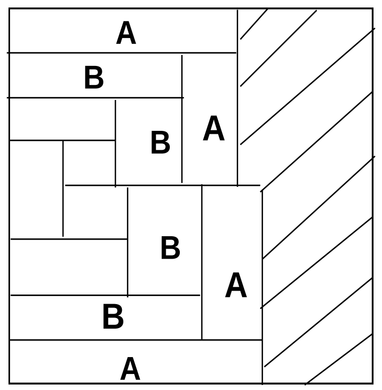


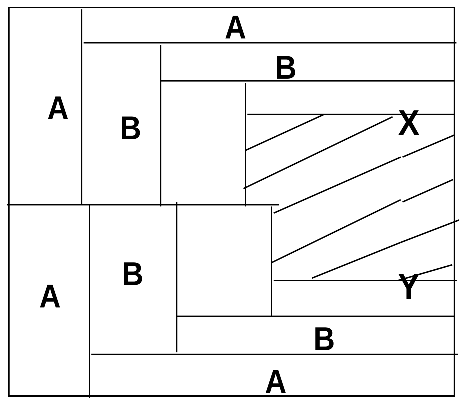


E10.2.1. THIRTEEN-AD TO TWELVE-ARCH

This next relationship starts with 13 Elements and finishes with 12 Elements, so below 2 is Order 1 less.

Note that if an Element is drawn at xy first, then the pattern of 13 Elements also fits making a Rectangle of the same Order. As usual this forms a series. When the A's are removed from the left giving 9 Elements, the matching pattern is as on the right with A's removed. Also A and B can be removed from both sides giving the Pentad to Diad-arch relationship (See E.)





E10.3. CONTINUING THE SERIES

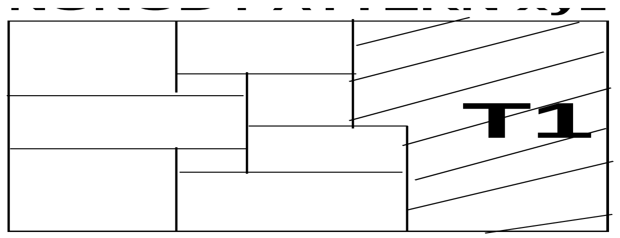
E10.4. ANOTHER SERIES

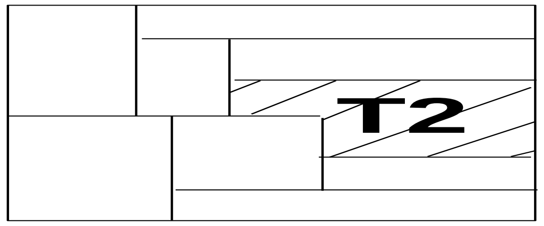
Refer E7.6. A further series can be obtained with increments of 4

E11. ENDINGS - 8TH SERIES

E11.1. NONOD TO EIGHT-ARCH SERIES

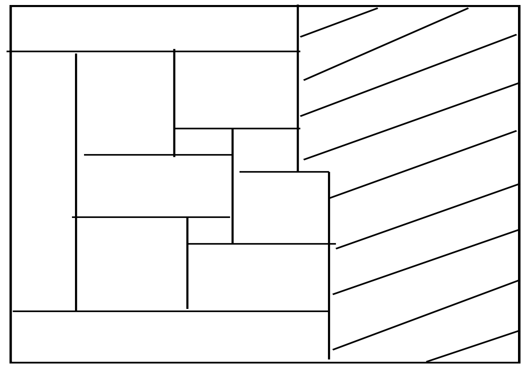
From 9 Elements to 8 with the Order reducing by 1. Yet another pattern which connects to the Eight-arch pattern.

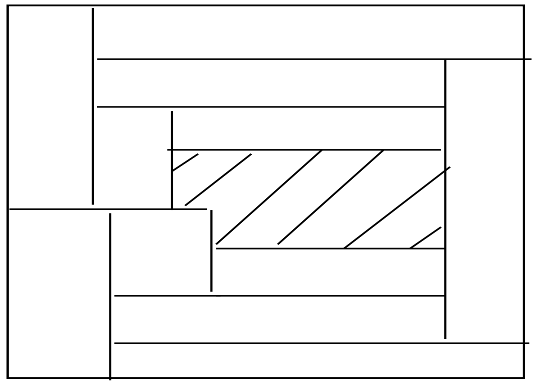




E11.2. NEXT IN SERIES

From 12 Elements to 11 with the Order reducing by 1.





E11.3. CONTINUING THE SERIES

15 to 14 Elements, 18 to 17, 21 to 20 ... the series goes on forever.

E11.4. ANOTHER SERIES

Refer E7.6. A further series can be obtained with increments of 4

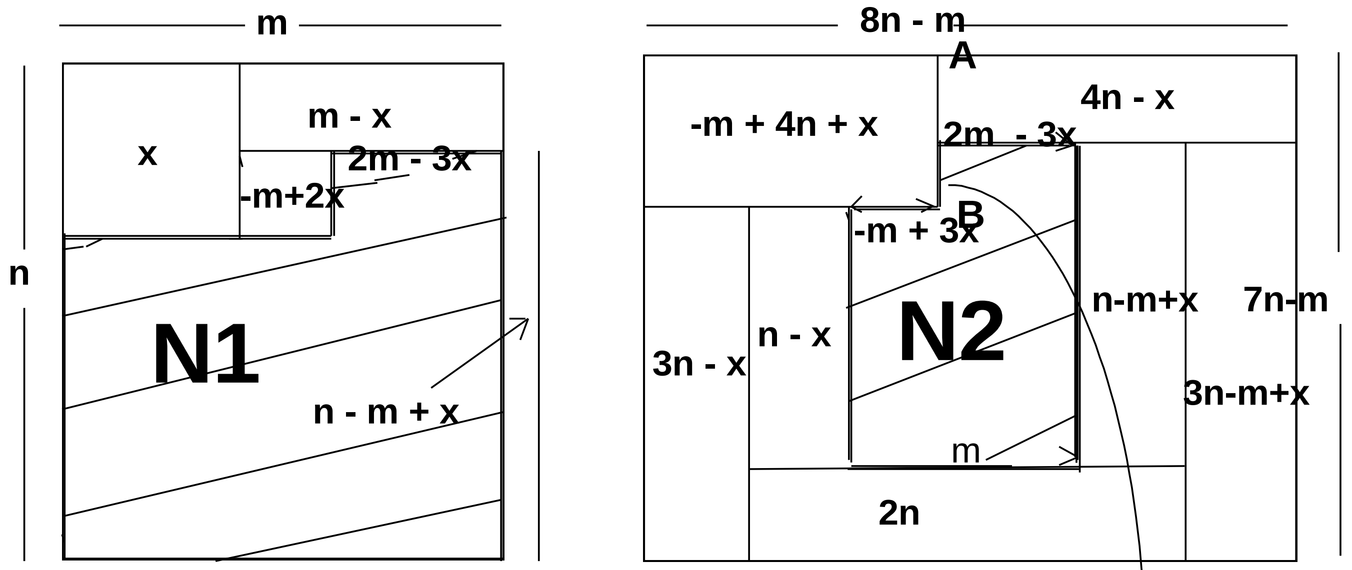
E12. ENDINGS - 9TH SERIES

E12.1. TRIAD ENDING TO DOUBLE ARCH - CODE N1 TO N2

The procedure which is here described is amazing since it can be applied to absolutely any solution containing a Triad - or in two ways with a two Triads solution.

E12.1.1. ALGEBRAIC PROOF

Below 1 shows the original Triad solution and below 2 the resultant one which is always S2223. The first solution of dimensions m x n is shown with the Triad at the top which often means it has to be shown vertically. The 3 Element Triad is stripped away and replaced by 7 Elements, so that the result has an Order of four greater.



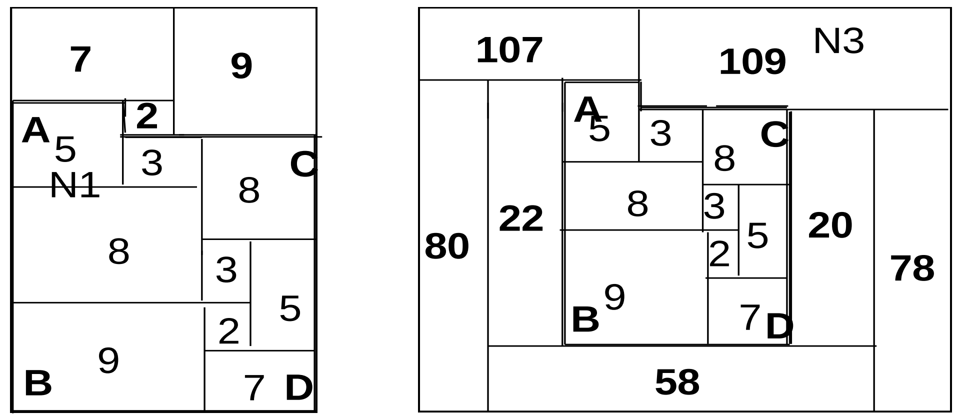
FROM [o] m x n TO [o+4] 8n - m x 7n - m

AT LINE AB (-m + 4n + x) EQUALS (-m +2x) + (4n - x) PROVING STRUCTURE

E12.1.2. EXAMPLE OF THIS PROCEDURE

It does not matter if the original solution is symmetric or not.

Below shows that the result is always asymmetric (symmetry 1) in any event.



2 7 & 9 ARE DELETED ON THE LEFT & 20 22 58 78 80 107 & 109 ADDED. NOTE THE NEED FOR 2+ ELEMENTS AT AB & CD FOR THE CONSTRUCTION TO WORK

FROM [12] 29 x 16 ¬ TO [16] 216 x 197

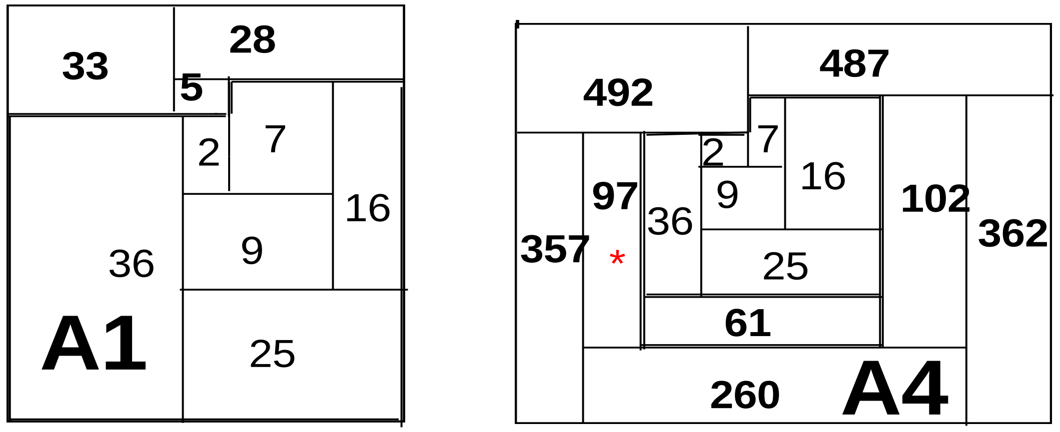
E12.1.3. OBSERVATIONS

1. No multiplying up of any Elements required
2. Works for all Triad solutions
3. Status of solution does not usually change, but note that if the original is imperfect only through the Triad part, the result is perfect.

4. No changes in the Reduction index - both 13 in above.
5. The result is always much larger than the original solution.

E12.1.4. ADDING A FURTHER ELEMENT FOR ANOTHER SOLUTION - CODE A1 - A4

Some Rectangles like [9] 69 x 61 below, fail to give a Valid Rectangle from the method given. However adding a further Element (61 in example below) and then proceeding as before [14] 979 x 849 is successfully created.
This system which adds 5 to the Order works for absolutely any Triad solution.

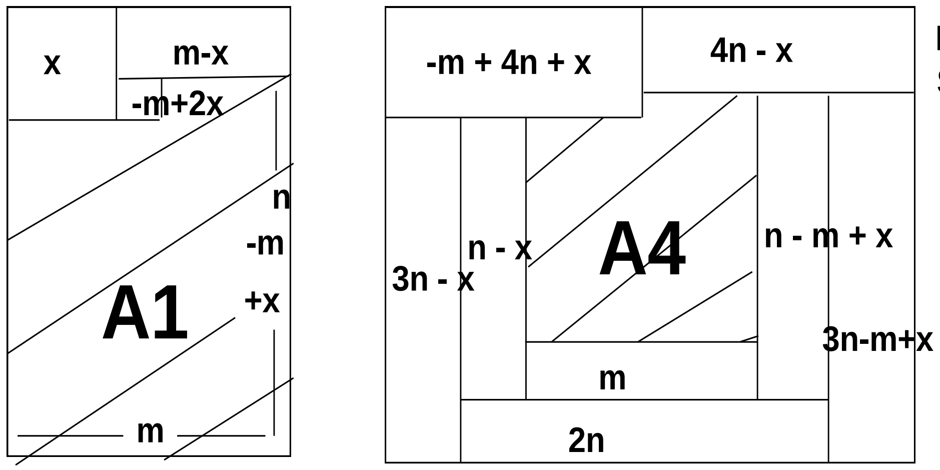


SIMILAR TO THE LAST PROCEDURE, EXCEPT THAT AN ELEMENT (61) IS ADDED FIRST TO PREVENT ELEMENT * BEING 36 AGAIN! THE ORDER RISES BY FIVE.

[9] 69 x 61¬ 3 elements dropped & 8 added - [14] 979 x 849

E12.1.5. DIMENSIONS OF THE REVISED STRUCTURE

Below is a chart showing that the original Rectangle [o] m x n becomes [o + 5] 8n + 7m x 7n + 6m.
The position mentioned about twins in the next section applies to both structures.



REVISED STRUCTURE

THE ELEMENT m IS INSERTED FIRST

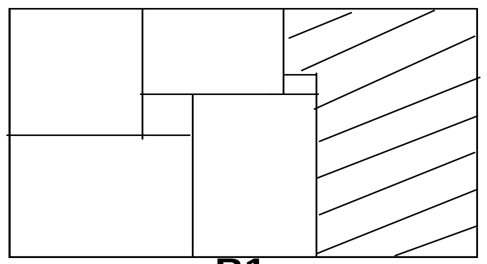
E12.1.6. TWINS ARISE FROM RECTANGLES WITH TWO TRIADS

The Rectangle as seen above increases from m x n to 8n -m x 7n - m.
This is a fixed amount and therefore not affected by what Triads are in the original. This leads to two important features.

1. If the original solution has two Triads the two resulting solutions must be Twins. 2. If two twin originals are taken each with at least one Triad, the resultant solutions must be Twins also!
An interesting example is from [13] 112 x 75 where m = 112 and n = 75 thus 8n - m x 7n - m is [17] 821 x 709.
Now [13] 112 x 75 has one Triad and gives [17] 821 x 709 (1). [13] 112 x 75 has two Triads and gives [17] 821 x 709 (2) and (3). Yes, three twins all different! For illustrations see twins section.

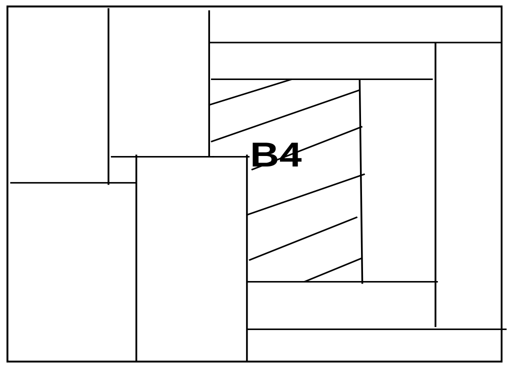
E12.2. DOUBLE PLUS TO DOUBLE FRAME - CODE B1 TO B4

Note alternative pattern with one less Element. The Formula is [o] m X n to [o+5] 11m + 4n X 8m + 3n.

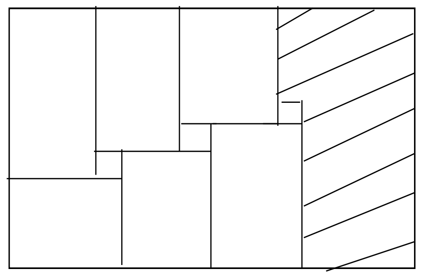


B1

DOUBLE +

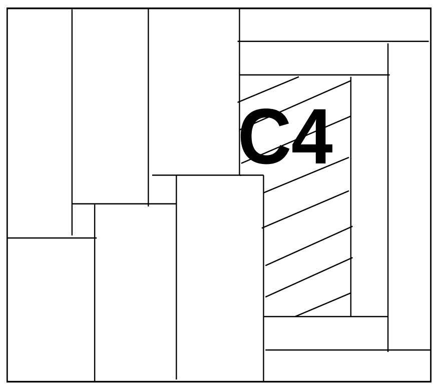


E12.3. TRIPLE PLUS TO DOUBLE FRAME



C1

DOUBLE +



E12.4. CONTINUANCE OF THE SERIES

The series continues with 4, 5, 6 Triads forever.

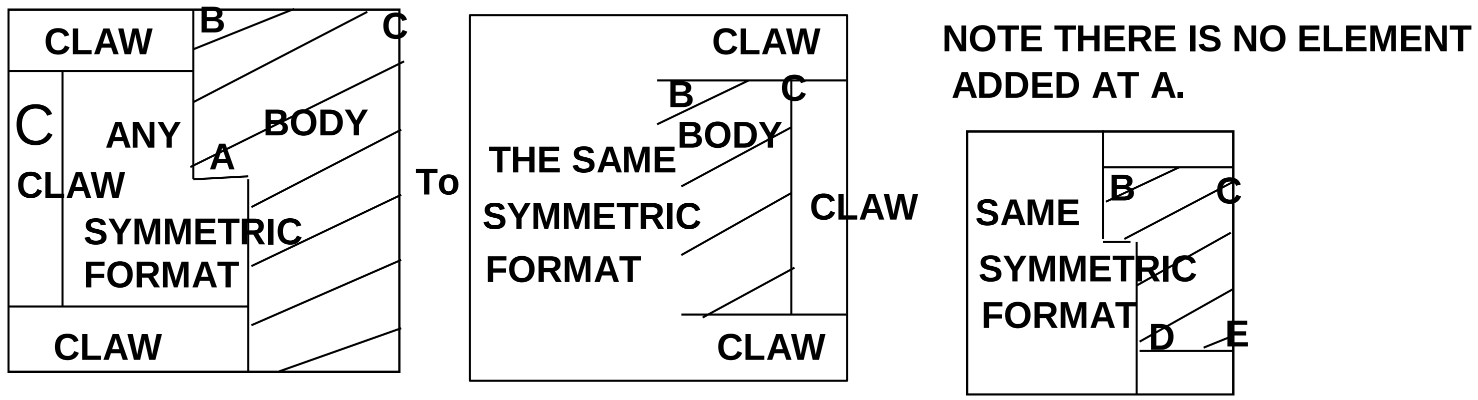
E12.5. ANOTHER SERIES

Refer E7.6. A further series can be obtained with increments of 4

E12.6. THE CLAW PRINCIPLES

A claw is three Elements drawn "|__| . This pattern is so important in linking together many of the sets of endings.

Firstly observe the following general pattern -

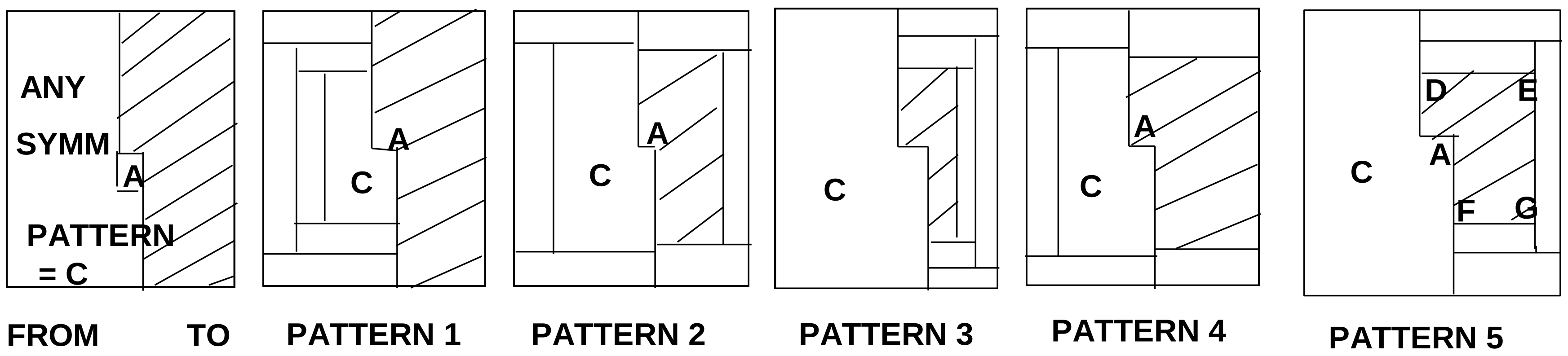


A good example of this is Diad to Pentad arch relationship discussed in E4.1.
1. Above 1 shows any symmetric format which includes a claw.
Now if the claw is removed from the left and added to the body on the right, a new Rectangle is obtained by revising the values in the symmetric format but retaining the same pattern - as above 2.
2. If both BC and DE are bounded by 2 or more Elements the pattern in above 3 also works (again the values in the symmetric format require revising, but its pattern is unchanged). The solution in above 2 has no change in Order, whilst the Order in Above 3 drops one.
3. Now, if the Element "C" in above 1 is removed, solutions for Above 2 and Above 3 (if 2+ Elements at BC) also happen to work! But the Order now rises by 1 and stays the same respectively.
4. Lastly, if a second claw is drawn round above 1 on the left and ditto round Above 2 and Above 3 at the right, the patterns still work! In fact 3, 4, 5... Etc. sets of claws can be employed.

E12.7. SEQUEL CLAW PRINCIPLE

If an extra Element is added to the symmetric format, then a different set of patterns apply!
Below 1 has a symmetric pattern - note the added Element A which does not appear in patterns 1-5 at position A.
Now patterns 1 2 and 3 can always be constructed from Below 1 - by adding two claws - 2 on left in pattern 1 - 1 left and 1 right in pattern 2 and - 2 on right in pattern 3. Where DE and FG both have 2+ Elements then patterns 4 and 5 are also possible.
Patterns 1 2 and 3 may be exchanged by substituting the claws (as in last section E12.6).
This section actually includes all that discussed in E12.6.

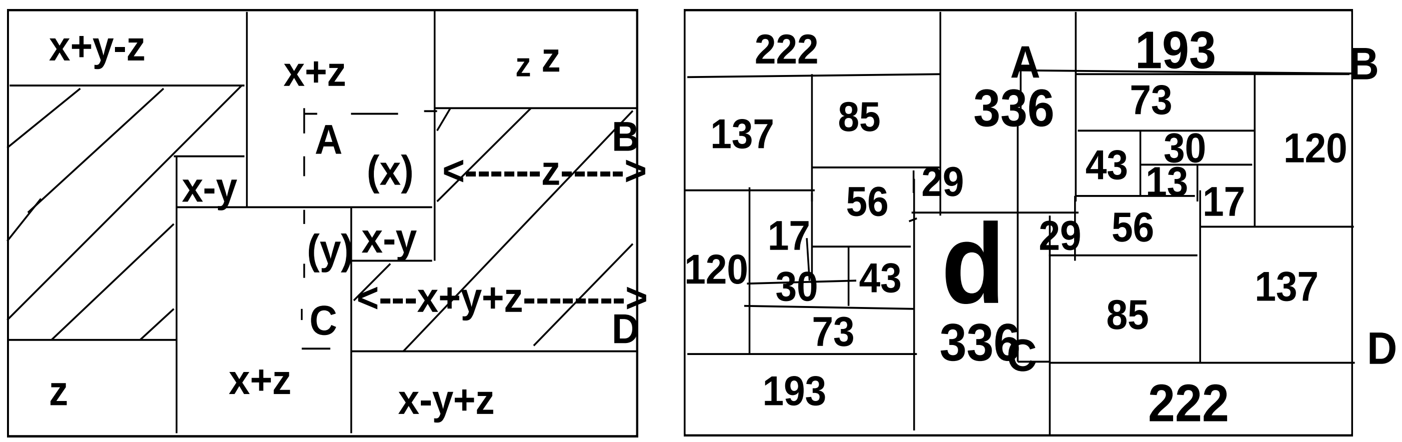
All these patterns were shown earlier with a Triad pattern at C. These patterns link up everything discussed in E4, E5 & E6.



E13. CREATING SOLUTIONS USING JOINS

E13.1. JOINING TO OBTAIN SYMMETRIC SOLUTIONS - CODE d

Consider a Squared-Rectangle ABCD below of dimensions x + z times x + y.
Elements of value z, x - y + z, x + z and another x + y can be drawn as shown. The Elements in the shaded area may be repeated upside-down with two further corner Elements to create a symmetric solution that fits.
Below 2 shows an actual symmetric solution created by using [12] 336 x 257 and adding the Elements 193 and 222 and 336 twice then adding the rest up-side-down on the left.
Although the solution chosen has a Triad end, this joining will also work with a Diad end.
(Note it is easy to confuse this construction with a different one later. In this type Elements are <u>added</u> to the original Rectangle, not stripped away)



ALGEBRAIC DIAGRAM & SYMMETRY 2 RECTANGLE (FROM [12] 336x257
SHOWN AS ABCD

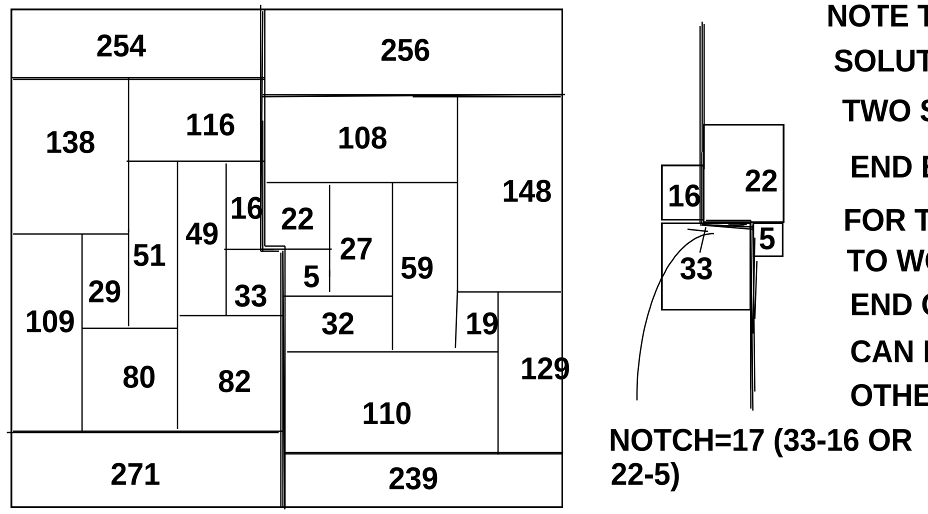
If the original solution only has 2 Elements at AB then a further Element at BD is necessary to make the construction work. The two end Elements at C and D are omitted and replaced by the 4 blue ones followed by a reversal at the left hand side. This discovery is important as large numbers of symmetric solutions can be produced in this way.

It always happens that the vertical side (672 above) is always twice the original horizontal side in the original (336 x 257, i.e. x + z becomes 2x + 2z.

E13.2. STEP JOINING

Using the idea in E. having a collection of worked Rectangles, it is possible to obtain asymmetric solutions and sometimes Perfect ones also. See Below for explanation:-

TWO SOLUTIONS PUT TOGETHER [24] 772X510



NOTE THAT THE ORIGINAL SOLUTIONS ARE THE TWO SECTIONS PLUS TWO END ELEMENTS.

FOR THIS CONSTRUCTION TO WORK ONLY THE DIAD END OF EACH SOLUTION CAN BE REMOVED & NO OTHER PATTERN!

The Above Perfect solution was achieved by choosing two Rectangles with Diad (never a triad) ends which had the same dimension, and also contained a single step of the same size, the step being between the two Diad Elements.

If any other step position is chosen the solution will fail. Now [12] 386 x 247 and [12] 386 x 277 both happened to contain Diads one of which is 17 more than the other, and the composition of the step is dissimilar. It also helped that 17 did not appear in both Rectangles.

Obviously most Rectangles constructed this way will prove imperfect. It should be noted that

1. Each Rectangle chosen must have sides of at least 2323
2. The Orders may be different.
3. There is no need for the vertical dimensions to be the same.
4. The notches must differ to prevent Compound solutions. (E13.2)

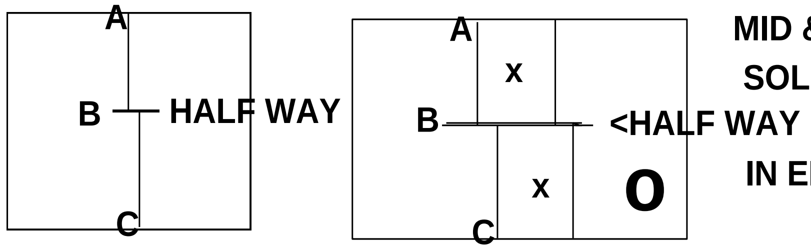
Perfect solutions are best chosen.

Extremely reduced solutions can be produced by this method - see section G. [28] 530 x 535 is Perfect and obtained from [14] 265 x 192 and [14] 265 x 198 using a notch of 12. This is a size typical to Order 13!

E13.3. MID AND MID-PLUS SOLUTIONS - CODE o

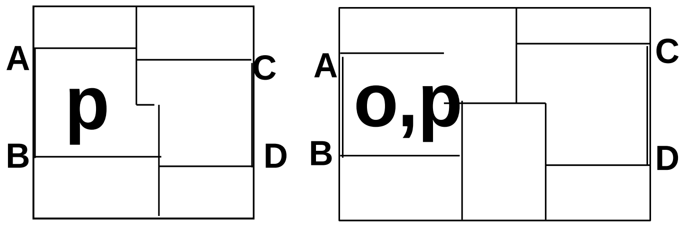
These solutions always contain a line which appears half way down the solution - known as Mid solutions.

If two equal extra Elements are added they are termed Mid-plus solutions.



MID & MIDPLUS
SOLUTIONS
AB = CD
IN EITHER CASE.

E13.4. MORE MID-PLUS SOLUTIONS



MID MIDPLUS

Previously mentioned Forms but with TWO ELEMENTS ADDED at AB & CD to form new solution mentioned in E8.4 & E8.3

Further solutions to E2.1 are readily obtained by adding an Element at one end as Below and finally at the other end. The construction is similar, but there is no need for the original solution to be S2323 or more.

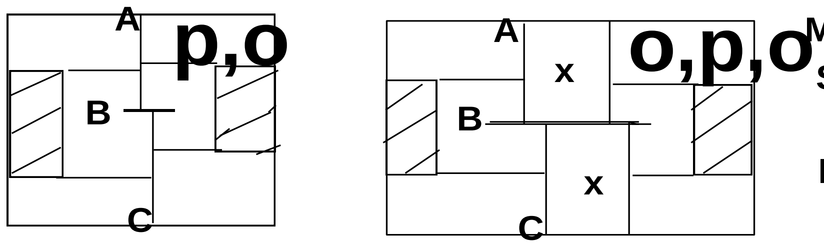
Two identical large Elements are added as above 2 & below 2. The right hand side is then added.

Formulae are [o] m X n to [2o + 2] 2m + n X 2m + 2n and [o] m X n to [2o + 4] 3m +2n x 2m +2n.

E13.5. MORE MID SOLUTIONS

Refer to E2.1 and Above 1. Some of the solutions in E8.4 do not need the two large Elements and can be left as below 1.

WITH 2 EXTRA ELEMENTS (SHADED)



FURTHER
MID & MIDPLUS
SOLUTIONS
AB=CD
IN EITHER CASE.

E13.6. TWO SOLUTIONS WITH THE SAME DIAD END

In compiling quantities of SR Solutions it is found that there are many instances where two rectangles have the same Diad end, e.g. and both occur in [13] x . For the following to work neither of the Diads must be a Triad (i.e. no Element connecting either Diad).

Otherwise there are no restrictions - which means the two starting solutions can have -

1. Different sizes except that one dimension will of course be identical.

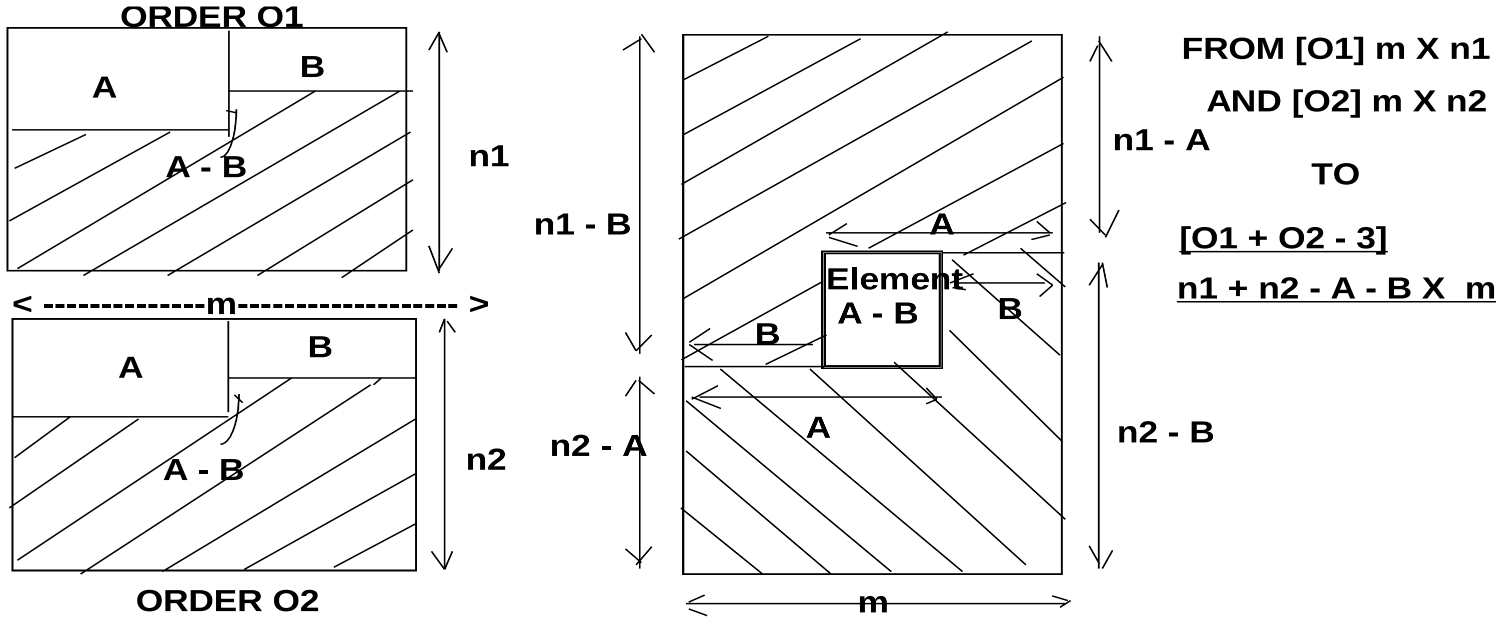
2. Different Orders.

3. Different Reductions.

4. Different Status - occasionally Non-zero solutions will occur however.

5. Both Diads on the shorter or longer sides, or one on each.

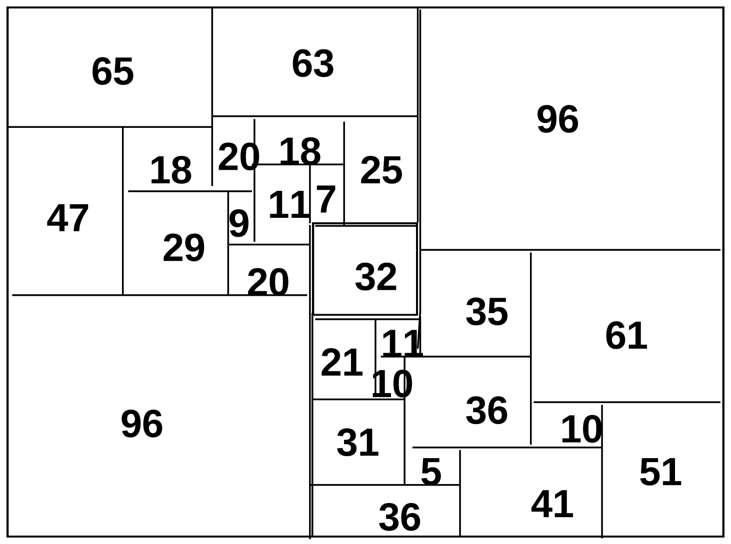
6. The two solutions can be even the same one used twice, but only symmetric solutions result.
7. Suitable twins can be used, but Twins are not essential.
8. Any combination of the above items.
This list is necessarily included as the form shown below is just one related to other Joins already described in this book.



Above shows that if the elements A and B are ignored and the top shaded area put upside-down, then the two shaded areas will fit together provided an extra element of size A - B is inserted as shown.

The formula <u>[O1 + O2 - 3] m X n1 + n2 - A - B</u> is true if m remains the greater side, but more often the result is <u>n1 + n2 - A - B X m</u>, so this has been shown above.

E13.6.1. EXAMPLE OF THIS JOIN

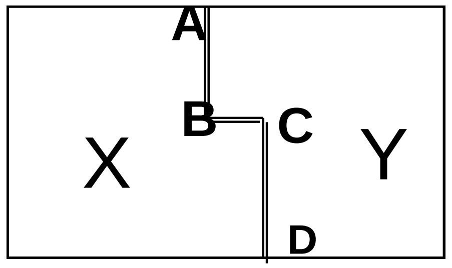


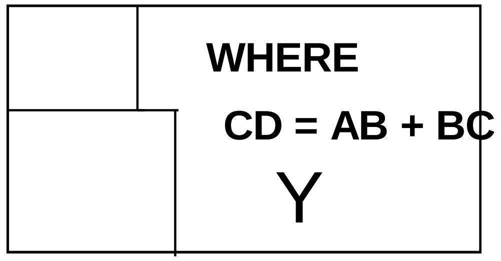
[27] 224 x 208 which has been constructed from [15] 216 x 208 (1) and [15] 216 x 208 (2) with Element 32 added. (The originals are seen if 32 is ignored and 88 and 120 added to the left & right hand portions respectively).

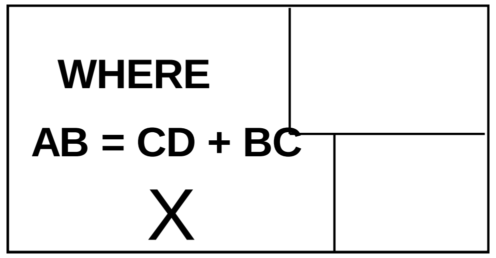
Note that this differs from the Step Joining previously shown in that there is a Central Element needed (32 in above) and that four added Elements are not needed.

E13.7. THE ABOVE IDEA IN REVERSE

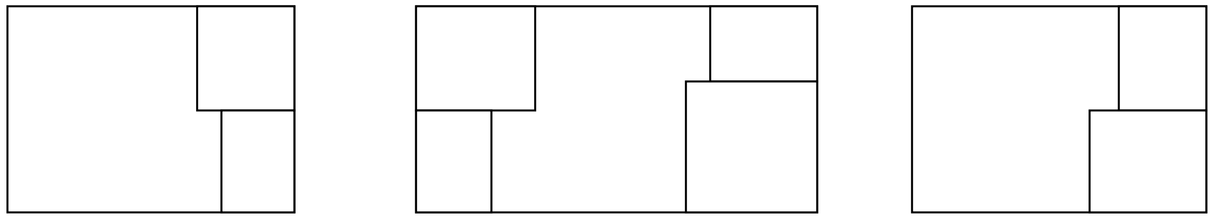
So whenever a solution is found with an internal divide as shown below such that the dividing line BC plus one of the others equals the third then solutions with Diad Ends can be worked back. Where CD = AB + BC or AB = CD + BC as in below 3 it is evident that a Diad will fit at one of the ends.







E13.8. USING THREE OR MORE SOLUTIONS TO FORM ONE LARGE SOLUTION



ENDS A & B DIAD ENDS A & B C & D DIAD ENDS C & D

The diagram above shows in theory how three Rectangles could be amalgamated into one large Solution. The 2nd Solution (of sides type S2x2x) contains four corners A & B (left) & C & D (right). Then if we have a 1st solution with A & B and a 3rd one with a Diad of C & D, then the solution is found by removing all 8 Elements of 4 Diads & inserting two internal ones instead. Such a solution is highly likely to be Imperfect in practice.

In theory any number of Solutions might be joined together. In practice however finding suitable Solutions would severely limit the number of Solutions which would be used. In theory solutions could be added infinitum but in practice would prove impossible.

E14. CREATING SYMMETRIC FROM OTHER SOLUTIONS

To make the following sections less confusing, these ones concern originals where the cover is stripped off and the earlier ones in E13 with sides added to the originals to obtain the resultant symmetric solution.

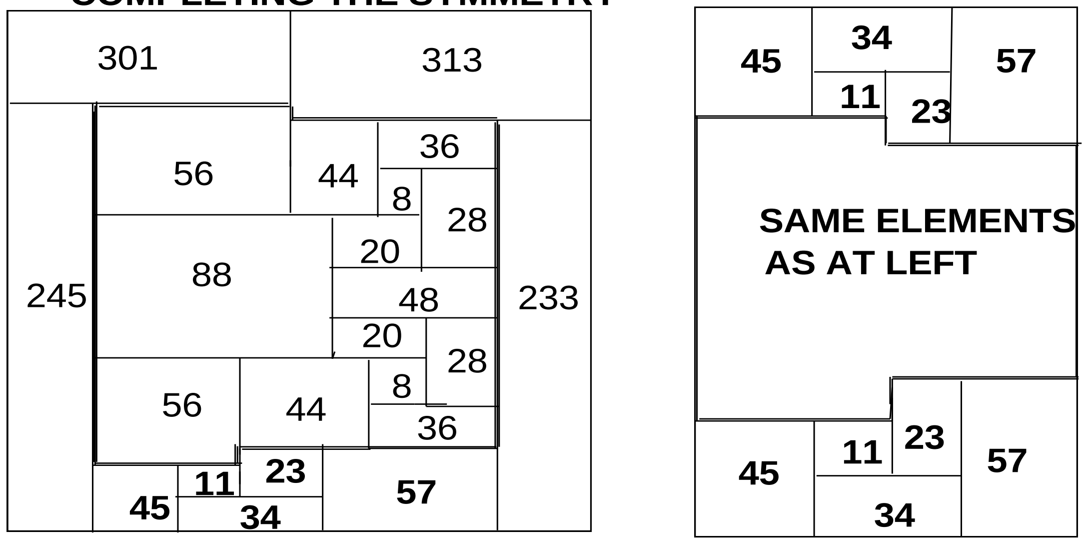
E14.1. DUPLICATED ELEMENTS IN PENTAD S222* SOLUTIONS

I noticed solutions of the type below 1 with a duplicated area of Elements.

This was puzzling until the relationship was made clear by below 2, where a second repeated Pentad has been added.

Note that below 2 is symmetry 3 (unlike [14] 116 x 104 Above.

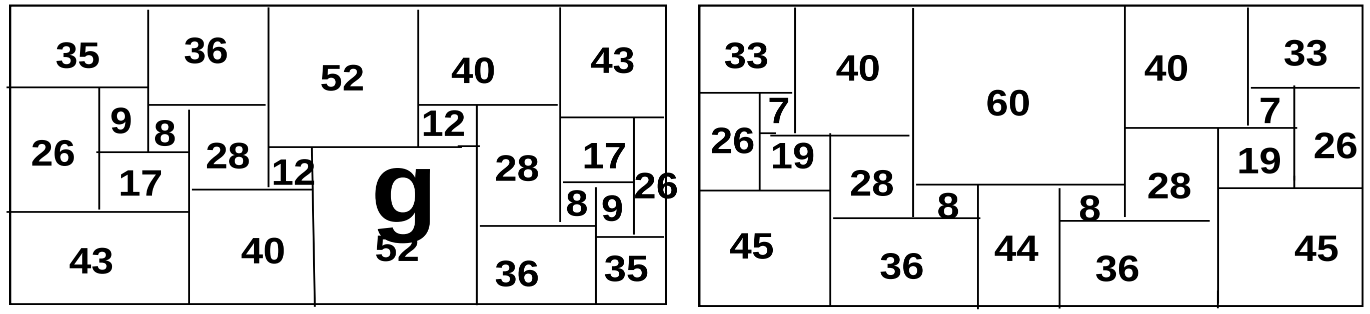
COMPLETING THE SYMMETRY



E14.2. FURTHER SOLUTIONS - SYMMETRIC

Below are two further solutions which can be obtained from the [12] 258 x 206 (1) and (2) and a range of other solutions are listed.

TWO MORE SOLUTIONS FROM [14] 116 x 104 - BOTH SYMMETRIC



[22] 206 x104 [20] 198 x 104

From the solutions [13] 258 x 206 (1) & [13] 258 x 206 (2) there is [14] 11 6x104

There are also $2223 solutions [14] 466 x 414 (1) & [14] 466 x 414 (2)

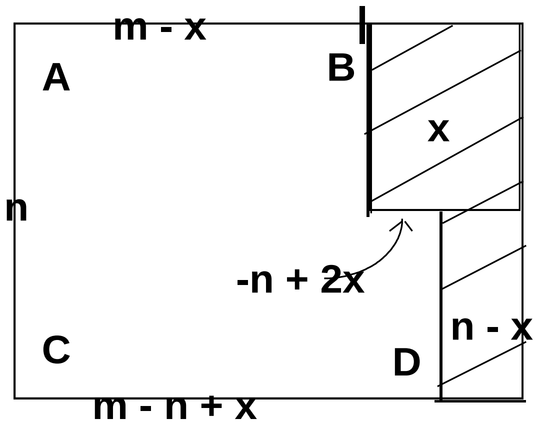
There are the above [22] 206 x 144 & [20] 198 x 104 symmetric.

Replacing a pentad with a 'cover' above [21] 438x285 & [19] 422x371!

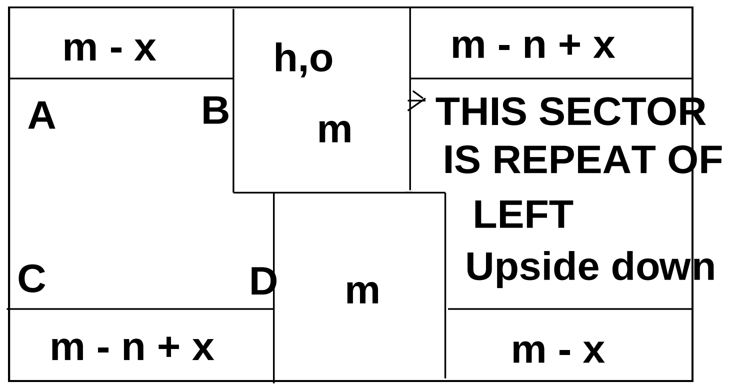
E14.3. DIAD ENDS TO SYMMETRIC SOLUTIONS

Different rules apply to Triad ends even though part of Triads are in fact Diads. Diads mean two Elements with <u>no Element adjoining both.</u>

If any solution with a Diad end is taken, it is possible to construct two symmetric solutions as shown where the algebraic proof is given.
Note the two added Elements (top and bottom), the Mid-plus Elements and the repeat on the right hand side.
With Diads the “m” Elements can be removed for a second solution.

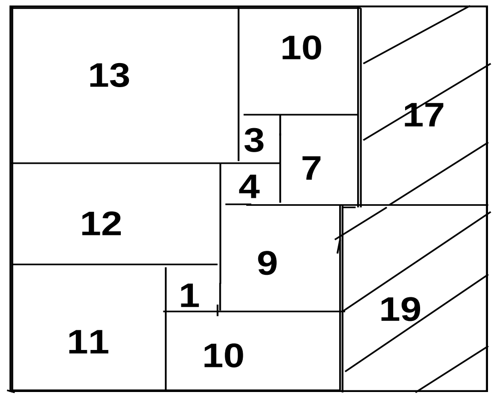


RECTANGLE [O] m X n
SHADED DIAD REMOVED

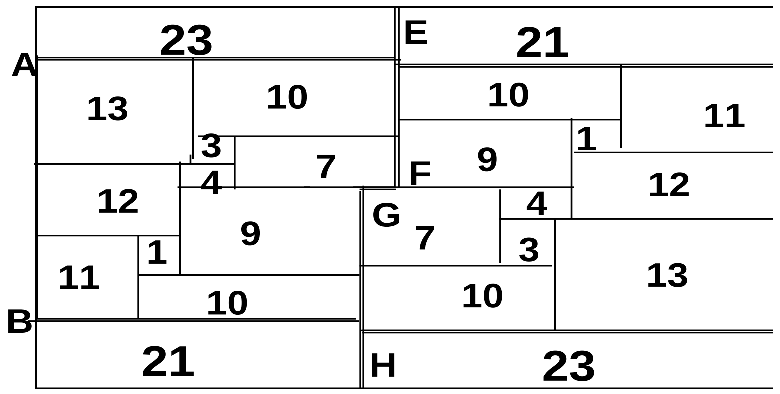


RECTANGLE [2o+2] 3m-n X 2m (MIDPLUS)
OR [2o] 2m-n X 2m (MID)

E14.4. AN EXAMPLE OF DIAD REMOVED SYMMETRIC



[12] 40 x 36 IMP



[24] 80 x 44 SYMMETRIC ¬ O

Further sol utions are obtained by adding Elements at AB & CD,
EF & GH or both

The Formula is [o] m X n to [2o] 2m X 2m - n after putting solution horizontal.

E14.5. ADDING 2 FURTHER END ELEMENTS x4

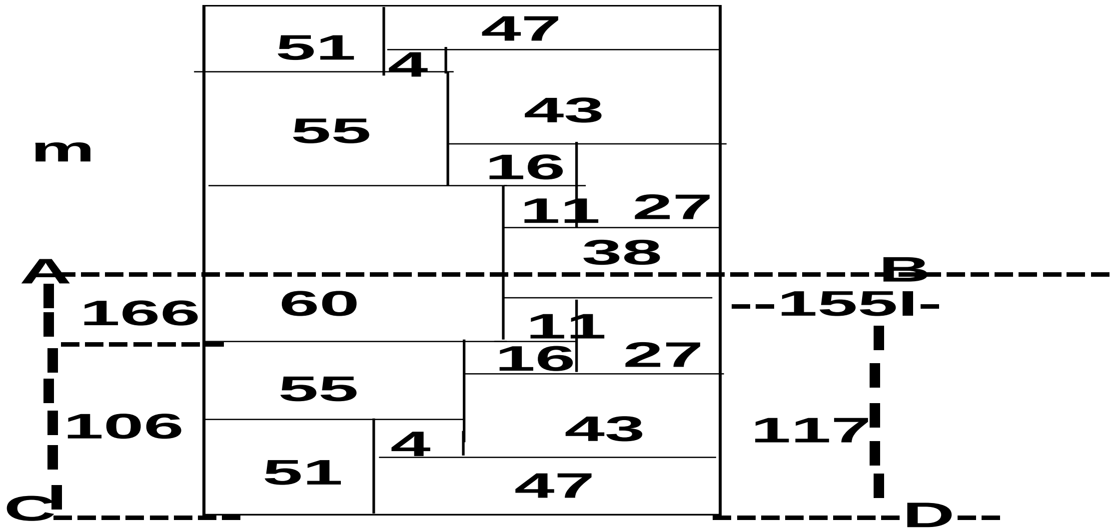
A further Element can be added at AB in above 2 and repeated on the right, making four symmetric 2 type solutions possible from the original.

E14.6. TRIAD ENDS TO SYMMETRIC SOLUTIONS x5

With Triad ends a Mid-plus solution as above 2 is also possible, but note there is no Valid Mid solution (due to connecting Elements in the Triad being adjacent).

E14.7. SYMMETRIC SOLUTIONS FROM TRIAD 2224+ TRIAD ARCH SOLUTIONS - CODE m

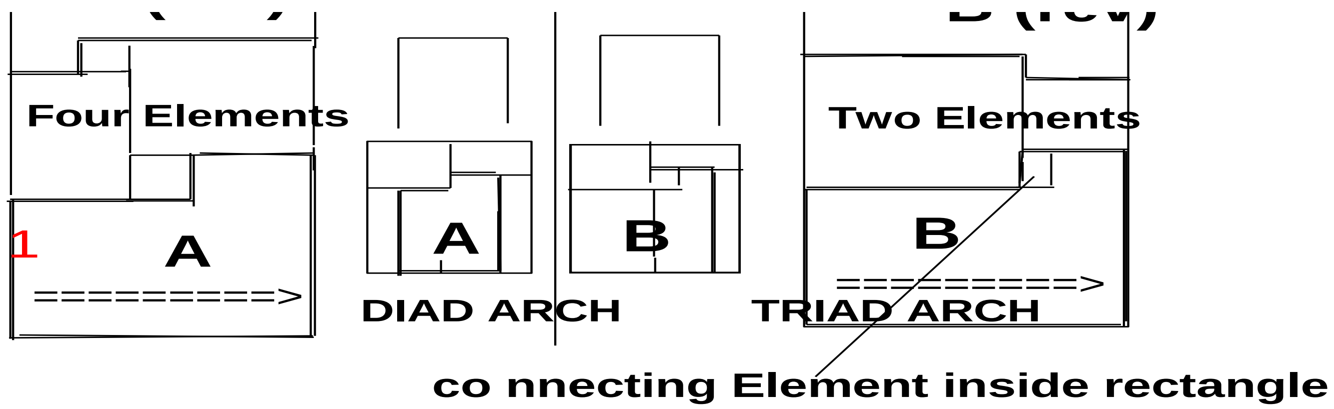
Below [12] 321 x 272 is shown as ABCD. The outer Elements 106, 166, 155 and 117 have been stripped away, and the Elements 60 and 38 added as shown.
The remaining Elements in ABCD have been repeated to form a symmetric solution.



Formula is [o] m X n to 2m - 2n X n.
Often symmetry 2 arises as in above 1, sometimes symmetry 3 as Above 2. Refer also to below where patterns 1 or 3 will apply.

E14.8. AS BEFORE BUT DIAD ARCH AND TRIAD ARCH

Above the S2224+ original solution has a Triad top to it. See Below 3 and 4. In the case of a Diad arch then the pattern below 1 and 2 applies and 4 Elements have to be added. Though the result is a Mid-plus solution it should be noted that no Valid Mid solution is possible.

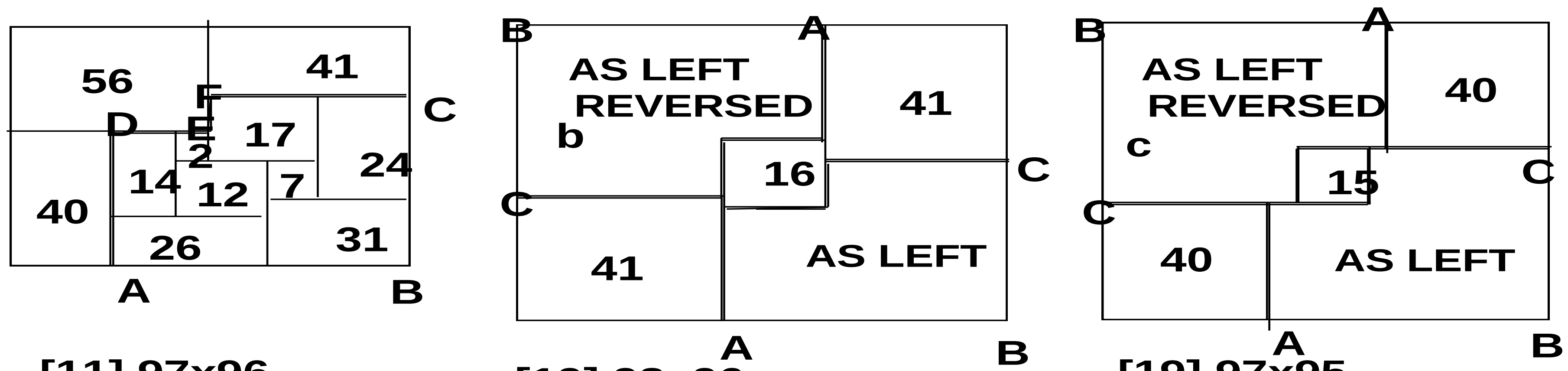


E14.9. REVERSING TO OBTAIN ASYMMETRIC SOLUTIONS

Obviously the procedure in this section may be reversed with symmetric solutions being examined to see if a smaller Order asymmetric Rectangle can be constructed.

E14.10. ANOTHER WAY OF CREATING SYMMETRIC SOLUTIONS - CODES b & c

This conversion involves solutions of sides S2-2-3+-3+ which have three corner Elements forming an “L”.
Below [11] 97x96 is shown and two symmetric solutions of Order 19 from it.



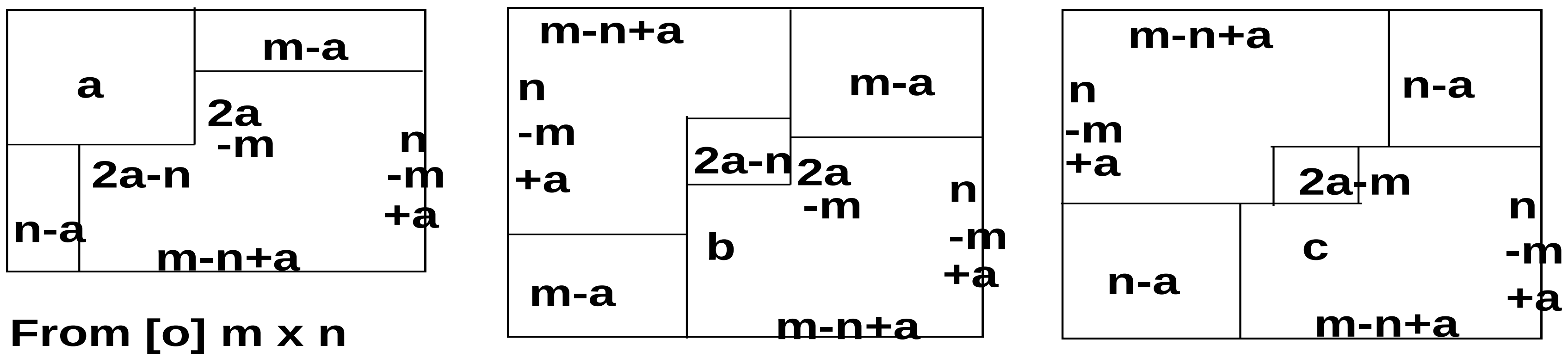
[11] 97x96 [19] 98x96 [19] 97x95

The original and resultant solutions are often fairly close in size, but the results are always symmetric, and xyz or greater, with sides of S3333 or greater. If either DE or EF are bounded by a single Element, one pattern above is not possible.

E14.11. ALGEBRAIC PROOF OF ABOVE

If the lines are carefully examined the expressions are found to agree in all directions to prove that the relationship is Valid.

The original Rectangle is given dimensions m x n and the largest Element called A. All else is calculated from this.
Note that A does not appear in any of the sizes.

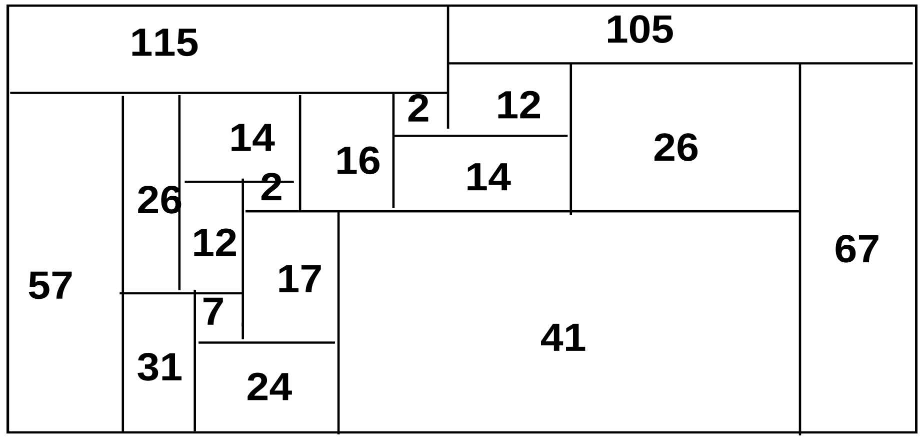


From [o] m x n To [2o - 3] 2m-n x n And [2o - 3] m x 2n-m

E14.12. OBTAINING SOME ASYMMETRIC SOLUTIONS

The solution [11] 97 x 96 happens to have a Pentad in it which reoccurs in [19] 98 x 96. Because of the Pentad to Diad-arch relationship in E--.-- it is possible to construct the asymmetric solution [18] 220 x 172 shown by axing the Pentad Elements and replacing with four arch ones (115 105 67 and 57).

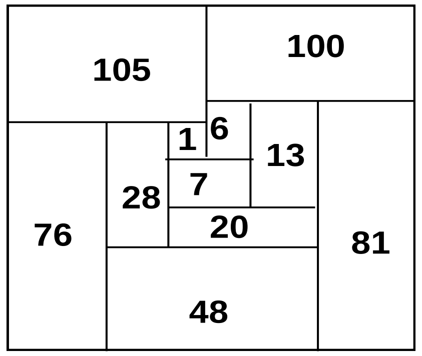
Asymmetric solutions can only be calculated if (1) a Pentad exists and (2) there is no single Element at either DE or EF.



[18] 220x172 IMPERFECT FROM [19] 98x96

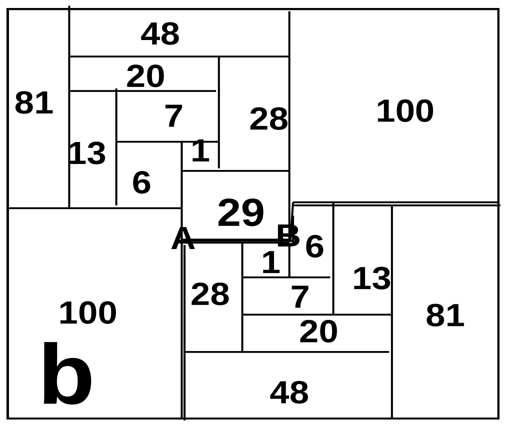
E14.13. EXTENSION OF CONVERSION ABOVE - CODES a AND b

Look at the line CB in the 2nd and 3rd diagram E14.11 representing 2 or more Elements.
Now what happens when CB is just one Element as in the case of a S2223 or S2224 original?
If the 2nd diagram is applied to [11] 205x181 below 2 is obtained with an added Element of 29. Fine! But applying the 3rd diagram under E14.11 an Element 5 has to be repeated for the conversion to work, and the result is Compound: although rather surprising the next section deals with a slightly different construction.
But now look at below 3 and observe the different position of the added Element 19!
Note also that the pair is symmetric twins. This property works with any S2223 to S2229 solution but only if there is not a single Element at AB and/or CD which either result in a one result or none, instead of two.
See also Twins Section G. The Formula for either is [o] m X n to [2o - 3] 2m - n X n.

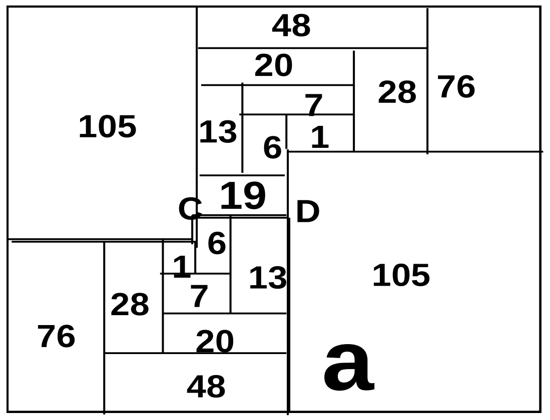


[11] 205 x 181

TO -



[19] 229 x 181 (1)



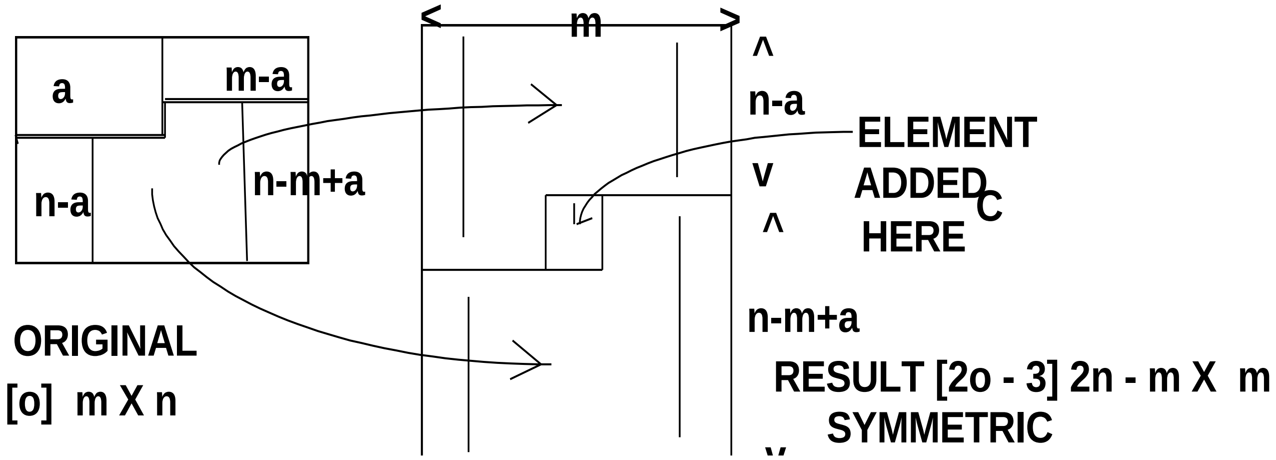
[19] 229 x 181 (2)

E14.14. ANOTHER SYMMETRIC JOIN

This seems a repeat of a previous join where sides are added, but in the following the two added Elements are already present in the original solution which has sides S222x. Note Below that a S2223 solution works equally well.

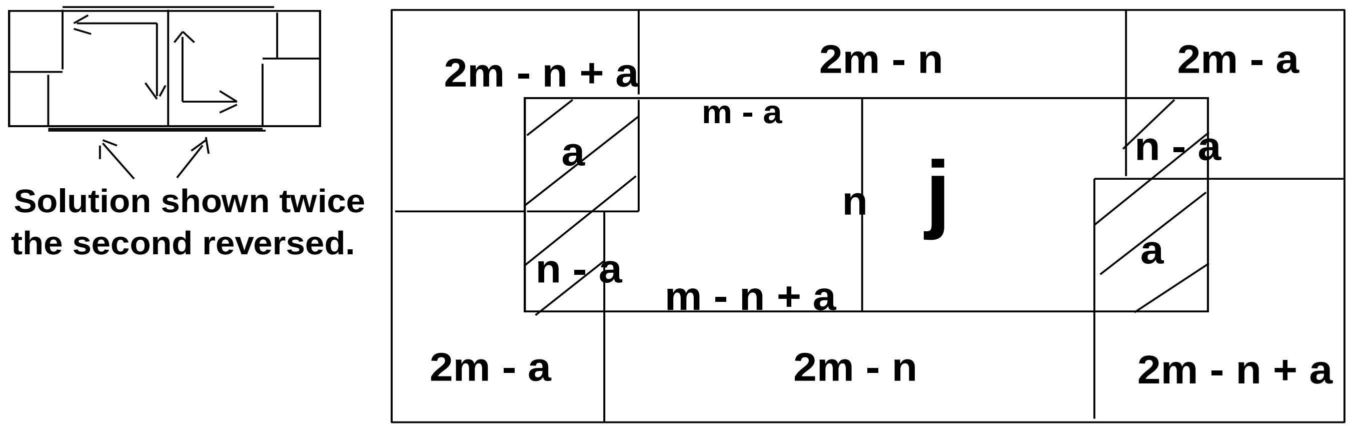
It is essential that there is no Element between A and m - A as this join never works with a Triad arch. This is because of the extra Element necessary.

The pattern works for any Diad arch of S2223 or S222x and the result is symmetric. An example is easy to construct and has not been shown. However the algebra here will be seen to agree.



E14.15. DOUBLE SYMMETRIC JOIN - CODE j

In E2.2 it was seen that if any Symmetry 2 solution of Sides 2 4+ 2 4+ was taken, a further Symmetry 2 solution could always be obtained by adding two side elements and suitably increasing the four end ones. So, if any solution containing a side of 2 is taken and then repeated upside-down on the right as Below 1, a Symmetry 2 solution is created albeit a Compound one. By adding two side elements and suitably increasing the four end ones, the new construction does fit -



From [o] m X n to [2o + 2] (6m - 2n) X (4m - n)
If element n is inserted first, a solution [2o + 3] (6m + n) X (4m + n) is also obtained!

E14.15.1. PROOF AND DIMENSIONS OF DOUBLE SYMMETRY JOIN

Close examination of above 2 will show that the algebra agrees in each direction in the corner elements.

Starting with the usual solution denoted by [o] m X n, further Symmetry solutions of [2o + 2] (6m - 2n) X (4m - n) and [2o + 3] (6m + n) X (4m + n) arise.

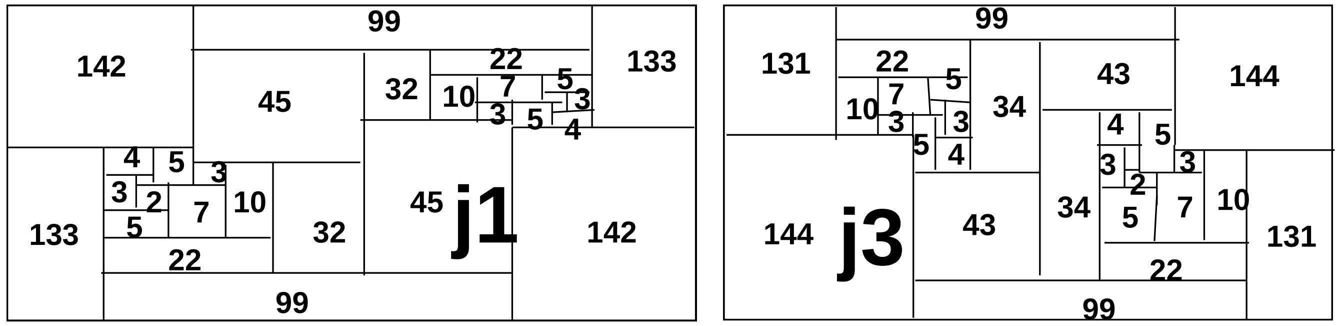
The second is found by adding an element n before adding the side elements. Both solutions are now Simple.

E14.16. FURTHER DOUBLE SYMMETRY JOINS

Of course a solution may have 2 opposite sides containing two elements, so two sets of solutions may be found.

The interesting thing is that they are always found to be twins! From the solution [14] 88 x 77 which has end elements 43 and 34 on one side and 45 and 32 on the other [30] 374 x 275 (1) and (2) are found - see below.

Actually there is a further solution also using the ends 43 and 45 as the solution is Sides S2223 (not shown).



[30] 374 x 275 (1) & (2) both constructed from [14] 88 x 77.

This type of construction is remarkable not so much from the type of Rectangle found, but the huge number of solutions it can produce as there are so many solutions with sides of 2.

Six (3 x 2 types) can be produced from any S222* solution alone - including two sets of twins.

Of all the Add-Ons and Joins possible, this one must produce the greatest number of new solutions from known ones!

E14.17. DOUBLE SYMMETRY JOINS USING PENTADS, OCTADS AND UNDECADS

If a solution has a Pentad end, then a similar construction with added side Elements and recalculated Pentad end is found to be possible. However up-rating of the elements by 2 or by 4 is often necessary to avoid fractions occurring.

The same is true of Octad and Undecad Endings except that Up-rating by 3 5 or 15 is often necessary in the first and up-rating by 56 or a factor of 56 in the second.

Note that solutions containing Triads can be used, but only the Diad part increases in size, not the whole Triad which is not a symmetrical construction on its own.

* LOOK AT OTHER CONSTRUCTIONS e.g. DOUBLE TRIAD ETC. - Do these work too?

E14.18. DOUBLE JOINS OF ASYMMETRIC TYPE

It is found possible to use two different solutions instead of the same one repeated, provided each has two end Elements exactly the same. Simple algebra proves that the Construction will always fit exactly.

The added side elements are always found to be equal as before, as are the two recalculated ends.

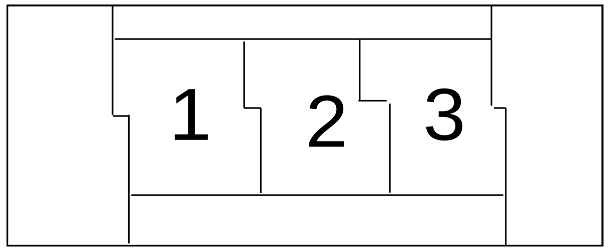
The Results though obviously always Imperfect, are Asymmetric and an improvement on the usual type.

The two solutions initially used do not have to be of the same Order.

E15. CREATING INFINITE SERIES

E15.1. REPEATED SYMMETRY 2

Another block addition which can be made is illustrated without examples below. the series shown can be produced ad-infinitum!



SYMMETRIC RECTANGLE
CREATED BY MULTIPLE ADDITIONS
OF ELEMENTS (1,2,3...)
THE ORDERS PROCEED BY
ARITHMETICAL PROGRESSION

E16. CREATING TRIAL & ERROR SOLUTIONS

E16.1. USING TRIAL & ERROR PROCESSES

Use of empirical methods can pay off in this subject. At first sight, using trial and error seems mad but features which simplify this procedure also increase the possibility of finding a Rectangle, although they are usually very elongated, and almost always Imperfect. The procedure is to -

1. Start at left end with an initial pattern.
2. Insert any numbers preferably small say between 10 and 50.
3. Calculate some Elements and proceed to right.
4. Continue until hopefully the structure ends in a Rectangle.
5. At any time the construction contains a single step. Sometimes the pattern fails and one gets stuck, but after a number of attempts, it is possible to get something. In the first 9 successful Rectangles found by me 7 were symmetric.

As the construction always ends in a Triad, the trick is to start the system without using a Triad as Below.

Also it can be helpful to have an odd lower dimension, such as 33 below.

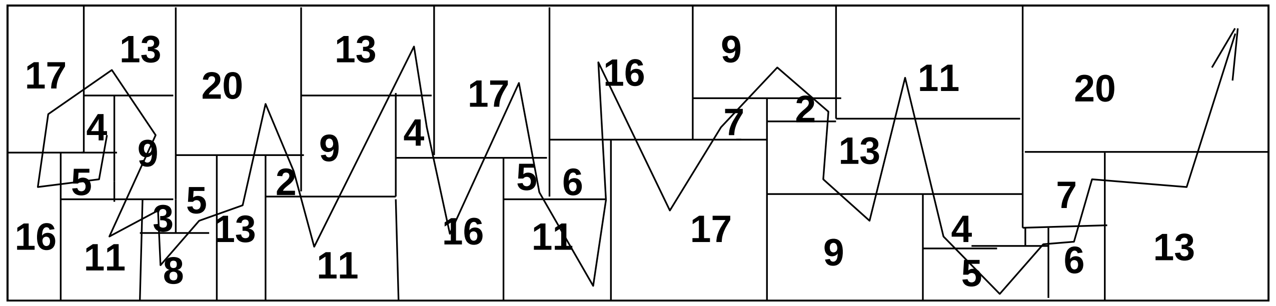
This helps to avoids Mid-plus and Symmetric solutions occurring too often.

E16.2. EXAMPLES FOUND BY TRIAL & ERROR

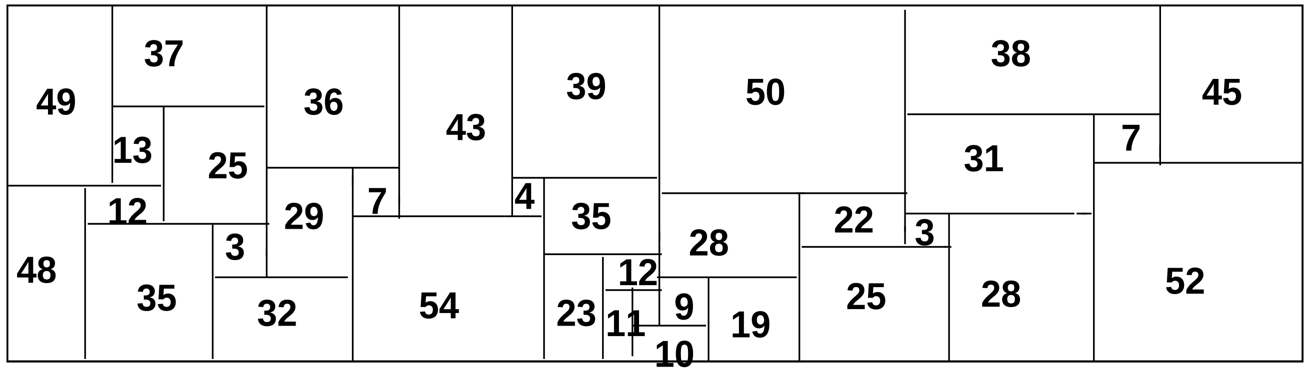
The trail line below shows how the Rectangle was constructed.

One drawback of this type of construction is that the Reduction index is never known without resorting to algebra.

4,5,9,13,17,16,11....& FOLLOW THE LINE THROUGH TO 20.



[38] 136 x 33 IMPERFECT FOUND BY TRIAL & ERROR!



[35] 337 x 97 IMPERFECT ENZ FOUND BY TRIAL & ERROR

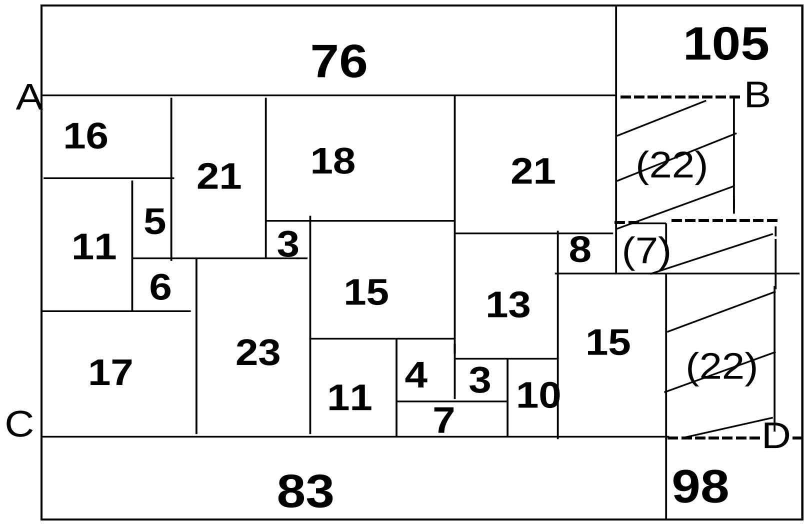
E16.3. CALCULATING SOME ASYMMETRIC RECTANGLES BY TRIAL

If the ending chosen is an even number and a series continued in the hope of two equal Elements occurring as below, then four external Elements can be added to obtain an asymmetric Rectangle.

Note that the connecting Element 7 is dropped as well as the two Mid Elements.

Again, the Reduction index is unknown, and pot luck is taken as to which Order will occur.

However this is an easy way to avoid nasty algebra!



ORIGINAL CONSTRUCTION
(ABCD)
SHADED ELEMENTS REMOVED
A "COVER" OF FOUR ELEMENTS
ADDED.
FOR THIS CONSTRUCTION TO
WORK AC MUST BE EVEN (44)

[23] 203 x 181

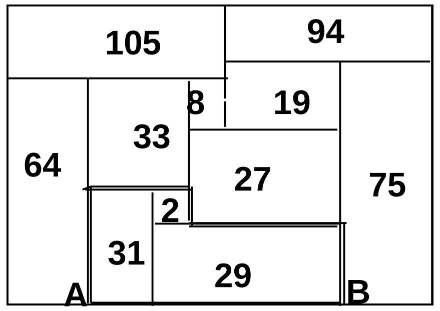
E16.4. A NEW SET OF LINKS

It was seen in Section E that an external Triad can always be converted into an Octad with recalculation.

But supposing the Triad is on the side between a S2224 Arch. Yes- the Link holds good! Recalculation is necessary in the Result (Below 2) and often all Elements will need to up-rated by 3 or 5 or 15. That is, unless AB happens to be divisible by 15 as in below 1 resulting in a solution with Order up five, yet slightly smaller than the original!

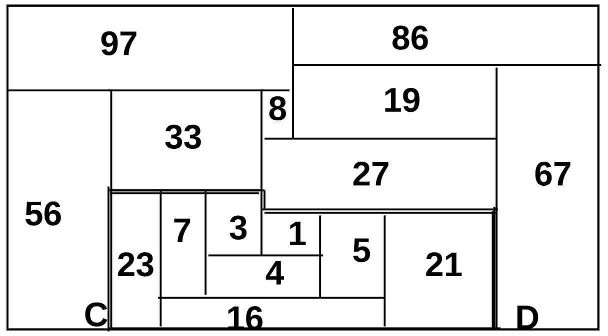
As the Result is still a Diad Arch it is always possible to replace it with a Pentad - see below 3, and this is shown as it is an incredibly reduced Order 17 solution - possibly the smallest Perfect Order 17 in <u>unit area</u>.

Note that the Centre Portion stays the same in each here - although in other solutions it keeps the same proportions but will be up-rated by 3 5 or 15. This is also true of the lines AB CD and EF (60 in size).



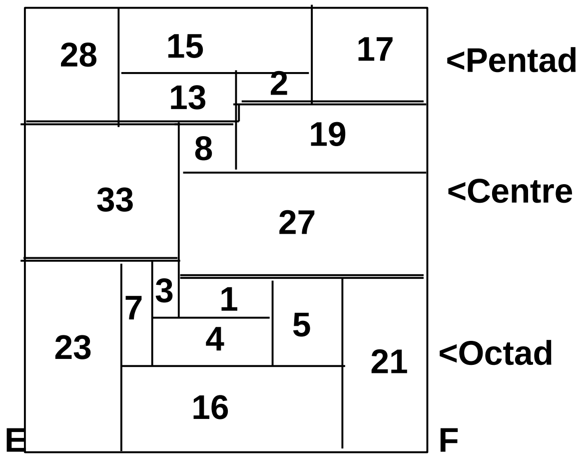
[11] 199 x 169 Perfect

Note the Triad lying between the s2224 Arch



[16] 183 x 153 Perfect

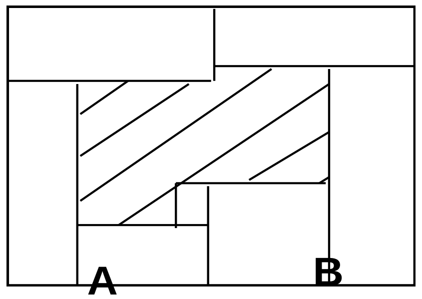
This is changed into an Octad between a s2225 Arch



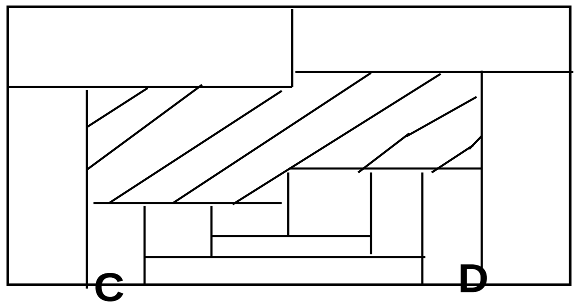
[17] 84 x 60 ¬ Perfect

E16.5. THE FORMULA

The algebra has not been shown, but the reader can verify if necessary that the Original [o] m X n does result in [o+5] 7m + 8n X -8m +23n.

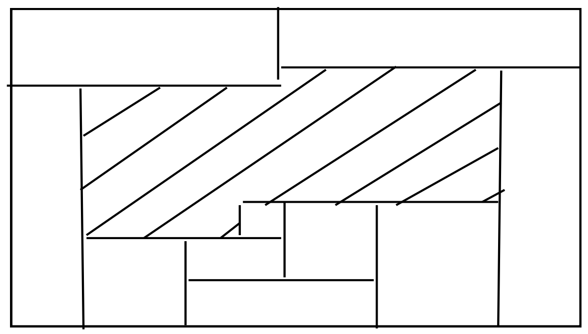


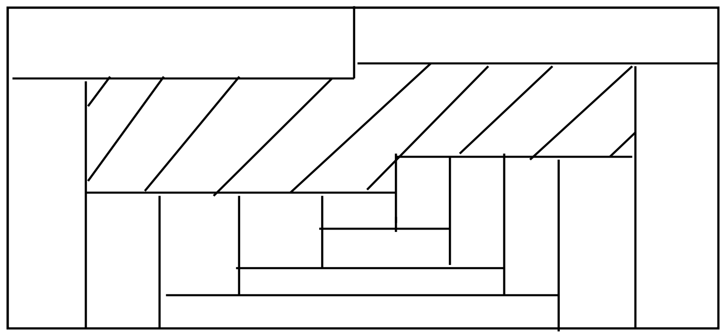
[o] m X n



to [o+5] 7m + 8n x -8m + 23n divided
by 1 or 3 or 5 or 15 according to Solution.

(Further solutions by adding Elements at AB & CD also possible)



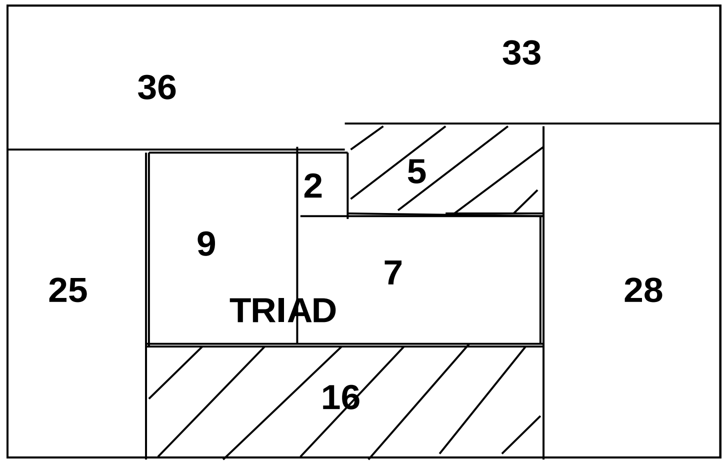


E16.6. TRIAD TO OCTAD IN s2223 SOLUTIONS

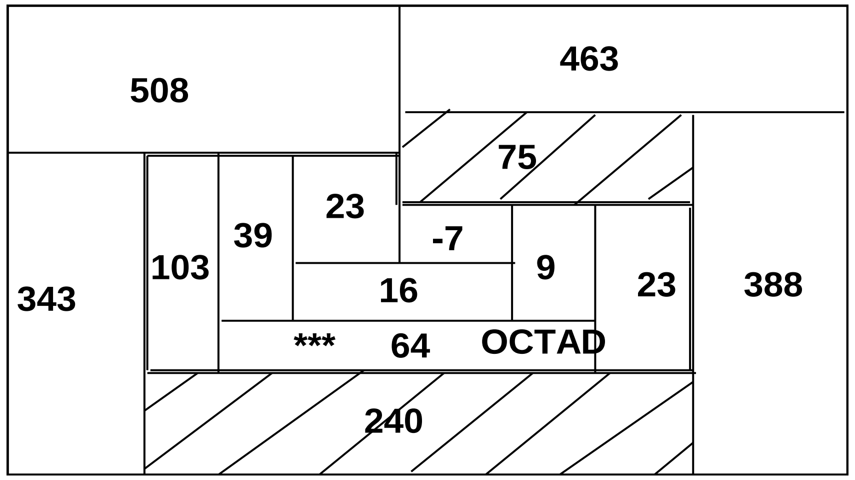
In view of the last Section this may seem unnecessary since an extra Element can be added at AB and CD in the above. However not all S2223 solutions can, in reverse, be reduced to S222X ones as in [9] 69 x 61 Below, and new solutions such as [14] 971 x 851 found.

This awkward Formula is actually easy to apply! Just multiply 69 and 61 by 15 and deduct 64 (*** in above 2) from each to get 971 x 851. As usual, the Dimensions will reduce 3 5 or 15 times in some solutions.

When division by 15 is possible, the Result is a little smaller than the Original despite the Order increasing by 5, and the Elements within the shaded areas for such solutions are identical, with the four corner Elements each slightly smaller.



[9] 69 x 61 to



[14] 971 x 851 Values of shaded elements up 15 times

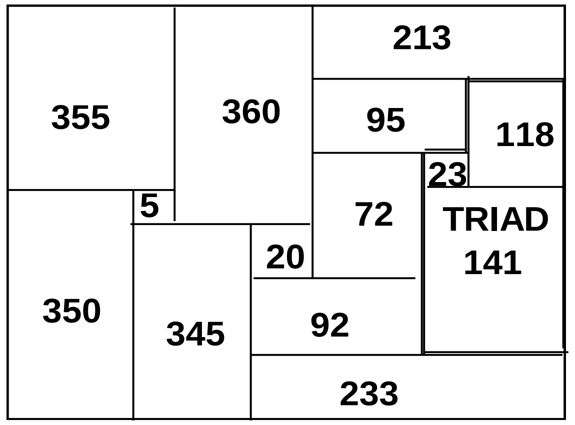
FORMULA [o] m X n to [o+5] 15m - 8/15 (m - n) X 15n - 8/15 (m - n) or divided by 3 5 or 15.

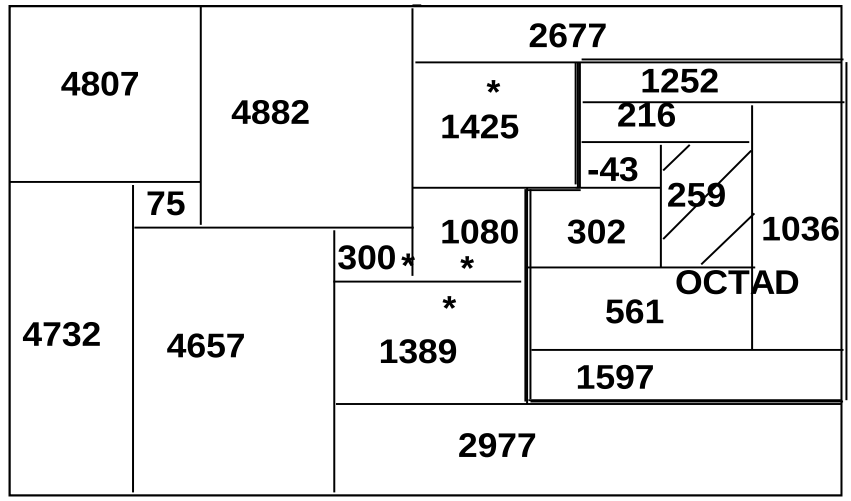
E16.7. OTHER LINKS SIMILAR TO ABOVE

Any regular pattern providing it has an extra linking Element nestling inside an Arch solution can be converted into another. e.g. Above 3 and 4 where an Arch Pentad + will convert into an Arch Undecad .

Similarly an Arch Double + can be converted to an Arch Double Claw as below. There are many other patterns rare enough to be ignored for the purposes of this book.

As always the proportions AB: BC: CD remain the same with AF smaller than AE. This link needs total recalculation but by arithmetic with a bit of algebra. How the Formula is found has not been shown, but is calculated using the facts that 12366 = 928 x 15 less 6 x 259 and 9569 = 705 x 15 less 4 x 259, 259 being shaded Below.





[14] 928 x 705 to [19] 12366 x 9539

Formula : [o] m X n to [o+5] 27m - 18n X 8m + 3n

TRIAD REPLACED BY OCTAD

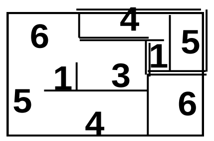
* These elements (but not the rest) are 15 times the respective ones on the left.

E16.8. SUBSTITUTING OCTADS FOR TRIADS IN OTHER PLACES

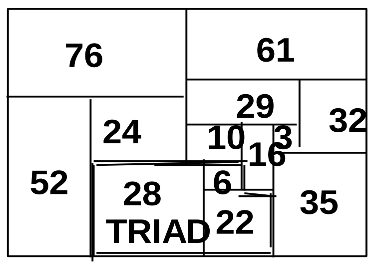
Most solutions contain several triples of Elements of Triad shape. Obviously replacing Triads with Octads will give normal Solutions, but often the solution will not fit from an Octad calculated with the same proportions, e.g. in below 1.

However the solution in below 2 does fit, though possibly by coincidence since other similar constructions do not.

Subject to further checking it seems that some degree of symmetry in the Rectangle is necessary for the solution to fit round the Octad calculated with the same proportions.

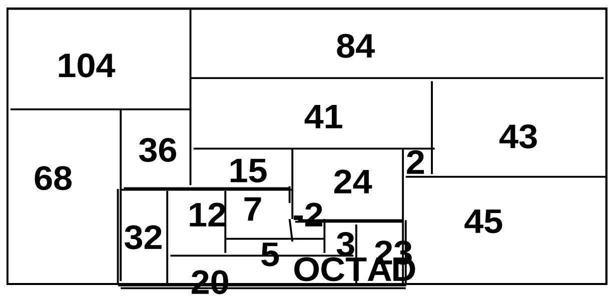


[9] 15 x 11



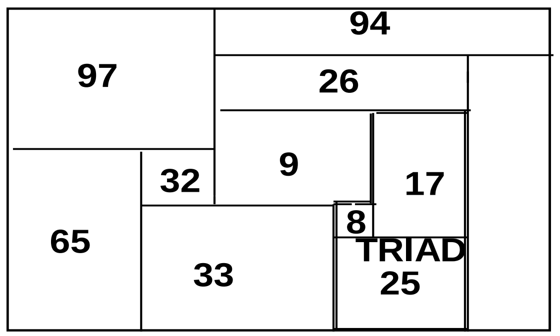
[13] 137 x 128

Replacement with Octad of same proportions does not work.

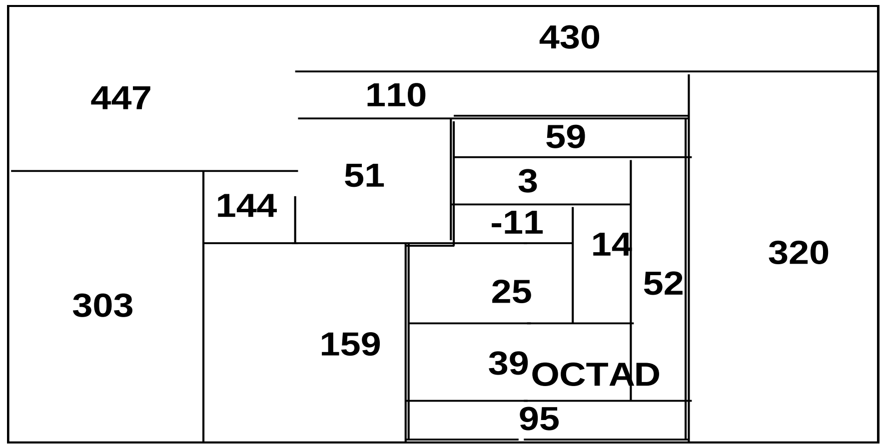


[18] 188 x 172 Does work from Octad having same proportions ie 34:6:16

This pattern below fits but I have not found any Formula for it:-



[11] 191 x 162 Note Triad chosen.



[16] 877 x 750

* More work needed to establish when patterns fit and when they do not. If they do is there a valid formula possible which will apply to other cases?

E17. WORKING OUT LINKING SOLUTIONS

E17.1. CONVERTING A SOLUTION INTO MORE SOLUTIONS

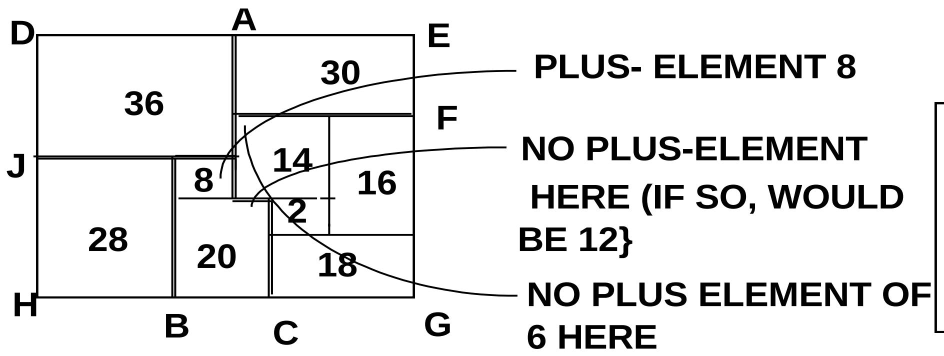
Using the ADDS, ENDS AND JOINS mentioned earlier it is possible to convert some solution into many more. However there are solutions of sides 3333 and greater which give no linking solutions at all, and others give very few.

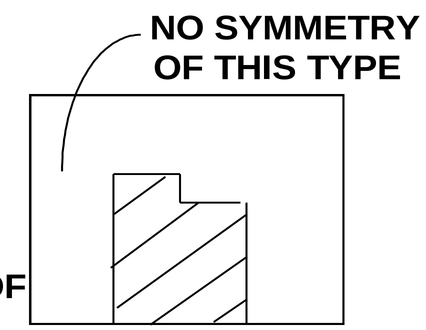
If Squared-squares are being sought - then surprisingly there are an unlimited amount of solutions possible from absolutely any given solution! None of these are however the most desired types.

If symmetric solutions are wanted - then again an unlimited amount of solutions are possible - all highly imperfect and mostly with high Orders. Although some symmetric types can be converted into slightly more desirable asymmetric ones, Add-ons tend to create symmetric solutions which don’t! (check this out).

E17.2. ANALYSING THE STARTING SOLUTION

What needs to be looked at to establish what links (adds, ends and joins) apply to a given Rectangle? Say [9] 66x64 Below.





1. We look for anything symmetric - an end, an arch, an internal bit or whatever. In Above 1, Elements 36 and 28 are a Diad (yes, this is symmetric!) also 30 and 36 are a Diad. Also AECG gives a Pentad. We have two Diads and one Pentad. However there is no arch type symmetry as above 2 in this solution [9] 66x64 (33x32)

2. Secondly, do plus Elements adjoin the symmetric portions? If so this increases the amount of linking solutions. Above, 8 adjoins one Diad, but there is no 6 adjoining the other and no 12 (i.e. 14-2) adjoining the Pentad.
We have one Diad, one Diad+ and one Pentad.
3. Now observe the remaining areas. Outside the Pentad is ADHC outside Diad DEFJ is JFGH. Outside the other Diad is AEGH. Does a single Element occur anywhere? In this solution there are in all three.
Using (s) for single we have 1 Diad(s) 1 Diad+(s) and 1 Pentad(s).

E17.3. ESTABLISHING THE LINKING SOLUTIONS

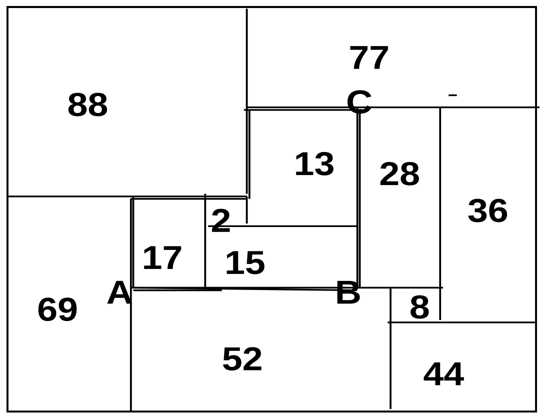
By reference to many parts of Section E -
Diad(s) Has no links apart from symmetric patterns.
Diad+(s) Has 3 links (would be 5 if not single). These happen to be [14] 480x431 [14] 488x403 and [14] 520x391
Pentad has a 1 link to a Diad arch (would be 2 if not single). This is [9] 69x61.
Now [9] 69x61 contains an arch (link already used) a Diad(s) and a Diad+(s) by the system indicated.
Only symmetric solutions arise from the Diad, but 3 more solutions from the Diad+(s). These happen to be [14] 915x13 [14] 260x197 and [14] 979x849.
More than 27 solutions can be found from [9] 66x64 with several Order 18 and 19.

E17.4. EFFECT ON THE ORDER IN LINKING SOLUTIONS

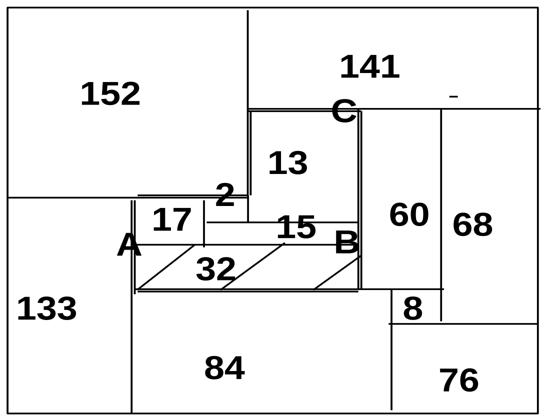
Results from given Rectangles vary widely, but frequently solutions with Orders up by five are found.
Other solutions may have same Orders, up by one or by four.
In symmetric or squares the Order often increases by two.
Obviously where links from a link from a link is taken the Orders can be much higher than the original solution.

E18. INTERNAL LINKS (& BRACKETS)

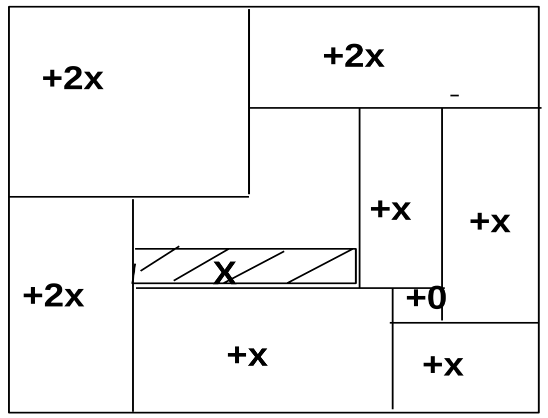
E18.1. AN INTERNAL LINK FOR S2233



[12] 165 x 157



[13] 293 x 285



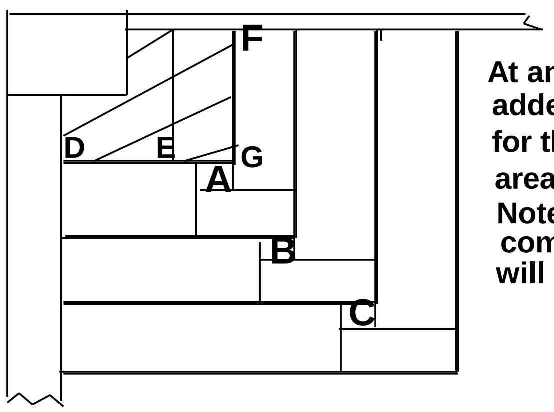
Effect on the outer Elements by adding an Element X as shown

Look at above 1 where the Elements bounded by AB & BC are identical with above 2. In above 2 the Element 32 has been added at AB and all the Outer Elements suitably increased some by 32 and some 64. The reason why this Add-On works is shown in above 3 where adding the Element X as shown increases some Elements by x and others by 2x. The whole Rectangle increases in size by 4x across and 4x down. Note that 165 - 157 is 8 and 293 - 285 is also 8 which is the value of the Element which remains unchanged in size.

For this Add-On to apply the Rectangle must contain the Elements in above 3. Also there must not be single Elements for either AB or BC.

E18.2. REPEATED INTERNAL LINKS FOR S2233

Similar Internal Links are possible by repeating the corners with repeated sets of four Elements in an L shape which will be referred to as Brackets.



At any stage, an Element can be added at the line DE without need for the Elements in the shaded area to change.
Note however that for each stage the composition of the shaded area will change.

So as well as with one Bracket, there is a linking solution also with two Brackets and with three, four and so on. In each case the link is created by adding the Element at the line DE.

Now the value of Element A is calculated as follows. Take DG and deduct FG and divide by four. So if FG = 28 and DG = 60 the difference is 32 and ÷ by 4, 8 is obtained.

The reader may be able to calculate from the above that if A is 8 then B will be 4 and C will be 2.
Or more importantly, <u>B is always half of A, and C is half of B and so on!</u>

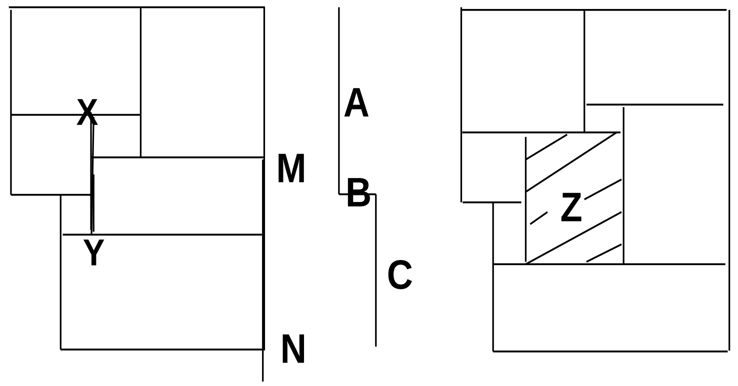
In calculating these type of Rectangles where x and y are sufficient to calculate the shaded area it seems necessary to use z and possibly more unknowns to calculate the series of Brackets. But since A, B and C etc. can be calculated by the simple formula mentioned so can the other Elements comprising the Brackets by simply using x and y alone. The only drawback is that fractions occur in halves, quarters etc. However if we replace x and y with 2x & 2y or 4x and 4y etc. According to the number of Brackets used this problem is removed. If A is, say, 6x + 4y in our calculation then be can halve this for B which will be 3x + 2y. C would then be half of this, and so on.

E19. NO SYMMETRY LINKS

E19.1. A DIFFERENT TYPE OF LINK WITH NO PATTERN SYMMETRY

Mentioned in Section C19.6 the patterns below each have a RATIO FORMULA of 5A = 7B + 3C and so are interchangeable. An additional Element is added at XY in the first to obtain the second. If A B C are made the same size it is found necessary for Z to be three-fifths of XY.

This means total recalculation is necessary and Up-rating as well to convert from one to the other. Note that both patterns can be turned upside-down for another very similar link in which case the RATIO FORMULA 5C = 7B + 5A.



There will undoubtedly be many other Links possible of similar kind, but what separates this from all previously shown is that the pattern lacks Symmetry! Having said this, if an Element is added at MN then the OCTAD pattern is obtained and the pattern then symmetric - if an Element is added at "Z" the NONUD pattern is obtained (also Symmetric) and these two are related.

*** INVESTIGATE Are there links for all patterns where XY separates two Elements one above & one below?

E20. SUMMARY OF ADD-ONS

The codes are shown in large size in some of the diagrams though out this section.

Sym 1 means the resulting solutions have no symmetry. Div 1 2 4 means that the formula for the new dimensions can sometimes be ÷ by 2 or 4.

Diad+ is the same as Triad.

Code	From -	To -	Order o -	Dims m x n -	Type of symmetry
A2	Diad+ A1	Octad	o+5	15m x 15n-2m	sym 1
A3	Diad+	Pentad-cloak	o+5	8m+8n x 5m+7n div 1 2 4	sym 1
A4	Diad+	Diad-6 cloak	o+5	7m+8n x 6m+7n	sym 1
A5	Diad+	Pentad-arch	o+4	8m x 7m-2n div 1 2 4	sym 1
A6	Diad+	Diad-5 cloak	o+4	8m-n x 7m-n	sym 1

F2 Pentad+ F1 Undecad o+5 div 1 ...56 sym 1
F3 Pentad+ Octad-cloak o+5 div 1 3 5 15 sym 1
F4 Pentad+ Pentad-6 cloak o+5 div 1 2 4 sym 1
F5 Pentad+ Octad-arch o+4 div 1 3 5 15 sym 1
F6 Pentad+ Pentad-5 cloak o+4 div 1 2 4 sym 1

B2 double+ B1 Undecid o+5 sym 1
B3 double+ Octid-cloak o+5 sym 1
B4 double+ double-6 cloak o+5 sym 1
B5 double+ Octid-arch o+4 sym 1
B6 double+ double-5 cloak o+4 sym 1

N2 Diad-arch N1 Pentad N2 o+1 2m-2n x m+n div 1 2 4 sym 1
N3 Diad-arch N1 Diad-cloak N3 o+1 4m-3n x 5m-4n sym 1
Septid sym 1
double-cloak sym 1
Octid sym 1

I .2.3.2.3. 2323 internal sym 1
I 2.3.2.4+ 2324 internal sym 1
I .2.4.2.4. 2424 internal sym 1

---

sym 1

Diad-arch Midplus 2o-4 sym 2
Diad-cloak Midplus 2o-6 sym 3
Triad-arch 2o-8 sym 2
Triad-cloak 2o-4 sym 3

Sym x|4+|x|4+ Sym x|3+|x|3+ o+2 sym 3

Diad Mid 2o+2 2m+n x 2m+2n sym 3
Diad Midplus 2o+4 3m+2n x 2m+2n sym 3
Diad Mid 2o+2 3m-n x 2m sym 3
Diad Midplus 2o 2m-n x 2m sym 3

---

a	2 2 3+ 3+	2o-3	sym 3
b	2 2 3+ 3+	2o-3	sym 3
	Diad/Triad-arch	2o-3	sym 3
(		2o-3	sym

F. SYMMETRIC SOLUTIONS

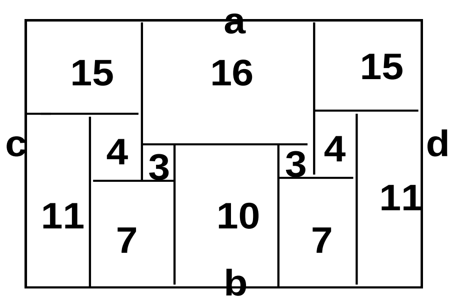
F1. TO F10. SYMMETRY 2 SOLUTIONS

F1.1. SYMMETRY 2 AND 3 SOLUTIONS

Two types of Twofold Symmetry were discussed in Section A and all aspects of these will be dealt with in Section G although references are throughout this Book. The Links affecting Symmetric solutions in Section E are referred to again here.

F1.2. EXAMPLE OF SYMMETRY 2 TYPE

In x Type symmetry, half the solution is upside-down in relation to the other, but in = Type, the halves are placed adjacent to each other. Whether the halves divide horizontally or vertically (as below) depends on the solution, but as solutions are often quite elongated, the vertical division is more common.



[12] 46 x 26 Symm 2
Note vertical ab divides this solution in two identical halves.
In some solutions the division is horizonal at cd.
This is the smallest Valid solution of this Symmetry type.

F1.3. CATEGORIES OF SYMMETRY 2 SOLUTIONS

No Perfect solutions are possible as by definition most elements are duplicated, and the solutions always highly imperfect.
Solution can, and often are Invalid. They may have adjacent Elements or contain zero Elements.
Squared-squares also exist, but not for smaller Orders, and not easy to find.

F1.4. QUANTITY OF SYMMETRY 2

In view of the huge quantity of all solutions possible to any given Order, the proportion of Symmetry 3 (= type) Solutions is <u>very small</u> and much smaller than would have been expected.
This is particularly true when all Invalid solutions are excluded.
Up to Order 14 only the following exist [12] 46 x 26 [14] 85 x 48 Are there any others?

F1.5. DIMENSIONS OF SYMMETRY 2

Once a Catalogue of Squared-Rectangles has been assembled, it is at once obvious that Symmetry 3 Solutions are <u>always small</u> where Reduced Dimensions are concerned.

F1.6. SIDE INDEXES FOR SYMMETRY 2

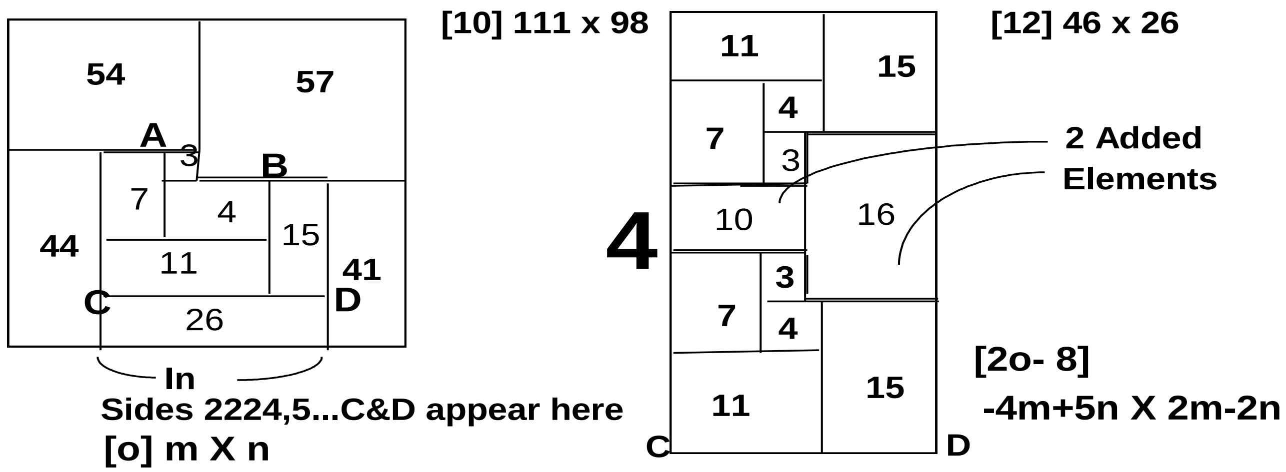
The Side Indexes which are possible for Valid Symmetry 3 solutions are restricted to s2323 s2325 s2327… s2525 S2527 S2529… S2727 S2729…. S3333 S3335 S3337... and some others. Other Side Indexes are also possible once Invalid Solutions are included.

F1.7. SYMMETRY 2 AND xyz UNKNOWNS

There appears to be an above average incidence of higher unknowns.
The above [12] 46 x 26 is xy. [14] 85 x 48 is xyz.

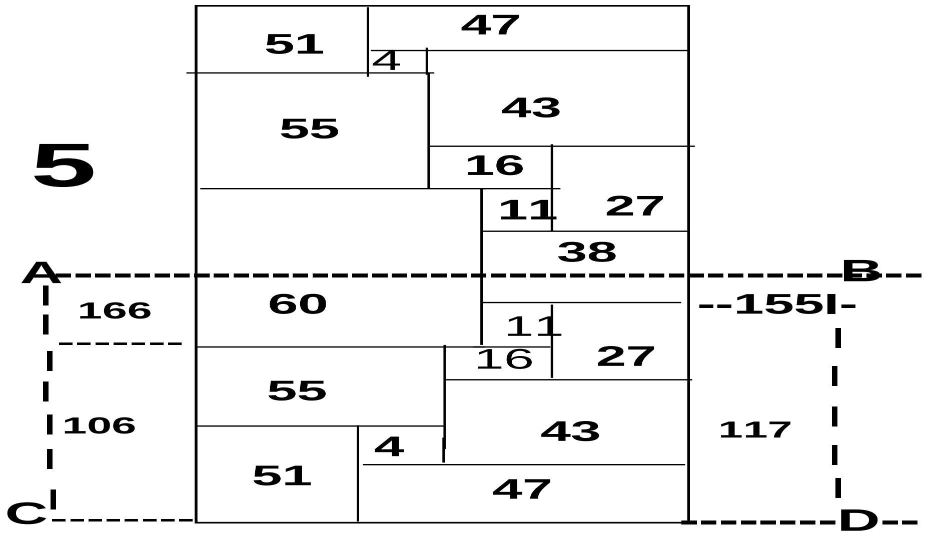
F2.1. FIRST WAY OF CONSTRUCTING SYMMETRY 2 SOLUTIONS - CODE 4

Once some Links (Section E) are known, the construction of these relatively elusive solutions becomes a lot easier and makes more sense. But it appears that some solutions are not easily linked and many may not have any possible link with other solutions.
In E14.8 any solution with sides s2223* which does not have a single Element bordering A and B Below can be converted in Symmetry 3. In this construction the Resultant solution must be substantially smaller than the Original despite an increase in Order. The Reduction in [12] 46 x 26 is 8. Sides S2224 S2225 S2226… are also possible - see next Section.



F3.1. SYMMETRIC SOLUTIONS FROM TRIAD 2224+ TRIAD ARCH SOLUTIONS - CODE 5

Below [12] 321 x 272 is shown as ABCD. The outer Elements 106, 166, 155 and 117 have been stripped away, and the Elements 60 and 38 added as shown.
The remaining Elements in ABCD have been repeated to form a symmetric solution. The Formula for this is [o] m X n to [2o - 6] n X 2m - 2n.



F3.2. SHOWING THE GENERAL FORMAT

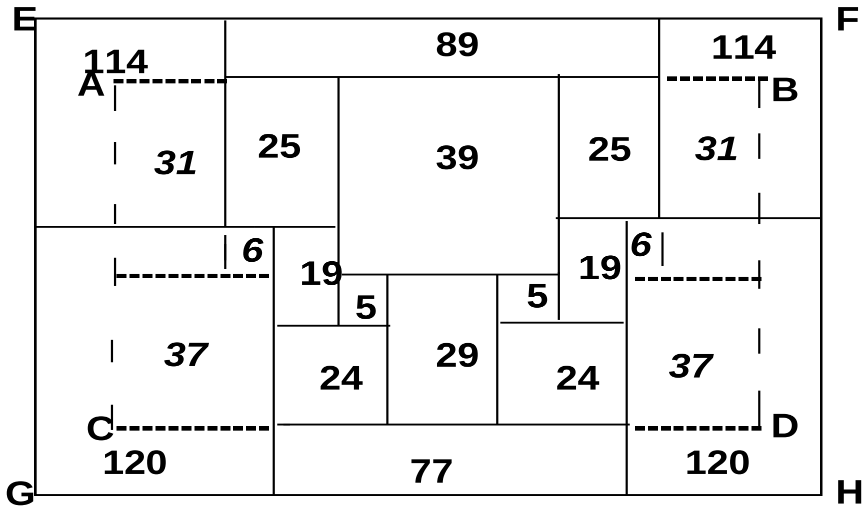
Whether or not the Original Solution is a Diad-arch or Triad-arch does not matter, but in the latter elements will appear as shown. Neither AB nor CD must form a single Element.

F3.3. SOLUTIONS FOUND FROM FIRST LINK

S2223 cases where Order [o] rises to [2o - 8] -
[10] 111 x 98 to [12] 46 x 26. [11] 205 x 181 to [14] 85 x48. [12] 113 x 98 to [16] 38 x 30
[12] 118 x 103 to [16] 43 x 30. [12] 162 x 142 to [16] 62 x 40. [12] 297 x 263 to [16] 127 x 68.
[12] 315 x 278 to [16] 130 x 74. [12] 339 x 299 to [16] 139 x 80. [12] 321 x 287 to [16] 151 x 68.
S2224+ cases where Order [o] rises to [2o - 6]
[12] 177 x 145 to [18] 145 x 64. [12] 326 x 267 to [18] 267 x 118. [12] 321 x 272 to [18] 272 x 98.
[12] 329 x 279 to [18] 279 x 100.

F4.1. SECOND WAY OF CONSTRUCTING SYMMETRY 2 SOLUTIONS - CODE 7

This is easily shown below. It requires a known Symmetry 2 Solution with Diad+ (or Triad) ends, but never works for Diad ends.



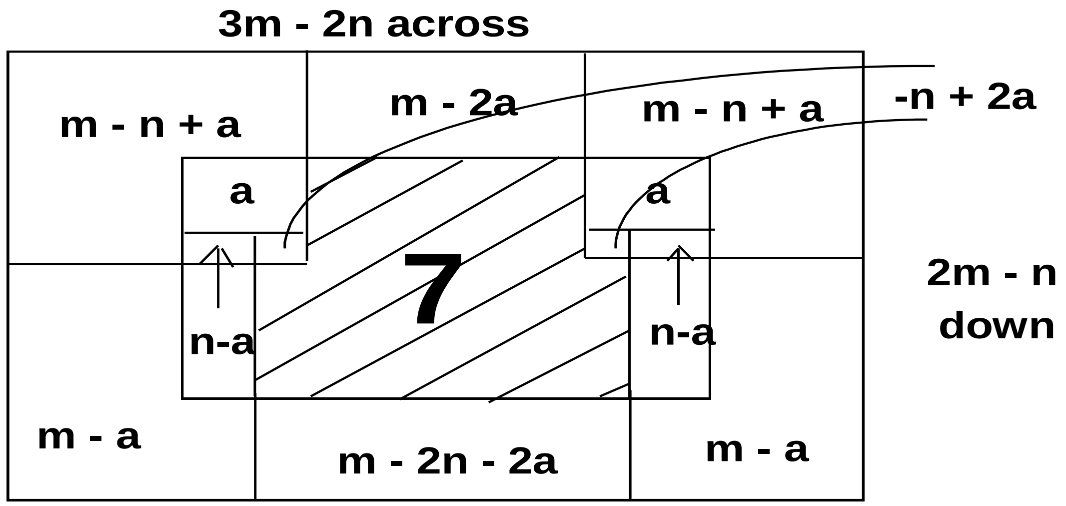
In [16] 151 x 68 the triads of 6 31 & 37 are stripped away & replaced with 114 (x2) 120 (x2) 89 & 77 to [16] 317 x 234.
The Resultant Rectangle has always Sides 23

SYMMETRY 2 SOLUTION TO ANOTHER

ABCD to EFGH

7

This construction is proved true by algebra as shown-



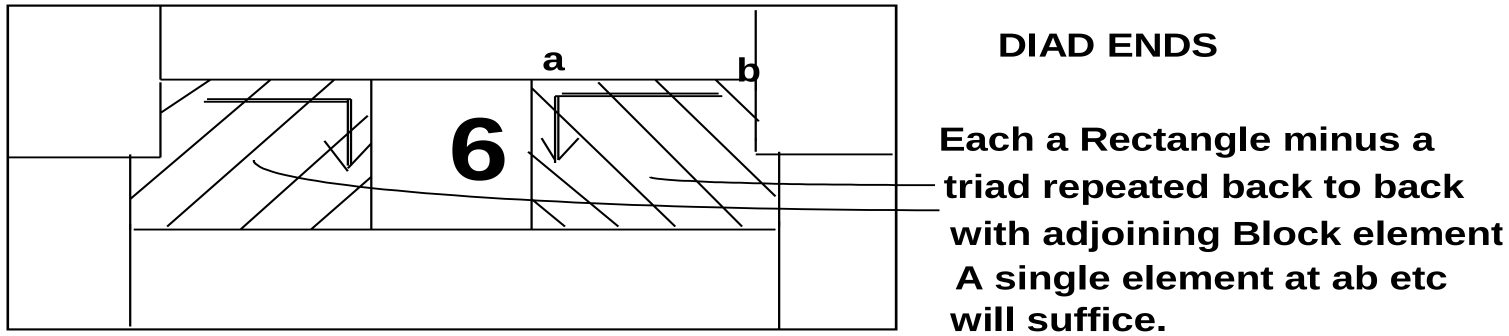
[o] m x n to [o] 3m - 2n by 2m - n of same Order but Sides 2323.

Solutions found from the above link are -
[16] 151 x 68 to [16] 317 x 234 (from [11] 185 x 151) [18] 145 x 64 to [18] 307 x 226 (from [12] 177 x 145)
[18] 272 x 98 to [18] 620 x 456 (from [12] 321 x 272) [18] 279 x 100 to [18] 628 x 452 (from [12] 329 x 279)
The resultant solutions are seen to be less reduced than the original Symmetry 3 ones.

F5.1. THIRD WAY OF CONSTRUCTING SYMMETRY 3 SOLUTIONS- CODE 6

Although related to above, this method involves taking a known Triad solution and arranging it as below. Note the middle Block Element which is added to prevent the solution being invalid.

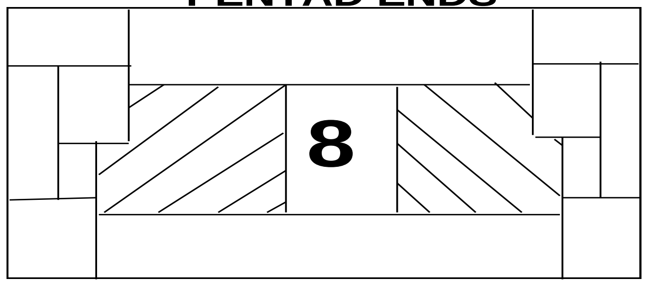
From the original Triad solution, the Formula is [o] m X n to [2o + 1] 6m + n X 4m + n where Triad is situated on left or right side, and [o] m X n to [2o+1] 6n + m X 4n + m (Triad at top).



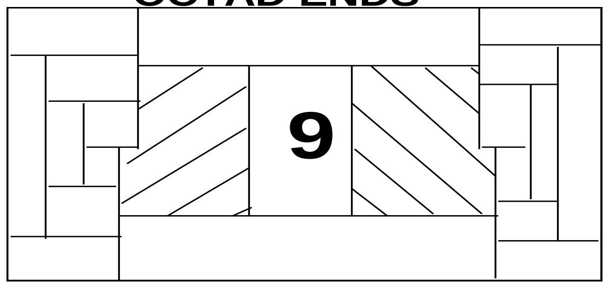
For similar patterns relating to Symmetry 3 please see F14.

The ends can be varied to Pentad or Octad as Below-

PENTAD ENDS



OCTAD ENDS



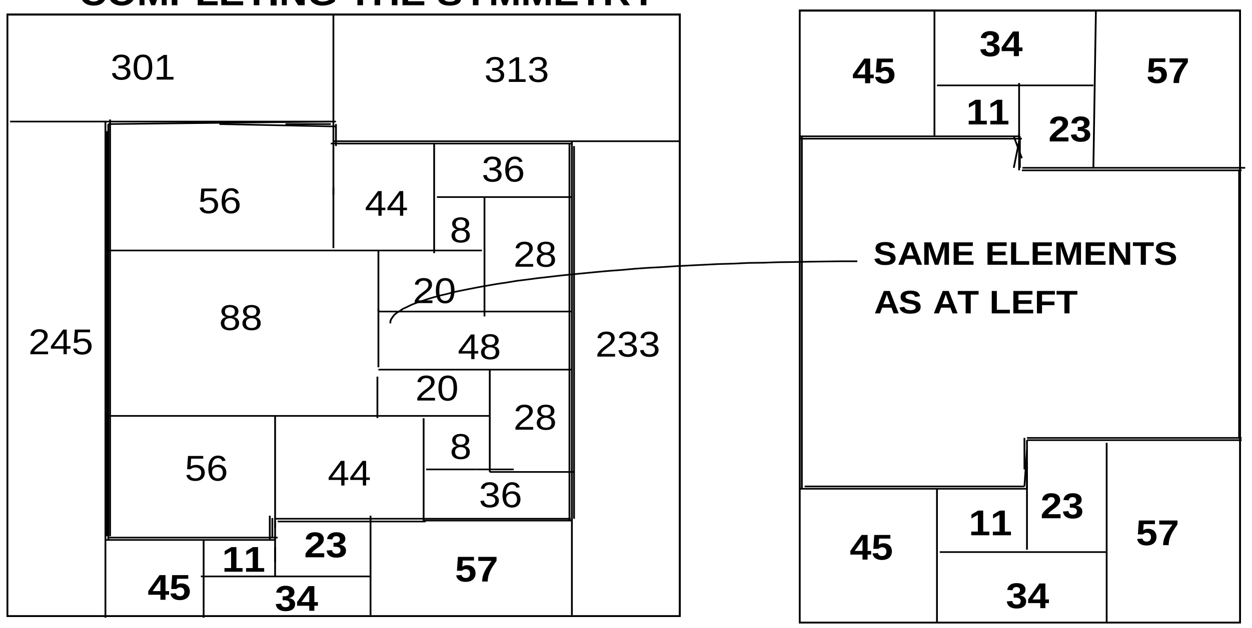
F5.2. DUPLICATED ELEMENTS IN PENTAD S222# SOLUTIONS

I noticed solutions of the type below 1 with a duplicated area of Elements.

This was puzzling until the relationship was made clear by below 2, where a second repeated Pentad has been added.

Note that Below 2 is symmetry 3 (unlike [14] 116 x 104 Above.

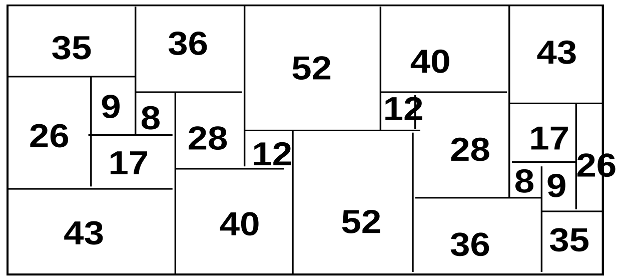
COMPLETING THE SYMMETRY

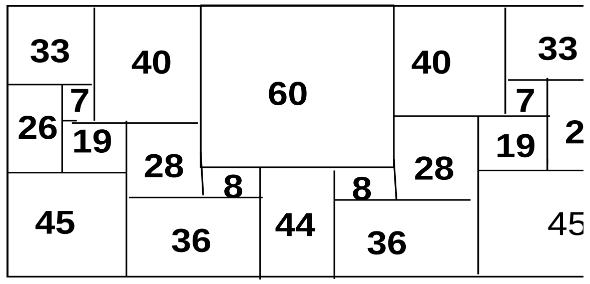


F6.1. FURTHER SOLUTIONS - SYMMETRIC

Below are two further solutions which can be obtained from the [12] 258 x 206 (1) and (2) and a range of other solutions are listed.

TWO MORE SOLUTIONS FROM [14] 116 x 104 - BOTH SYMMETRIC





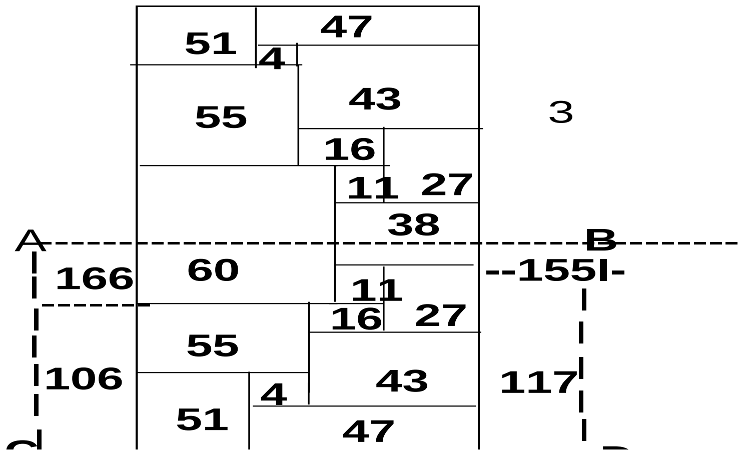
[22] 206 x104

[20] 198 x 104

From the solutions [13] 258 x 206 (1) & [13] 258 x 206 (2) there is [14] 11 6x10
There are also $2223 solutions [14] 466 x 414 (1) & [14] 466 x 414 (2)
There are the above [22] 206 x 144 & [20] 198 x 104 symmetric.
Replacing a pentad with a 'cover' above [21] 438x285 & [19] 422x371!

F7.1. SYMMETRIC SOLUTIONS FROM TRIAD S2224+ TRIAD ARCH SOLUTIONS - CODE 3

Below [12] 321 x 272 is shown as ABCD. The outer Elements 106, 166, 155 and 117 have been stripped away, and the Elements 60 and 38 added as shown.
The remaining Elements in ABCD have been repeated to form a symmetric solution.
The Formula to be calculated.



F8.1. TO F9. UNUSED AT MOMENT

Often symmetry 2 arises as in above 1, sometimes symmetry 3 as Above 2.
Refer also to below where patterns 1 or 3 will apply.

F10. TO G36. SYMMETRY 3 SOLUTIONS F10. SYMMETRY GENERAL PRINCIPLES

F10. LABELLING OF SYMMETRY 3 LINKS & RELATIONSHIP TO SYMMETRY 1 LINKS

To distinguish the Codes used for links the following has been adopted
1. All Asymmetric i.e. Symmetry 1 commences with a CAPITAL Codes A B C D to Z. 2. All Symmetry 2 Links commence with a NUMBER Codes 0 to 9

3. All Symmetry 3 Links commence with a lowercase Codes a to z. Codes a to m (inclusive) refer to Asymmetric forms converted to Symmetry 3 whereas n to z refer to Symmetry 3 to Symmetry 3 forms.
4. All Links resulting in Squares have Codes which commence with S.
(This is regardless of whether the Square is Symmetry 1, 2 or 4 and / or whether Original is a Rectangle or Square).
The Reader could check out that each of the Symmetry 3 Links "a" to "q" only (but not "r" to "z"), can in fact also be legitimately used as Asymmetric (Symmetry 1) Links, but only o p and q have been previously referred to in Section E - as I2 I6 and IA.
However, it is always easier to have or find suitable Original Solutions for these Links which are Symmetric. Generally TWO Asymmetric Solutions have to be known or found before any of these otherwise Symmetric Links can be employed - and the problem is that both have to comply with strict conditions and so not readily known or found, e.g. where two different Rectangles each with Diads of 35 and 59 only are required. This limits the usefulness and occurrence of Asymmetric type Results - which is why I have left them out in Section E. They are potentially useful - but in practice not that much.

F10.1. SYMMETRY 3, x TYPE

Symmetry 2 Solutions always consist of two halves, one being reversed and upside-down to the other - not adjacent as in Symmetry 3. Unlike Symmetry 2, Symmetry 3 solutions can be halved diagonally or halved vertically or halved horizontally and the elements comprising each half are always identical.

F10.2. CATEGORIES OF SYMMETRY 3 SOLUTIONS

By definition, Perfect solutions are impossible, all Solutions being highly imperfect. Many solutions are also invalid with either Zero Elements, adjacent Elements or both.
Squared-squares are rarely met with but also exist. For Even Orders every element is duplicated once, but in Odd Orders there is a single central Element with all others duplicated.

F10.3 QUANTITY OF SYMMETRY 3 SOLUTIONS

Once Invalid Solutions are ignored, Symmetry 3 solutions are proportionately unusual.
However they are much more common than Symmetry 3 ones.

F10.4 DIMENSIONS OF SYMMETRY 3 SOLUTIONS

As in Symmetry 3, the Dimensions of Rectangles are always well reduced and some reasons for this are seen in the type of Links which give rise to these Rectangles.

F10.5. xyz IN SYMMETRY 3 SOLUTIONS

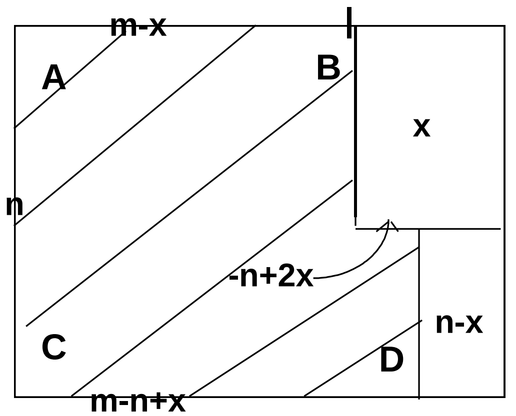
Solutions may be xy, xyz, xyza ... with a slight bias toward more Unknowns than average for Asymmetric ones.

F10.6. SIDES INDEXES IN SYMMETRY 3 SOLUTIONS

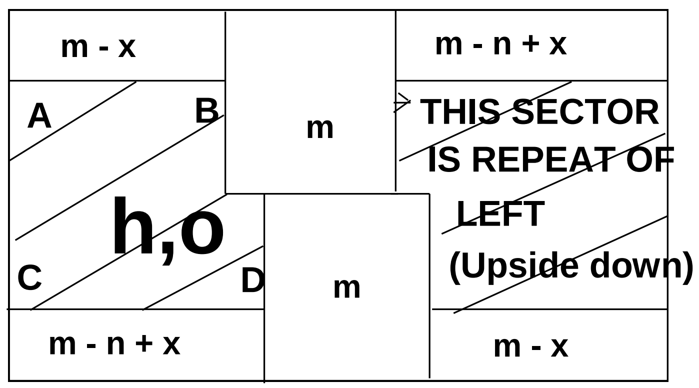
To make the following sections less confusing, these ones concern originals where the cover is stripped off and the earlier ones in E13 with sides added to the originals to obtain the resultant symmetric solution.

F10.7. DIAD ENDS TO SYMMETRIC SOLUTIONS - CODE h

Different rules apply to Triad ends even though part of a Triad is a Diad. Diads mean two Elements with no Element adjoining both.
If any solution with a Diad end is taken, it is possible to construct two symmetric solutions as below where the algebraic proof is given. Note the two added Elements (top and bottom), the Mid-plus Elements and the repeat on the right hand side. With Diads the "m" Elements can be removed for a second solution.

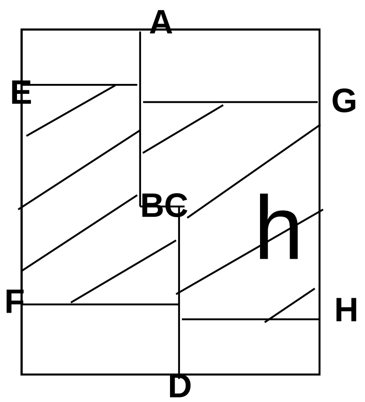


RECTANGLE [o] m X n
SHADED DIAD REMOVED

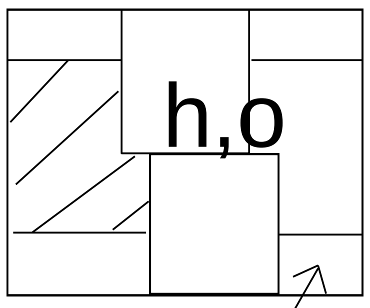


TO [2o+2] 3m - n X 2m (MIDPLUS)
or [2o] 2m - n X 2m (MID)

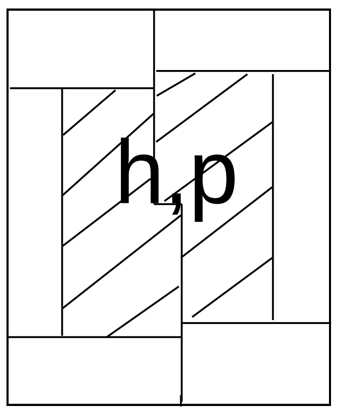
These Links are mentioned again in G11 and G12 q.v. as they involve two types of links.
A further Element can be added at AB in above 2 and repeated on the right, making four symmetric 3 type solutions possible from the original.
In some solutions, elements can be added internally and externally alternately for ever more.
In others, four patterns only are possible as shown



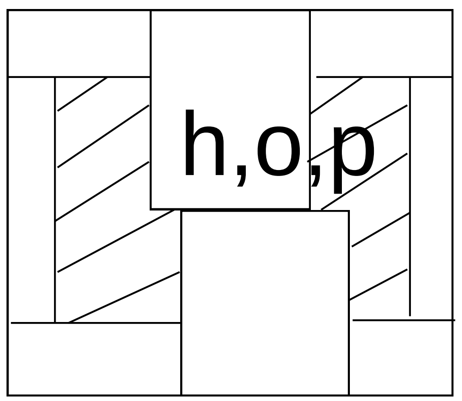
ANY MID SOLUTION
of s2323 (only this pattern possible)
or s2 4+ 2 4+ (all patterns possible)



AB & CD added
s 2+ 3 2+ 3



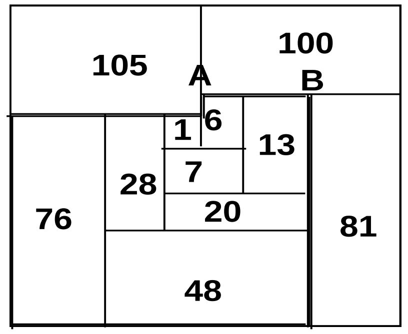
EF & GH added
s2323



AB/CD & EF/GH added
s3333

F11. SYMMETRY TYPE a

F11.1. This Link A and the two which follow (B and C) are somewhat similar and also rather confusing! However each involves a central added Element in a different position. A B and C all involve Original Solutions of type s222X but will equally work with type s22XX also. The Result is of an order twice as large minus three, e.g. Order 9 to Order 15 (2 x 9 - 3). Not every s222X or s22XX solution is suitable for this Link as AB cannot be a single element otherwise the Result is Invalid.

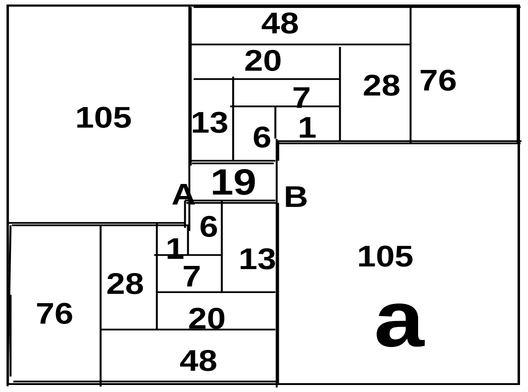


[11] 225 x 181
[o] m X n

TO-

In Link A there must not be a single element at AB as an Element is added here. The pattern is repeated upside-down & reversed at top right, and two Elements of same value added >>>

[2o-3] 2m-n X n



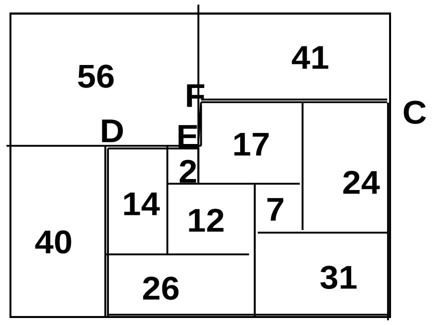
[19] 22 9 x181

F12. SYMMETRY TYPE b

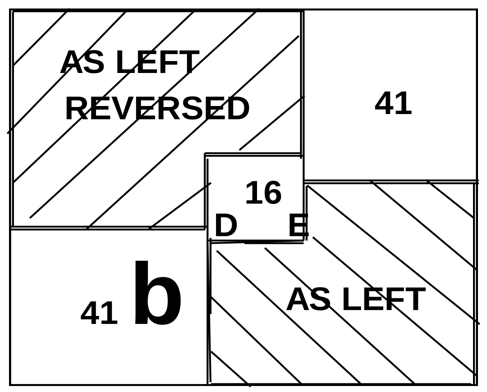
F12.1. ANOTHER WAY OF CREATING SYMMETRIC SOLUTIONS

This conversion involves solutions of sides S2 2 3+ 3+ which have three corner Elements forming an “L”.
Below [11] 97 x 96 is shown and two symmetric solutions of Order 19 from it, the second of which is Link C described in the next Section.

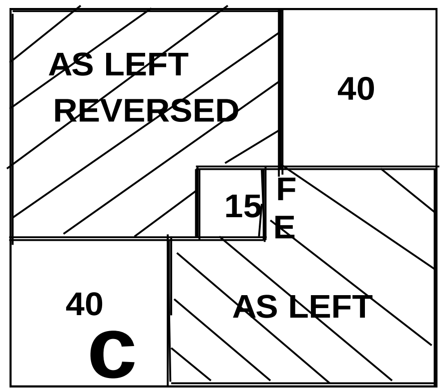
Although the dimensions for Links B and C are often close in size they are different, and not Twins as is the case for Links A and B. Compare with Below2. The Formula is [o] m X n to 2m - n X n.



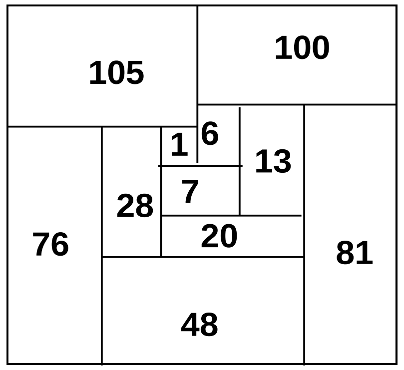
[11] 97x96



[19] 98x96

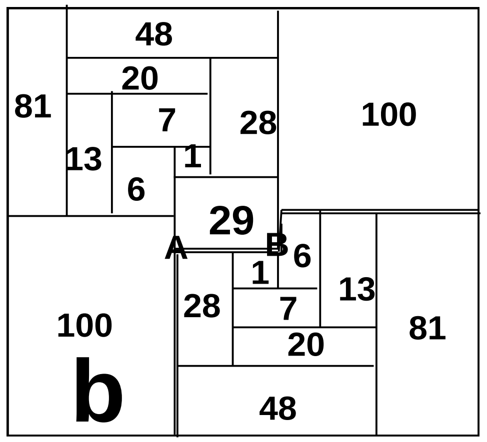


[19] 97x95

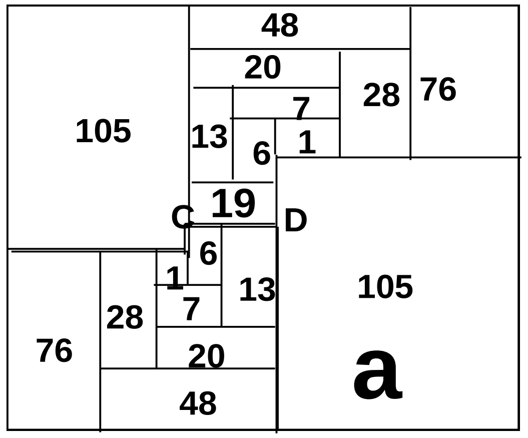


[11] 205 x 181

TO-



[19] 229 x 181 (1)



[19] 229 x 181 (2)

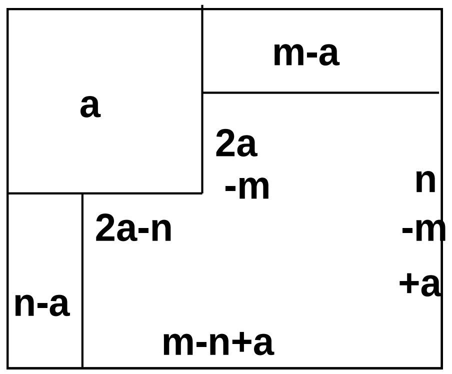
The original and resultant solutions are often fairly close in size, but the results are always symmetric, and xyz or greater, with sides S3333 or greater. If either DE or EF are bounded by a single Element, one pattern Above is not possible.

F12.2. ALGEBRAIC PROOF OF ABOVE

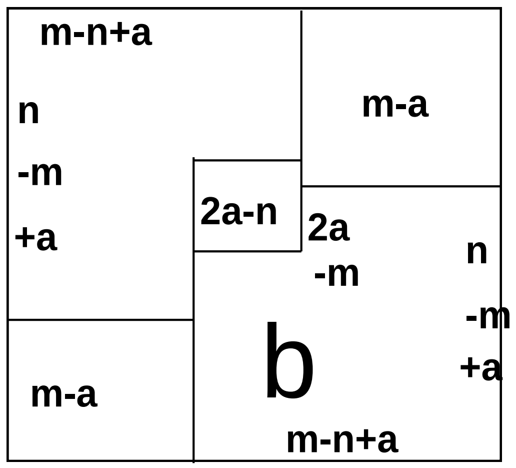
If the lines are carefully examined the expressions are found to agree in all directions to prove that the relationship is Valid.

The original Rectangle is given dimensions m x n and the largest Element called a.

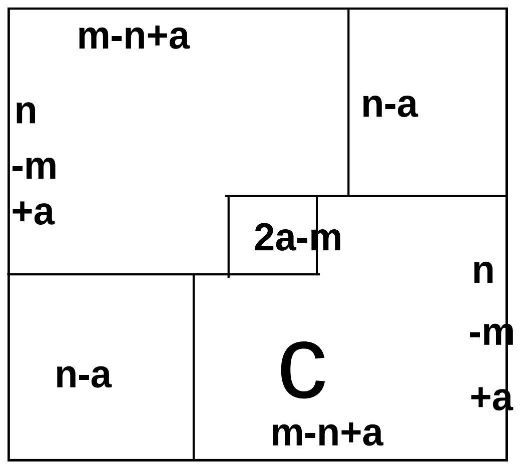
Everything else is calculated from this. Note that a disappears in the final dimensions.



DIMENSIONS m x n



DIMENSIONS 2m-n x n



DIMENSIONS m x 2n-m

F12.3. EXTENSION OF CONVERSION ABOVE

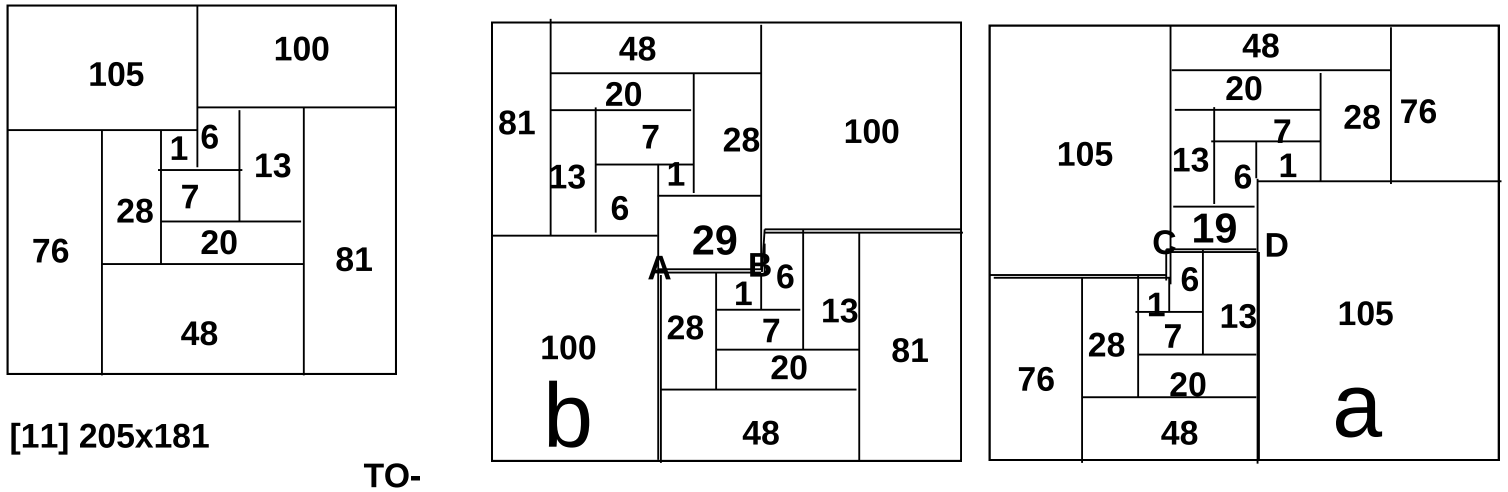
Look at the line CB in the 2nd and 3rd diagram representing 2 or more Elements. Now what happens when CB is just one Element as in the case of a S2223 or S2224 original?

If the 2nd diagram is applied to [11] 205 x 181 below 2 is obtained with an added Element of 29. Fine!

But applying the 3rd diagram under xxxxxxx an Element 5 has to be repeated for the conversion to work, and the result is Compound: although rather surprising the next section deals with a slightly different construction.

But now look at below 3 and observe the different position of the added Element 19!

Note also that the pair are symmetric twins. This property works with any S2223 to S2229 solution but only if there is not a single Element at AB and/or CD which either results in a one result or none, instead of two. See also Twins Section H.



[11] 205x181 TO- [19] 229x181 (1) [19] 229x181 (2)

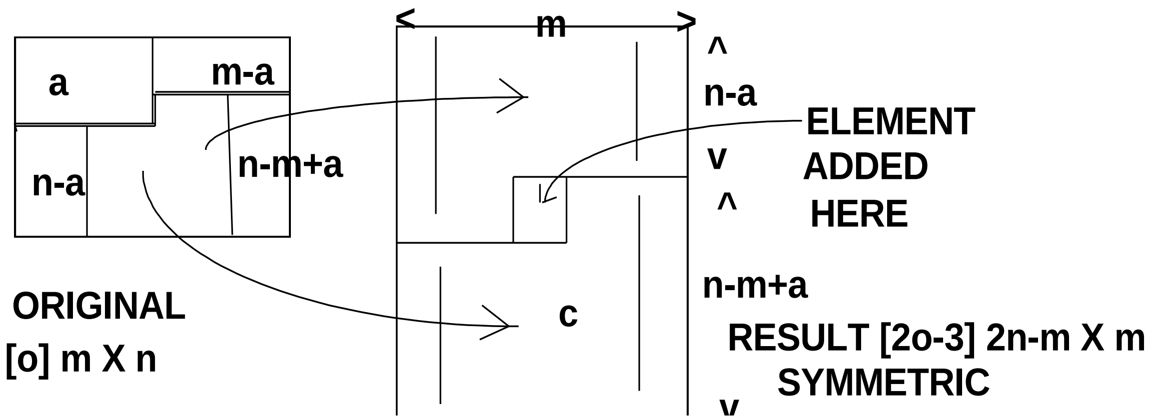
F13. SYMMETRY TYPE c

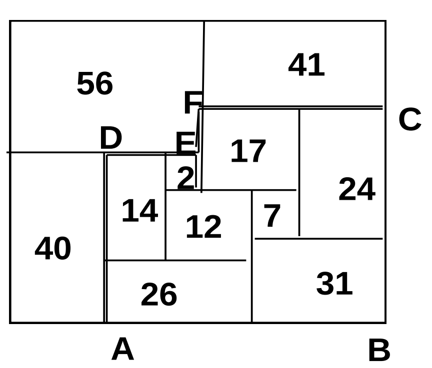
F13.1. ANOTHER SYMMETRIC JOIN - CODE c

This seems a repeat of a previous join where sides are added, but in the following the two added Elements are already present in the original solution which has sides S222x. Note Below that a S2223 solution works equally well.

It is essential that there is no Element between A and m - A as this join never works with a Triad arch.

This is because of the extra Element necessary. The pattern works for any Diad arch of S2223 or S222x and the result is symmetric. An example is easy to construct and has not been shown. However the algebra here will be seen to agree.



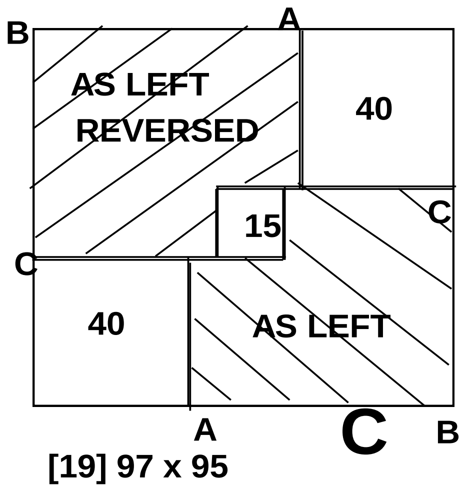


[o] m X n

[11] 97 x 96

An Element is added at EF and the shaded pattern repeated but reversed at top left. Finally two corner Elements of equal value added >>>

[2o - 3] -m + 2n X m



[19] 97 x 95

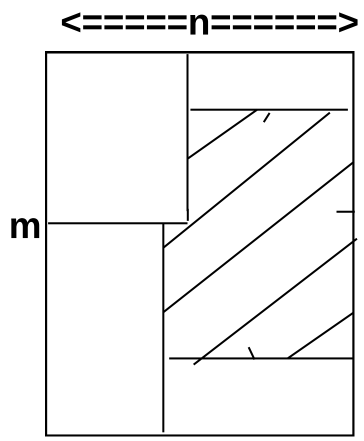
F14. SYMMETRIC TYPE d

In Below 2 the pattern is vertical so m and n have been reversed. This link will not work with Diad+ Arch (i.e. Triad Arch) solutions. The resulting Mid-plus solution cannot be converted into a Mid Solution without being Invalid.

The resulting Order is 2o - 4 so if o is 9 the result is Order 14. The Resultant solution is at least S2 5+ 2 5+.

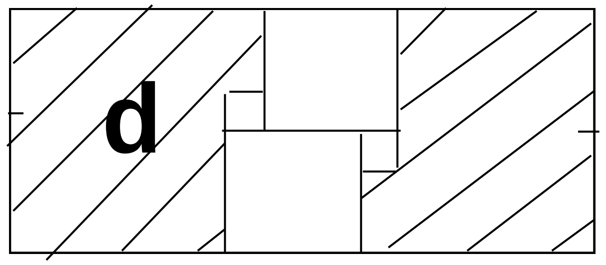
For Resulting solutions of S3535 (from Origin of S2225) the minimum Unknowns are at least three (xyz).

For Resulting solutions of S4545 (from Origin of S2226) the minimum Unknowns are at least four (xyza).



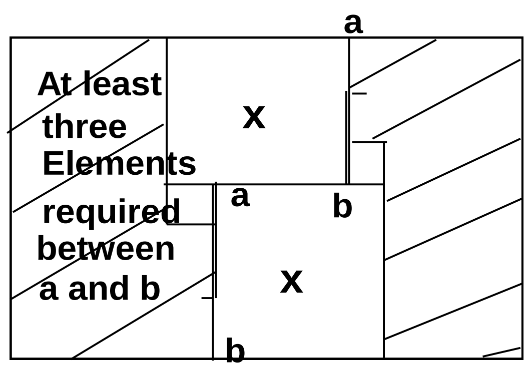
Diad Arch (Not Diad+)

Midplus (No Mid solution possible}



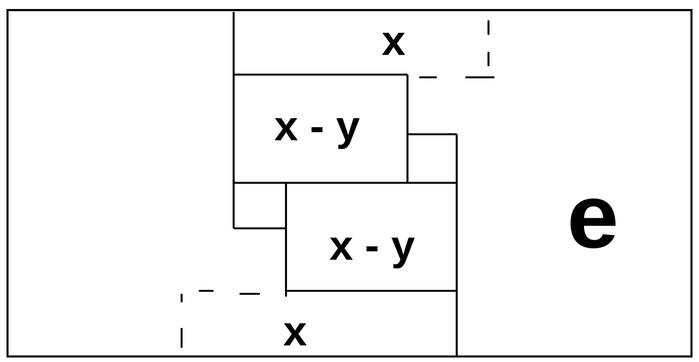
[o] m X n to [2o - 4] m X 2m - 2n

F15. SYMMETRIC TYPE e



As in [16] 85 x 48

[o] m X n to [o] X

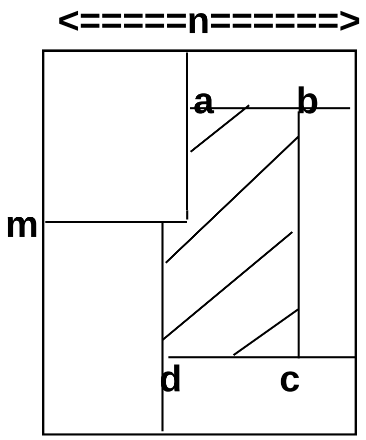


As in [16] 94 x 70 (same Order)

This link requires a Symmetry 3 solution of Mid type with Elements adjoining the Mid part, but there must be three or more Elements bordering x. It is a less common Link. The Formula is not yet calculated.

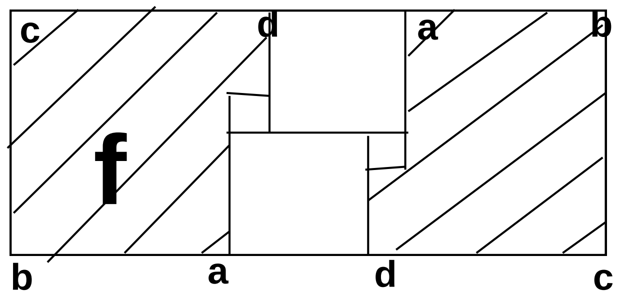
F16. SYMMETRIC TYPE f

This is very like Type d but the Original Solution is S2223.
This results in a different formula, and it is pointed out that ab and or cd may border one Element only which is untrue in Type d.



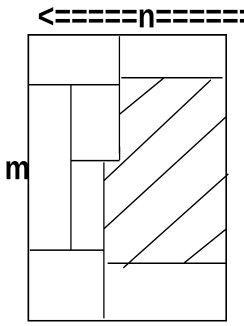
Diad Cloak (Not Diad+)

Midplus (No Mid solution possible}



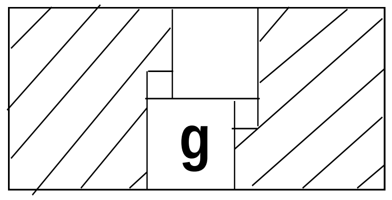
[o] m X n to [2o-6] -4m+5n X 2m-2n

F17. SYMMETRIC TYPE g



Pentad Arch (Not Diad+)

Midplus (No Mid solution possible}



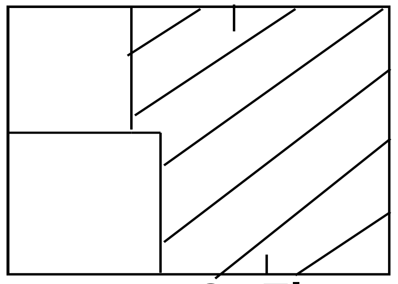
[o] m X n to [2o - 6] -2m +8n X 4m

In the above Link any Pentad Arch provided that it is not a Pentad Arch + will suffice. Note that the Result does not contain any Pentad, and therefore customary multiplying or dividing the result by 3 5 or 15 is <u>not</u> necessary.
Note also that above 1 is vertical in shape and above 2 horizontal, hence the switching of m and n in Above 1.

F18. SYMMETRIC TYPE h

This is an amazing Link since all it requires in the Original solution is <u>one Diad</u>. Some solutions contain three Diads (Below 3) from which three separate solutions, two of them Twins, can be found.
It should be noted that the Diad is not part of a Triad, or the Resultant solution becomes Invalid with two adjacent Elements, but see Link i.
However the source for new solutions is large.

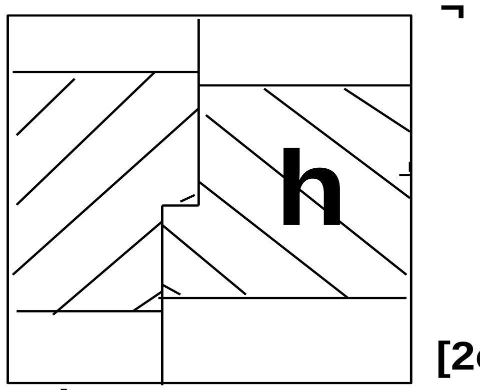
<2+ Elements>



< 2+ Elements

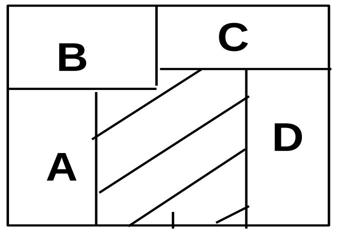
[o] m X n

DIAD ONLY (no Element at *)



[2o] 2m X 2m - n

when turned 90 deg.

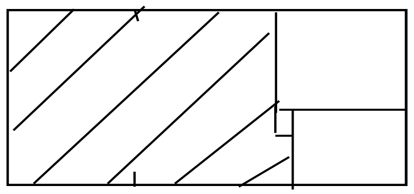


Left A & B } TWINS
Right C & D
Top B & C 3rd Sol.

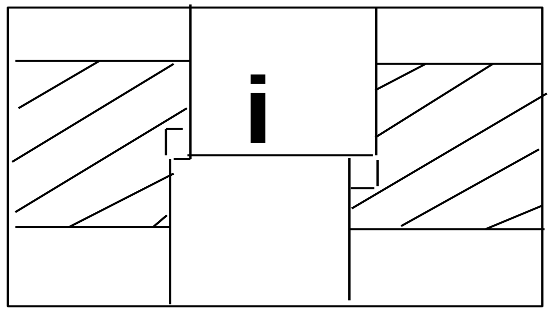
Showing three Sols possible from some Rects with 3 diff lots of diads (not triads)

F19. SYMMETRIC TYPE I

To avoid adjacent elements when Triads are used instead of Diads in Type h, adjacent elements which would give Invalid solutions are avoided by creating a Mid-plus Solution with two added Elements as it Below 2.
Note that the Elements forming the Triads remain in the Result. Initially it seems that this Link will be the source of many new Solutions, but this is not so.
The reason is that ab and cd must both border at least two elements, and this restricts the quantity of possible Origins.



[o] m X n TRIAD ONLY

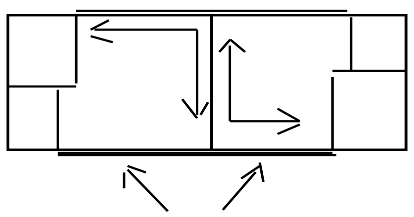


[2o+2] 3m - n X 2m

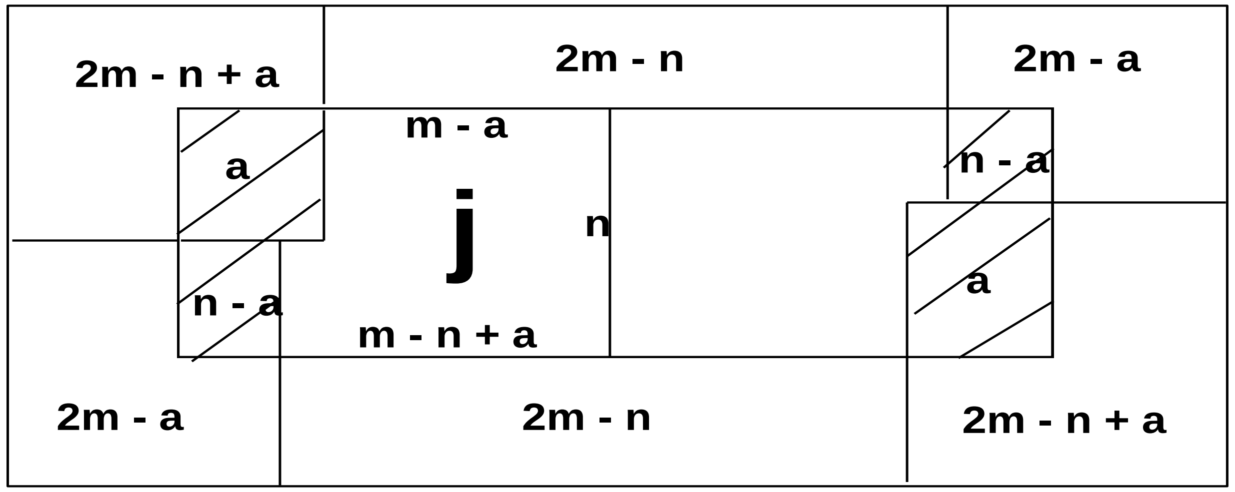
F20. SYMMETRIC TYPE j

F20.1. DOUBLE SYMMETRIC JOIN

In E2.2 it was seen that if any Symmetry 2 solution of S2 4+ 2 4+ was taken, a further Symmetry 2 solution could always be obtained by adding two side elements and suitably increasing the four end ones. So, if any solution containing a side of 2 is taken and then repeated upside-down on the right as Below 1, a Symmetry 2 solution is created albeit a Compound one.
By adding two side elements and suitably increasing the four end ones, the new construction does fit -



Solution shown twice
the second reversed.

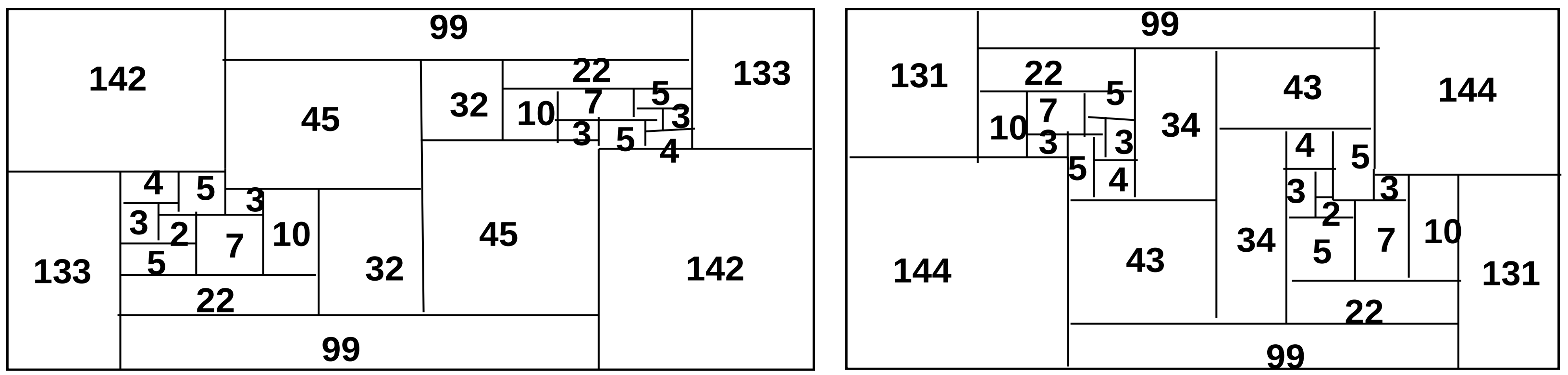


From [o] m X n to [2o + 2] (6m - 2n) X (4m - n)
If element n is inserted first, a solution [2o + 3] (6m + n) X (4m + n) is also obtained!

F20.2. FURTHER DOUBLE SYMMETRY JOINS

Of course a solution may have 2 opposite sides containing two elements, so two sets of solutions may be found. Interestingly they are always found to be twins! From the solution [14] 88 x 77 which has end elements 43 and 34 on one side and 45 and 32 on the other [30] 374 x 275 (1) and (2) are found - see below.

Actually there is a further solution also using the ends 43 and 45 as the solution is Sides S2223 (not shown).



[30] 374 x 275 (1) & (2) both constructed from [14] 88 x 77.

This type of construction is remarkable not so much from the class of Rectangle found but the huge number of solutions it can produce, as there are so many solutions with sides of 2. Six (3 x 2 types) can be produced from any 222X solution alone - including two sets of twins. Of all the Add-Ons and Joins possible, this one must produce the greatest number of new solutions from known ones!

F20.3. DOUBLE SYMMETRY JOINS USING PENTADS, OCTADS AND UNDECADS

If a solution has a Pentad end, then a similar construction with added side Elements and recalculated Pentad end is found to be possible. However up-rating of the elements by 2 or by 4 is often necessary to avoid fractions occurring.

The same is true of Octad and Undecad Endings except that up-rating by 3 5 or 15 is often necessary in the first and up-rating by 56 or a factor of 56 in the second.

Note that solutions containing Triads can be used, but only the Diad part increases in size, not the whole Triad which is not a symmetrical construction on its own.

******* LOOK AT OTHER CONSTRUCTIONS e.g. DOUBLE TRIAD etc. - do these work too?

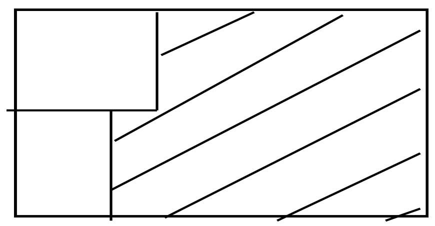
F20.4. PROOF AND DIMENSIONS OF DOUBLE SYMMETRY JOIN

Close examination of above 2 will show that the algebra agrees in each direction in the corner Elements.

Starting with the usual solution denoted by [o] m X n, further Symmetry solutions of [2o + 2] (6m - 2n) X (4m - n)) and [2o + 3] (6m + n) X (4m + n) arise.

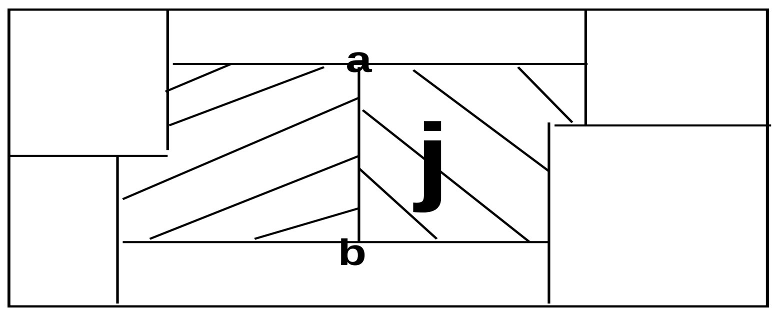
The second is found by adding an Element n before adding the side Elements.

Both solutions are now Simple. See later for a further Link by adding an internal Element at ab.



DIAD (TRIAD OK)

[o] m X n

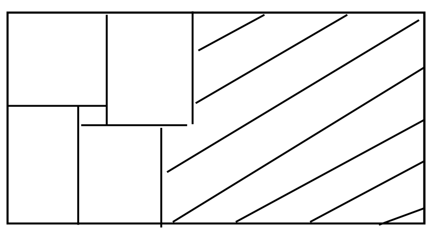


[2o+2] 3m + n X 2m + n

Note: for 2223+ origins three separate solutions are possible - two of ther twins. For 2323 etc two solutions.

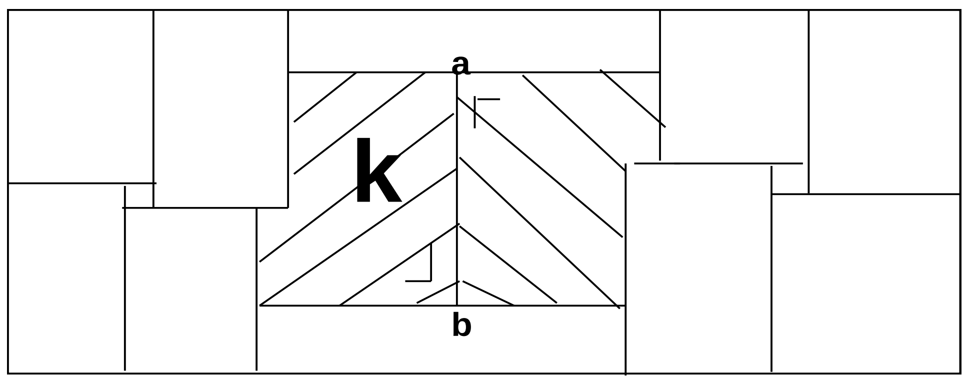
F21. SYMMETRIC TYPE k

For any solution containing a Double or Double + End, the pattern below can be constructed. The Result has Sides 2525. See later for a further Link by adding an internal Element at ab.



DOUBLE (DOUBLE+ OK)

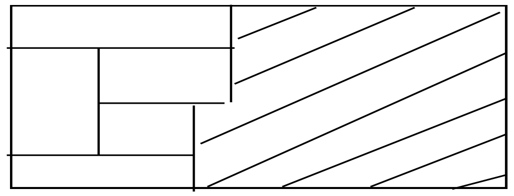
[o] m X n



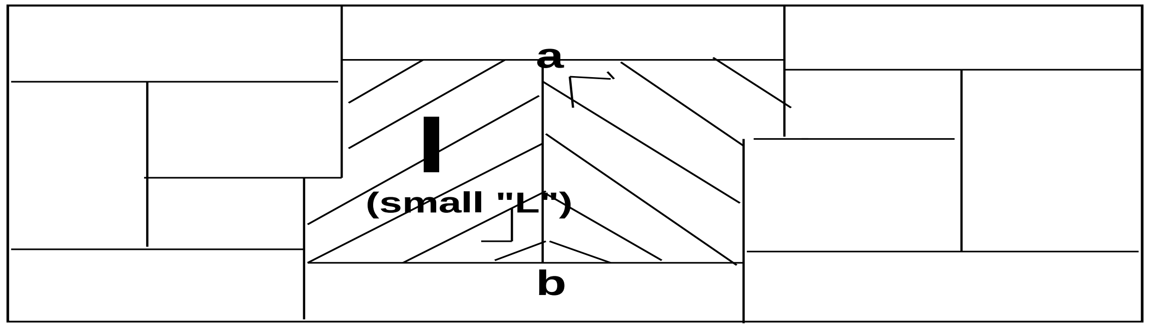
[2o+2] 20m-18n X 4m-3n s2525

F22. SYMMETRIC TYPE l

Similar to Type L this Link can be constructed from any solution containing a Pentad or Pentad +. The Pentad portions have to be recalculated by algebra having been increased in size. See later for further Link by adding one Element at ab.

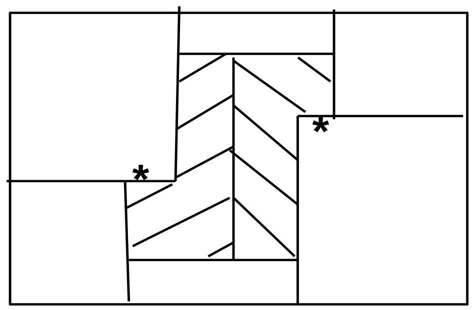


PENTAD (PENTAD+ OK)
[o] m X n

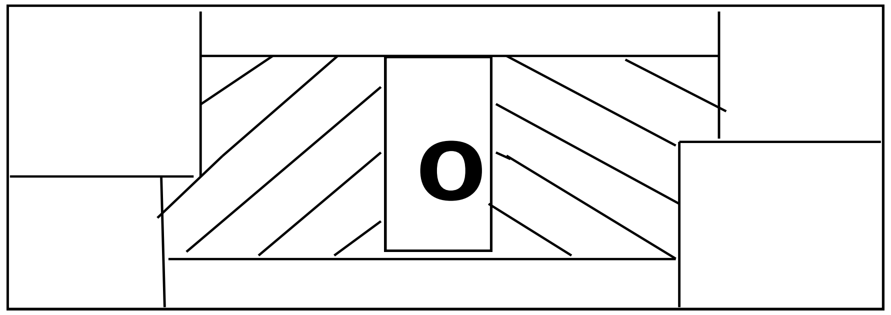


[2o+2] -16m + 24n X 4m - 0.5s3333

F23. SYMM TO SYMM TYPE o



[o] m X n
Symmetric Divide



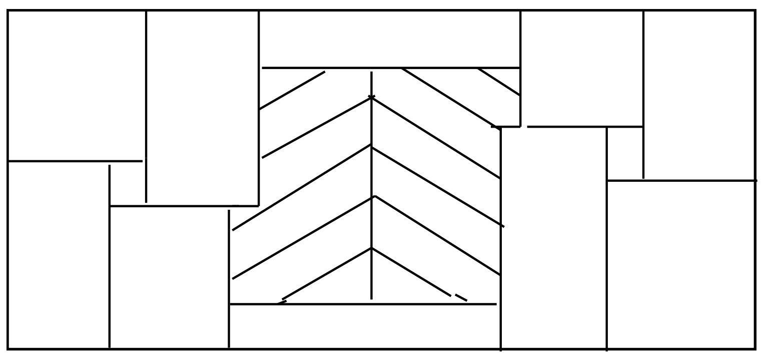
[o+1] 7m - 6n X 4m - 3n
Symmetric Block

This and Types 4 and 5 following are related, and simply have one internal Element added, turning the "Divide" solution into a "Block" one.

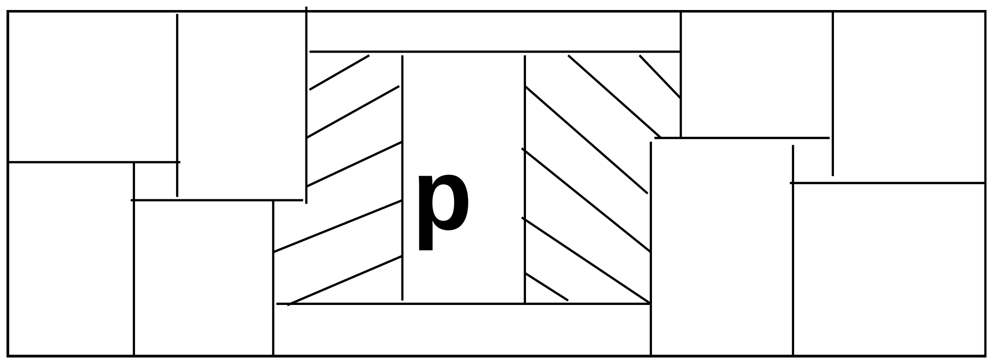
A similar Asymmetric Type was shown in Section E with the same formula, but here the two shaded areas are the same (one reversed and upside-down against the other).

An example of above 1 is easily found. Simply take any solution containing a Diad (and a Triad is admissible).
Display it twice with the Diads removed. Add elements at top and bottom. Then replace the Diads suitable increased in size - they are always found to fit.
In S222* solutions three pairs of solutions can be found using this link, two pairs of which are Twins and the other pair distinct.

F24. SYMM TO SYMM TYPE p



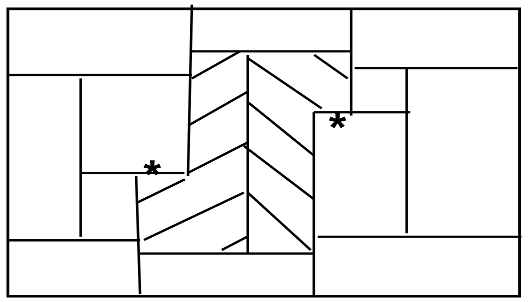
[o] m X n Symmetric s2525 Divide



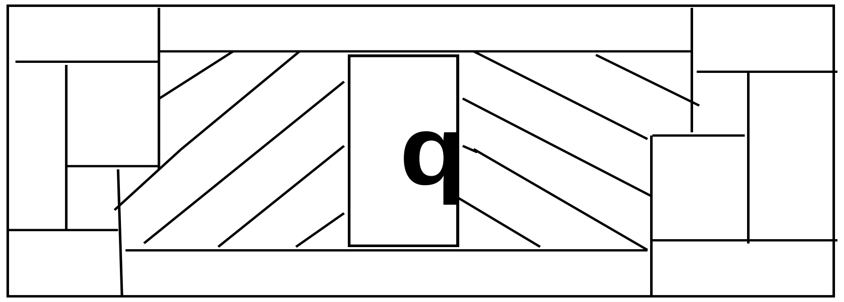
[o+1] X Symmetric s2525 Block

The remarks in G26 apply here except that a solution with either a Double or Double+ end is needed to construct a Solution of the type above 1. The link is therefore less common than Type 3. However from solutions with two Double/+ Ends, two sets of further solutions are possible being two pairs of Twins. The Formula is not yet calculated.

F25. SYMM TO SYMM TYPE q



[o] m X n
Symmetric Divide



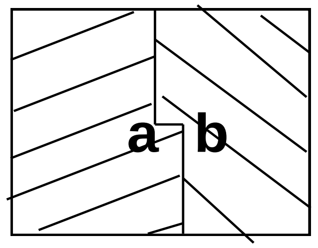
[o+1] X
Symmetric Block

Again the remarks in G27 apply except that this type occurs less often.
Above 1 can be calculated from any Pentad or Pentad + solution. If the solution has two Pentad or two Pentad + Ends, two pairs can be calculated from this link, forming two sets of Twins. The Formula is not yet calculated.

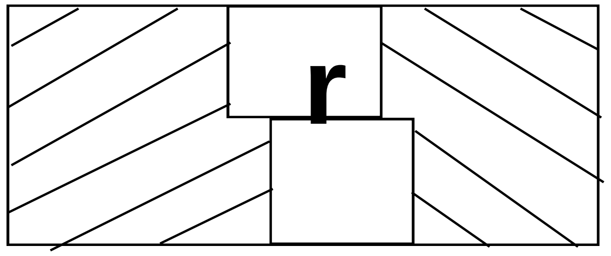
F26. SYMM TO SYMM TYPE r

F26.1. TYPE CODE r

Those links which involve a Symmetric Solution being converted to a different Symmetric Solution are now considered in Types o to y.
This Type and Type p following, were mentioned earlier in G10 as from some solutions an everlasting series is obtained and others have just 2 or 4 solutions.
This link requires a symmetric solution of Mid type, although if an asymmetric solution is known, the Add-on is just as valid. A Mid solution may be obtained through using various other Links.
Types d f g and i are Mid-plus only, and no help here.
But Types h v w and x are Mid and can always be converted to Mid-plus.

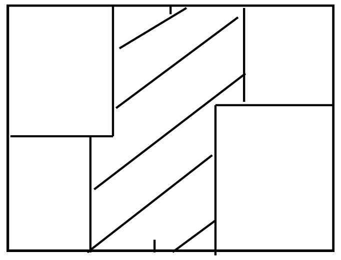


[o] m X n
Symmetric with ab midway between top & bottom



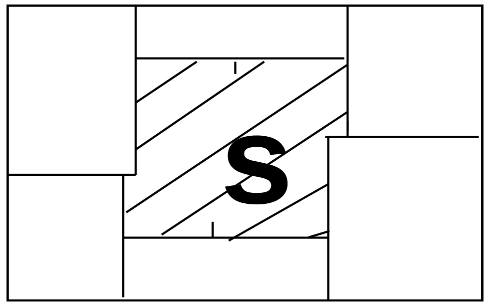
[o+2] m + 0.5n X n
(Note: Link also works with asymmetric solutions with same formula)

F27. SYMM TO SYMM TYPE s



[o] m X n

Any s2 4+ 2 4+
Symmetric solution

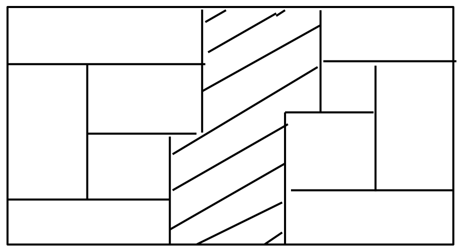


[o+2] 3m - 2n X 2m - n

(Note: reverse not possible for s2323 solutions)
GENERAL FORM
- A VERY COMMON TYPE OF LINK

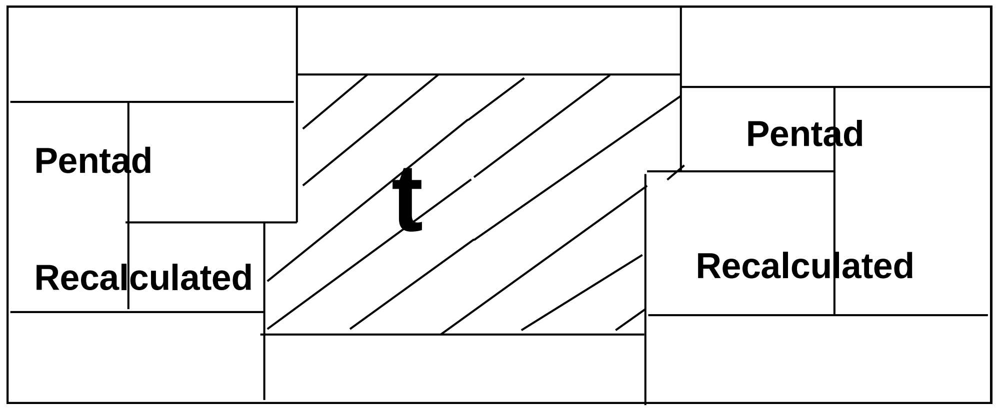
This common link is possible for absolutely all Symmetric solutions of S2424 S2525 S2626... with no exceptions whatever, and so is a source of many new solutions.
Although Diad ends are shown above, the link holds true for Triads also.
If the two elements left and right are replaced with three (or more), then this type of link fails to apply, unless the pattern containing the three (or more) happens to be symmetric in itself - e.g. pentad, octad etc. - see next Section.

F28. SYMM TO SYMM TYPE t



PENTAD ENDS (Symm)

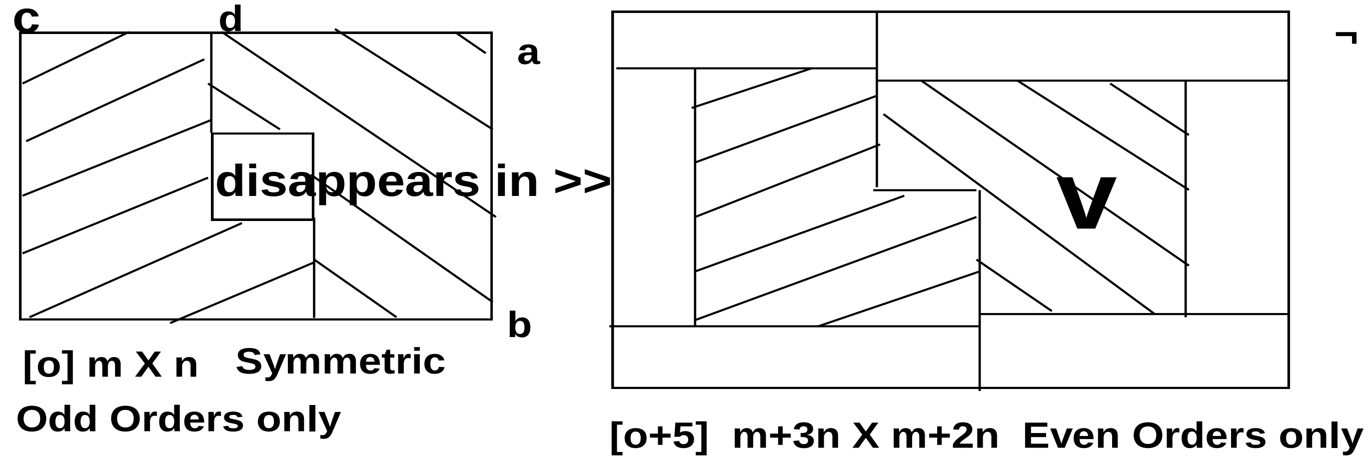
[o] m X n



[o+2] 2.5m - 9/8n X 2m - 0.5n s3333

Similar to Link 1 and as already stated, if the ends are symmetric (in this case Pentad) then a link can be made.
However the whole Pentad part has to be recalculated with a certain amount of algebra required.
The result may have to be scaled up or down by 3, 5 or 15 in some cases.

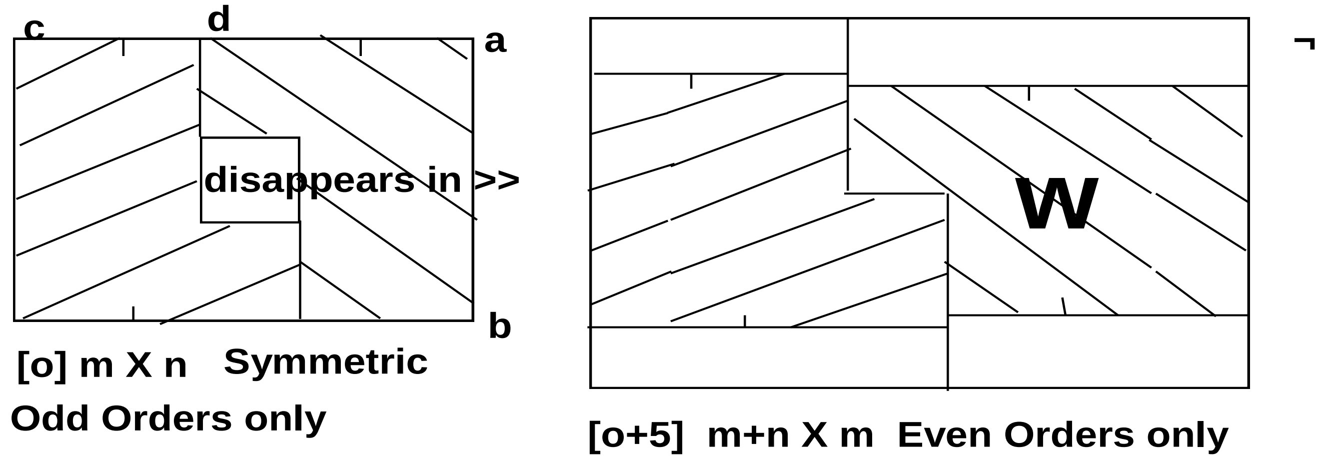
F29. SYMM TO SYMM TYPE v



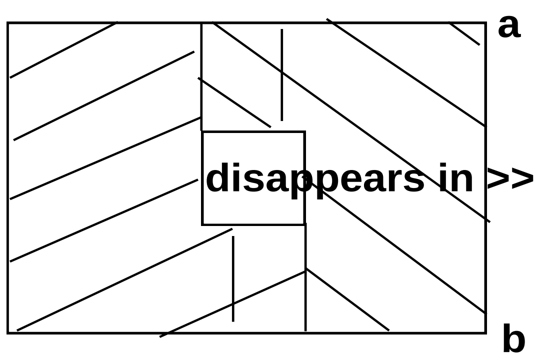
For this link to apply, a symmetric Odd order solution is required, but the solution must have a single Step dividing it in two as above 1.
The central Element is 'dropped', and the right half slipped down with six elements added as in Above 2.
Unlike the next Link ad and / or cd may be bordered by a single element or greater.
The Resultant Solution is always Even Ordered.

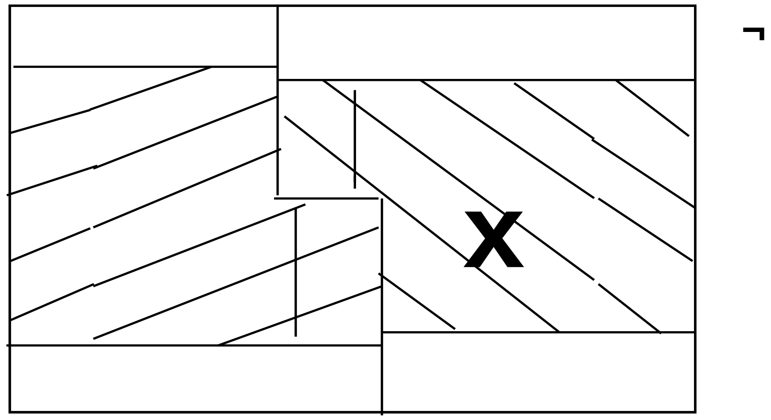
F30. SYMM TO SYMM TYPE w

This seems identical to Type 7, but in this Link cd and da must both be bordered by 2 or more elements.
This means below 1 must be at least S2525 S626 etc. and so occurs less often.
The Origin is Symmetric of Odd Order and the Result Even.
As in G30 there must be a single step only dividing the Origin in two, for this Link to apply.



F31. SYMM TO SYMM TYPE x



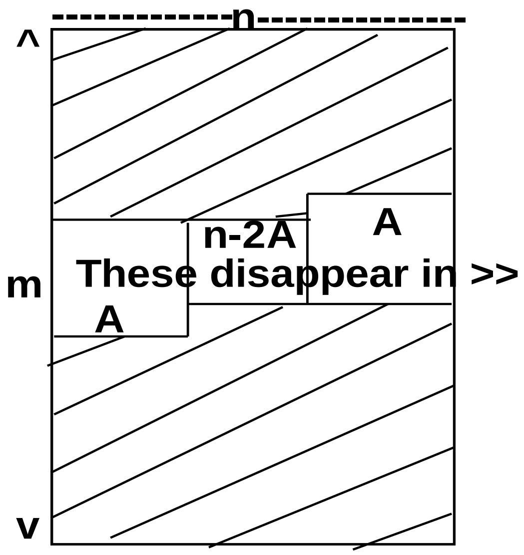


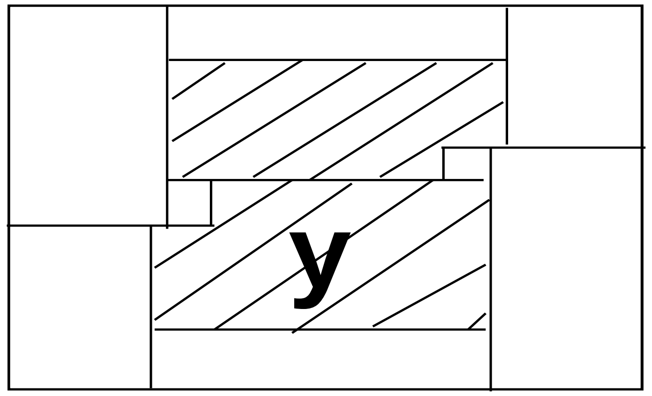
[o] m X n Symmetric
Odd Orders only
Note ab can be single Element

[o+5] m+5n X m+2n Even Orders only

Really a corollary of previous links, this one differs in having three Elements as shown in above 1.

F32. SYMM TO SYMM TYPE y





[o] m X n

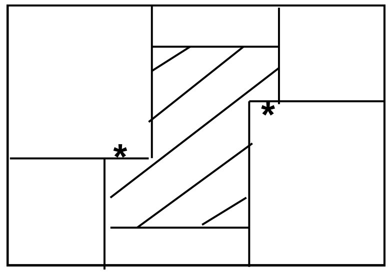
[o+5] m + 3n - A X m + n + 2A

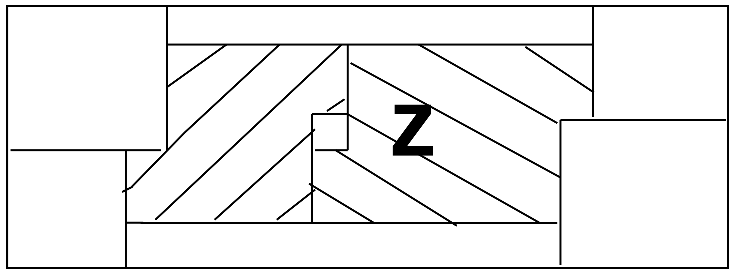
This Link happens to require a Symmetric solution with three elements through the middle in the pattern of above 1.
Most Links can be expressed in terms of the dimensions m and n only, but the formula here also requires the middle Element A

F33. SYMMETRIC TYPE z

F33. EVERLASTING SERIES - CODE z

In any S2323 solution with Diad ends, provided that the ends are not Triads, an everlasting series can be constructed as below 2.
For this to work notice the extra element required between each pair of shaded areas, i.e. the next size up shows three shaded areas with a single Elements between each two.
Although interesting, the solutions are not particularly that desirable the Elements being heavily duplicated.
The series, coded z1, z2, z3, z4... has Orders Odd - Even - Odd - Even....





======>>>

[o] m X n

* there must not be elements here, otherwise no restrictions

Everlasting series --- [2o - 5] 4m - 3n X 2m-n Code z1
[2o +4] 7m - 6n X 4m-3n Code z2
[2o +13] 10m - 9n X 6m - 5n Code z3
& so on, in arithmetical progression.

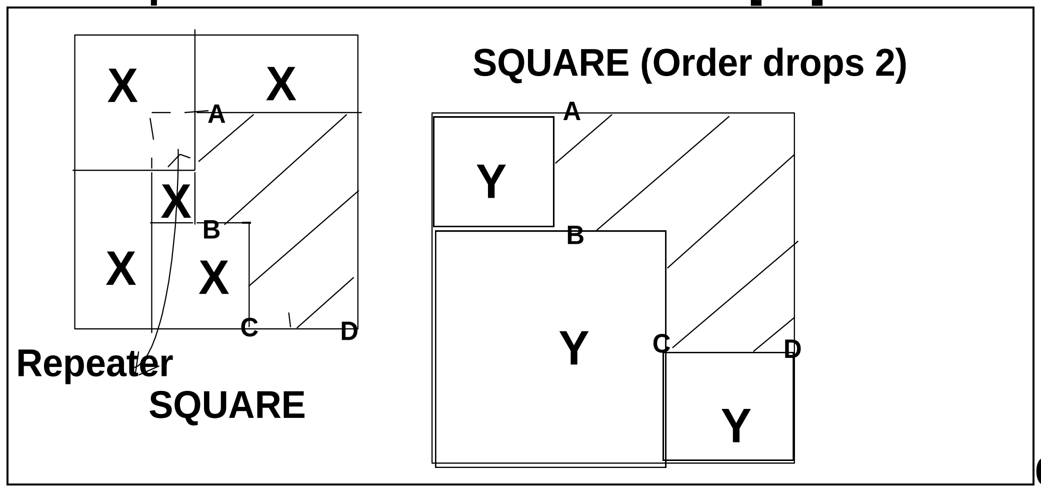
F34 LINKS INVOLVING A REPEATER

F34. GENERAL

These links involve Solutions which have a Corner Repeater which may be arranged horizontally or vertically.
Usually there is little point in calculating these Adjacent Solutions. However many Imperfect Squared-Squares of Sides 22** give this type of Solution if the 2 largest Elements are removed and the Element in between suitably reduced.
Algebra proving these Links is not shown.

F34.1. SQUARE TO SQUARE - Type R1

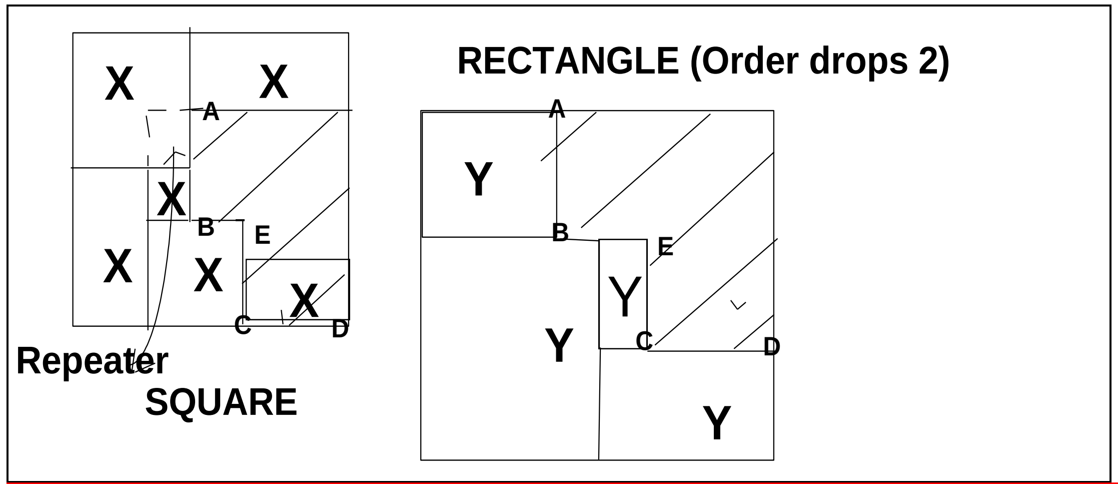
In Below 1 a Repeater could be drawn where shown. Both diagrams are Squares. Elements "X" are replaced by Elements "Y" in below 2. CD must span more than one Element. [20] 135 x 135 can be converted to [18] 99 x 99 using this Link.



d

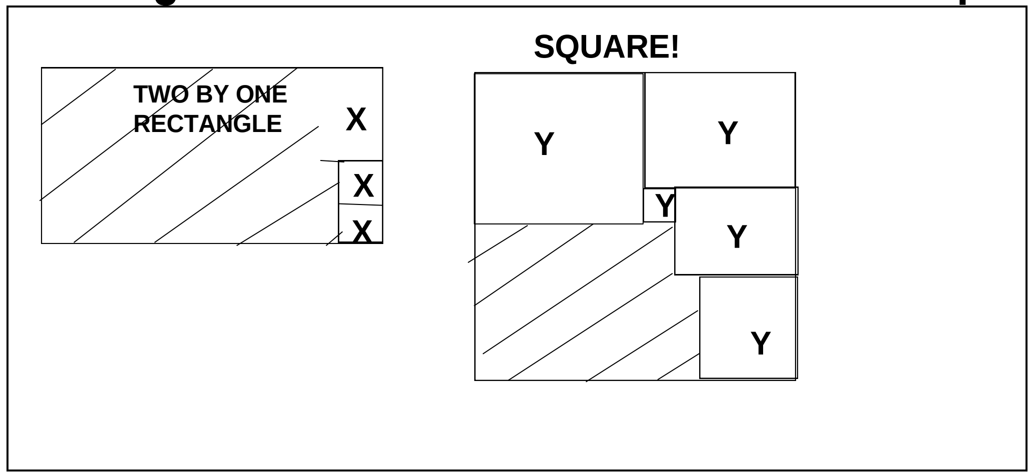
F34.2. SQUARE TO RECTANGLE - TYPE R2

Similar to last but with a Single Element at CD, Elements "X" are replaced by Elements "Y" and the Order drops 2.
An Example is Imperfect Square [17] 64 x 64 to Rectangle [15] 44 x 40.



F34.3. TWO BY ONE RECTANGLE WITH REPEATER TO SQUARE

A most interesting Link, quite simple in construction which has been used to create many spectacular Perfect Simple Squared-Squares of high Orders. What is far from easy is finding a Two by One Rectangle which is Perfect apart from the Repeater Elements. My attempt to continue the Rectangle with further Elements has failed to produce a normal Rectangle without Repeaters.



G. DIMENSIONS

G1. SIDES

G1.1. SIDES INDEX

Solutions can be recorded according to the sides Index. A solution may have sides 2223 2322 2232 3222 but the smallest amount of S2223 is shown for convenience. Note that sides such as S2354 and S2345 are different but that
S2453 is corrected to S2354.

G1.2. SIDES INDEXES AND MINIMUM AMOUNT OF UNKNOWNS

Without explaining why -
1. For sides S2223 to S2999 etc the amount of Unknowns required is at least 2.
2. Sides S3333 to S3399... 3 Unknowns or more.
3. S4444 to S4999... 4 Unknowns or more.
4. S5555 to S5999 5 Unknowns minimum and so on...

Note that the amount of Unknowns is not constant for a given sides Index.
However for sides S4457 for example, it must be 4 or greater and never be 2 or 3.
Likewise S3333 can never be xy.

G1.3. RANGE OF SIDE INDEXES FOR AN ORDER

As the Order increases, the range of Sides Indexes does also.
Up to and including Order 13 the following Sides are found to apply:
S2223 S2224 S2225 & S2226. S2233 S2234 S2235. S2323 S2324 S2325. S2333 S2334. S2424 S2425. S2434 & S3333.
Sides S3333 and larger do not apply to Orders 7 to 12.

G2. REDUCED AND FULL SIDES

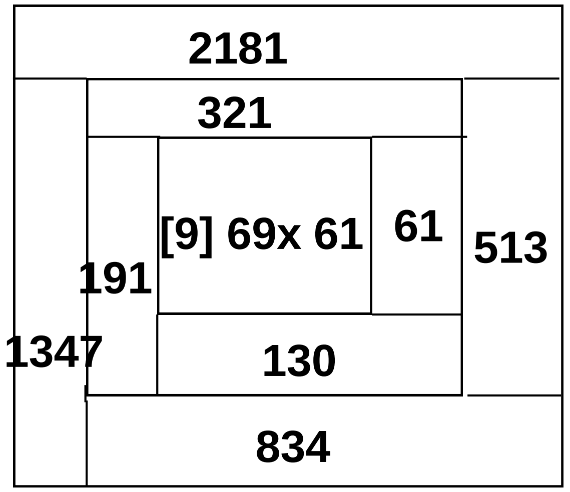
G2.1. SEMI-PERIMETERS AND DIMENSION OF SIDES

In calculation low Order solutions it is striking how often the Semi-perimeter of one Order is repeated as the larger side of the next Order. For example, 209 occurs as a frequent SP in Order 10 and a frequent side in Order 11.
This is no coincidence. A starting clue to this is to add a single Element at one end. [9] 69 x 61 becomes [10] 130 x 61 or 130 x 69 and in all Singlend solutions the side is clearly the Semi-perimeter of the smaller solution.

But the real proof is seen in a Smith diagram network which has a fixed Complexity no matter which choice of poles is chosen. This includes choosing poles 1 apart only which provides an amount of Singlend solutions.
If the end Element is removed, the Semi-perimeter obtained is clearly equal to the original Complexity.

G2.2. SERIES OF SIDES

Consider the series below, from which the following solutions connect- [9] 69 x 61 [10] 130 x 79 [11] 191 x 162 and 191 x 177 [12] 321 x 272 & 321 x 287 [13] 513 x 415 ...



SERIES FROM [9] 69 x 61
9-61 10-130 11-321 12-513 13-834 14-1347
15-2181 & Then may be continued -
16-3528 17-5709 18-9237 19-14946 20-24183
21-30129 22-63212 23-102441 24-165753
25-268194 26-433947 27-702141 & so on.

G2.3. FULL AND REDUCED DIMENSIONS

When solutions are calculated using xyz… these Unknowns can often be cancelled down by a common factor called the Reduction Index. Thus each solution has both full and reduced dimensions, though where the common factor is only one, both are of course the same.

G2.4. FULL DIMENSIONS, TENDENCY TO TYPICAL SIZES PER ORDER

Whereas the reduced dimensions for any given Order vary dramatically (e.g. 6 x 5 to 130 x 179 in Order 10), the full dimensions for a given Order tend towards a typical range of size.
there are 11 solutions for Order 10 the full dimensions being 114 x 95, 128 x 96, 110 x 99, 121 x 88, 120 x 104 (Invalid solutions) and 114 x 110, 130 x 94, 105 x 104, 111 x 98, 115 x 94 and 130 x 79.
Note there is a wider variation in the smaller than the larger dimensions, 79 to 110 and 105 to 130, but the overall range is not great.
The range widens proportionally with higher Orders.

G2.5. A ROUGH TABLE OF TYPICAL SIZES

7 22	9 65	11 180	13 520	15 1500	17 5000	19 13000
8 40	10 100	12 320	14 950	16 2200	18 8500	20 21000

G2.6. LARGEST AND SMALLEST SOLUTIONS FOR AN ORDER DEFINED

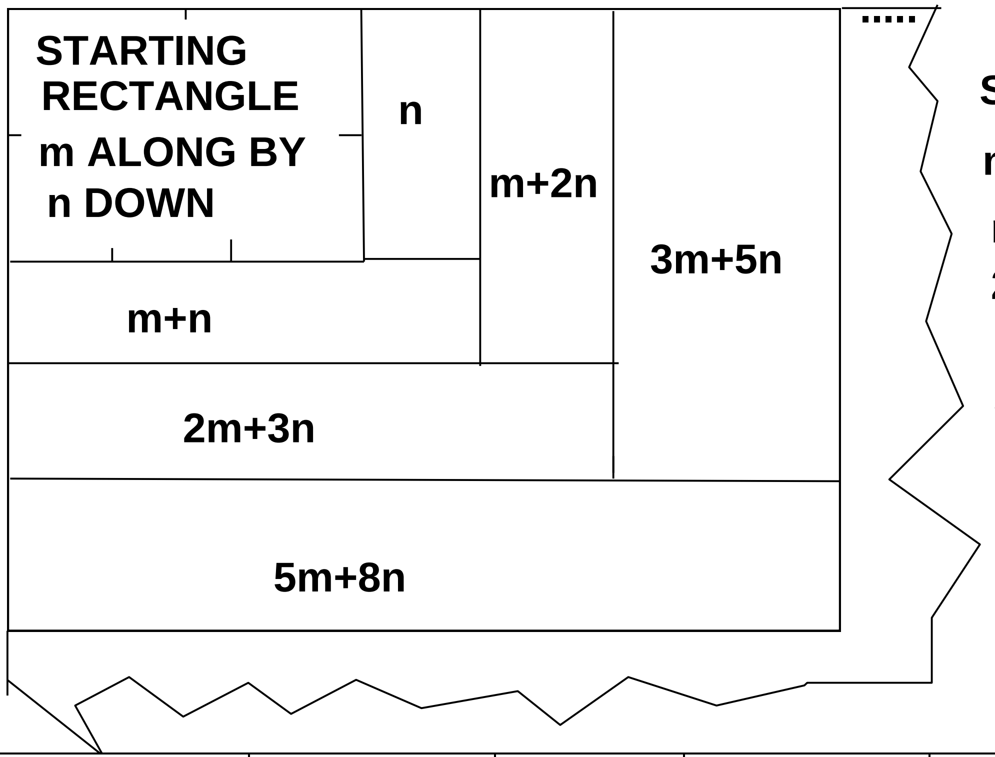
These can have different meanings as follows

1. Largest / smallest larger dimension of the reduced solutions.
[10] 130 x 79 and 6 x 5 for example.
2. Largest / smallest larger dimension of the full size solutions.
[10] 130 x 79 and 105 x 104 for example.
3. Largest / smallest according to the semi-perimeters of the solutions.
[10] SP 224 down to 209.
4. Largest / smallest unit area for the full sizes of the solutions. 12540 for [10] 114 x 110 (57 x 55) and 10270 for [10] 130 x 79. This last solution is the smallest in 4. but the largest in 1. and 2. hence the need for clarification!

G2.7. GROWTH AND FIBONACCI SERIES

In this study the Fibonacci series of numbers is relevant, namely 1 1 2 3 5 8 13 21 34 55 89 144… where each amount is the sum of the previous two.

Below shows a growth series from a solution m x n where the Semi-perimeter, adding single end n, the Semi-perimeter becomes m + 2n.



GROWTH SERIES
SEMI-PERIMETER

m + n
m + 2n
2m + 3n
3m + 5n
5m + 8n
8m + 13n....

[7] 24 x 21	x1 x1	8 45	[8] 40 x 35	x2 x3	11 185
	x1 x2	9 66		x3 x2	11 190
	x2 x1	9 69	[9] 66 x 65	x1 x1	10 121
	x2 x3	10 111		x1 x2	11 176
	x3 x2	10 114		x2 x1	11 187

	x3 x5	11 177	[9] 66 x 64	x1 x1	10 130
	x5 x3	11 183	= 33 x 32	x1 x2	11 194
[8] 45 x 30	x1 x1	9 75		x2 x1	11 196
	x1 x2	10 105	[9] 69 x 61	x1 x1	10 130
	x2 x1	10 120		x1 x2	11 191
	x2 x3	11 180		x2 x1	11 199
	x3 x2	11 195	[9] 75 x 55	x1 x1	10 130
[8] 40 x 35	x1 x1	9 75	= 15 x 11	x1 x2	11 185
	x1 x2	10 110		x2 x1	11 205
	x2 x1	10 115	[10] var.	x1 x1	11 209, 224

G2.8. FIBONACCI SERIES - THE RATIO

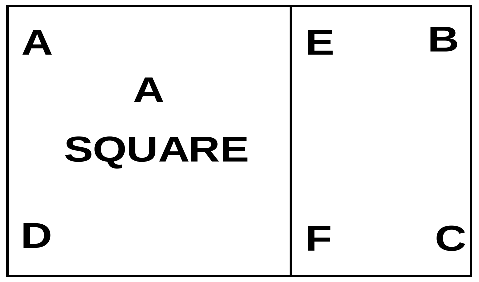
The series 0, 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377…. is extremely interesting, and has some unexpected features. Firstly, if any number is divided by the number before the following results occur -

NO.	F number	RATIO	NO.	F number	RATIO
1	1	1.0000000:	2	2	2.0000000: (2 div 1)
3	3	1.5000000: (3 div2)	4	5	1.6666666: (5 div 3)
5	8	1.6000000: (8 div5)	6	13	1.6250000: (13 div 8)
7	21	1.6153846:	8	34	1.6190476:
9	55	1.6176470: note	10	89	1.6181818: note
11	144	1.6179775: the	12	233	1.6180555: the
13	377	1.6180257: increase	14	610	1.6180371: decrease
15	987	1.6180327: in the	16	1,597	1.6180344: in the
17	2,584	1.6180338: series	18	4,181	1.6180340: series
19	6,765	1.6180339:	20	10,946	1.6180339:
21	17,711	1.6180339:	22	28,657	1.6180339:
23	46,368	1.6180339:	24	75,025	1.6180339:
25	121,393	1.6180339:	26	196,418	1.6180339:

G2.9. THE FIBONACCI RATIO 1.6180339….

The above shows that the ratios tend to, but are never exactly the ratio of 1.6180339… an amount impossible to show exactly, and is never actually reached no matter how many in the series are calculated.

Note that the ratios form a series which wavers up and down alternately, so that only the left the series actual gradually increases and on the right decreases.
This suggests it may be more sensible to consider this as two series - one on the increase and one on the decrease.
Why this strange amount? Is there anything special about 1.6180339 etc?
Yes! It is that number which when reciprocated reduces the result by one, in other words 1 ÷ by 1.6180339.. gives 0.6180339..!
Note also that the ratio between an F number and that two below tends to 2.1680339 .. (e.g. 121393 / 46368).
This is not as surprising at it first seems as the ratios for differences of 0,1,2,3,4 are 1.0000000 1.6180339 2.6180339 4.2360679 and 6.8541019 each being the sum of the previous two.
Below shows a solution of proportions 1.6180339... to 1. No matter how many squares are deducted the elongation always remains 0.6180339.. and 1.6180339 if the longer dimension is divided by the shorter.



AB div BC IS 1.6180339.. WHEN A SQUARE (ADFE) IS TAKEN
BC div BE IS ALSO 1.6180339...!
BC div AB IS 0.6180339.. WHEN A SQUARE IS TAKEN OFF,
BE div BC IS ALSO 0.6180339...!

G2.10. ANOTHER RELATIONSHIP IN THE FIBONACCI SERIES

If any number in the series is taken and multiplied by the one, two below and the square of the one below is deducted the result is 1 or -1! e.g... 5 x 2 - 3 x 3 =1 13 x 34 - 21 x 21 = 1 5 x 5 - 3 x 8=1 13 x 13 - 8 x 21 = 1 34 x 34 - 21 x 55 = 1 and so on. The series alternates, and again may be better considered as two series.

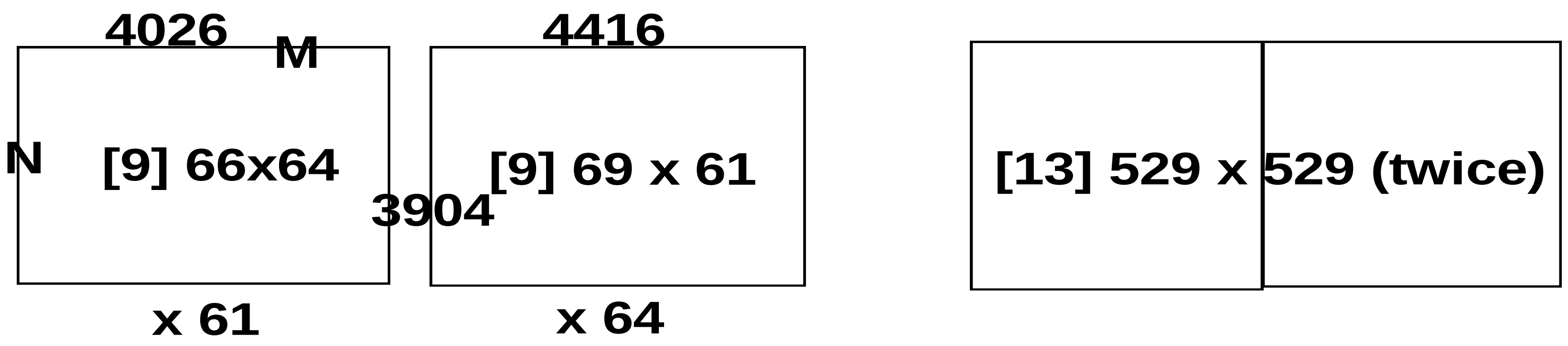
G2.11. FIBONACCI SERIES RELATED TO GROWTH

Without going into detail, some plants have leaves which follow the series.
Some flowers have petals which spiral 34 in one direction and 55 in the other,
34 and 55 being Fibonacci numbers! Others have similar patterns with other F numbers.

G2.12. FULL DIMENSIONS FOR TWO RECTANGLES SIDE BY SIDE

The purpose of this Section is to observe what happens when two known Full Dimension Rectangles of known Orders are put together and then up-rated to form one Rectangle (Compound of course) of double the Order. What are the Full Dimensions of the resulting Solution? Looking at an actual case with two Order 9 Solutions, 135 (i.e. 66 +69) may seem to be the Reduced greater Dimension - but no - since the Reduced lesser Dimension cannot be 64 and 61 together at the same time. But if we make the lesser Dimension 64 x 61 in the first half and 61 x 64 in the second each will be 3904 and the same. Then of course the Greater Dimension in the first becomes 66 x 61 and the second 61 x 64 which is 4026 + 4416 which is 8442. So the Rectangle is [18] 8442 x 3904 Compound. As this Rectangle being quite elongated it becomes

evident that 3904 is not a very typical Dimension size - in fact rather smaller. Nevertheless we may find 8442 a much more typical Full size for Order 18.
This indicates that if we multiply the unit area of one of the Order 9 Rectangles and then multiply by 2 we should have a representative Full size for an Order 18 Solution. 66 x 64 x 2 gives 8440 and 60 x 61 x 2 gives 8418.
Now look at below 2 where [13] 23 x 23 Imperfect is illustrated as Full Size [13] 529 x 529. Now 529 x 529 is 279841 and twice this is 559,682 so our Full size Rectangle is [26] 559,682 x 279,841! Again, the larger 559,682 is much more typical than 279,841.



What is the point of all this then? Well, if a Full Size Rectangle as [O] M x N is taken, then it means that an Order 2 * O solution will have a Full Dimension *somewhere in the region* of The Square of Half of (M + N) times two. In the case of the Order 18 Solutions this is 65 x 65 x 2 (left) or 65 x 65 x 2 (right) which is 8,450 (compare 8442 8440 and 8418 earlier which are not greatly different). This Rule is a general observation rather than an exact science. In reality the possible ranges of Upper Dimensions are very great - even in Orders of say 20.
M + N is of course the Semi-Perimeter, and half this is the Average of the two sides. So putting it another way;
2 x (SP x half) squared gives an indication of the sort of size of the Rectangle if the Order is Doubled.

G3. DIMENSION TYPES

G3.1. GROUPINGS OF MAIN DIMENSIONS

The Above table is constructed thus “[7] 24 x 21 x3 x2” means a side 24 x 3 + 21 x 2 = 114 is found for Order = 10.
In fact there are 2 examples [10] 114 x 95 (6 x 5) and 114 x 110 (57 x 55).
The Order numbers are shown and up to Order 11 include.
All Valid Order 10 solutions have Semi-perimeters of either 209 or 224 and there is no point in writing these in 11 times!
Inspection of all Order 11 solutions reveals that the upper dimensions are almost all found in this table. Order 11 solutions group as follows:-
1. From [10] m x n as 1m + 1n or if you prefer, the semi-perimeter. 1st degree
2. From [9] m x n as 1m + 2n and 2m + 1n. 2nd degree
3. From [8] m x n as 2m + 3n and 3m + 2n. 3rd degree
4. From [7] m x n as 3m + 5n and 5m + 3n. 4th degree .... and so on... using the Fibonacci series 1, 1, 2, 3, 5, 8, 13, 21, 34....

G3.2. THE GROUPINGS RELATE TO THE DOMINANT DIMENSION

Squared-Rectangles have an upper and a lower dimension and usually, but not always, the upper one acts as the Dominant dimension also. When more of the above series are calculated, numbers occur duplicated, and it is no longer clear as to which dimensions arise from which series. A means of finding how they divide is necessary.

G3.3. FINDING WHICH DEGREE APPLIES TO A DOMINANT DIMENSION

I have found an apparent correlation between 1st degree dimensions and Triads which could be deducted without a Singlend solution arising, but then saw more 1st degree examples where this is untrue. A better and more accurate division is needed. Upon examining the horizontal smith diagrams for a number of solutions where the degree is known, a pattern started to emerge.

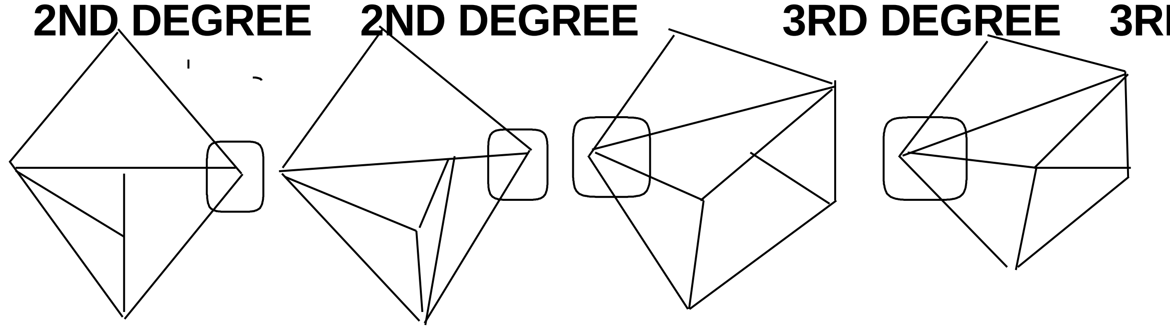
G3.4. MINIMUM 3 ELEMENTS AT TOP & BOTTOM REQUIRED FOR 1ST DEGREE

If 3 or more wires radiate from both poles and there are 2 wires between the poles at least once, then the Dominant dimension is 1st degree. This is so regardless of the existence or otherwise of triads. Thus all S2323, S2324 S2333 (horizontal) etc. solutions are this type.

G3.5. DEGREES FOR 2-2-3+-3+ HORIZONTAL SOLUTIONS.

Below are some smith diagrams for 2nd and 3rd degree. Only the pattern is necessary and arrows and Elements are not shown.

2ND DEGREE 2ND DEGREE 3RD DEGREE 3RD DEGREE



[11] 205X181 [11] 115X94 [11] 195X191 [11] 57X55

3 WIRES JOIN AT BOXES 4 WIRES JOIN AT BOXES.

NOTE TWO WIRES ONLY AT TOP POLE.

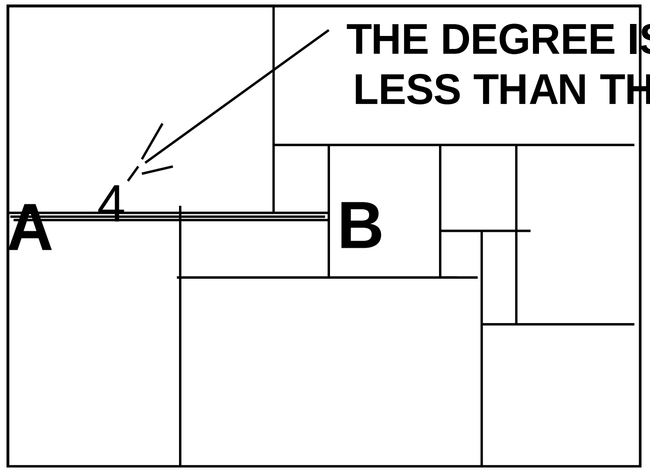
2 WIRES BETWEEN POLES WHERE BOXES SHOWN

Above shows what appears to be the relationship. For 2nd degree and beyond there are only 2 Elements at the top and down at least one side. A number of lines join where the boxes have been drawn and the degree is one less than this number.

G3.6. FINDING DEGREES BY INSPECTION OF SOLUTIONS

In all Solutions with sides other than S223+3+, the horizontal dimensions appear to be 1st degree. But in the case of S223+3+ it depends on the number of bordering Elements between A and B.

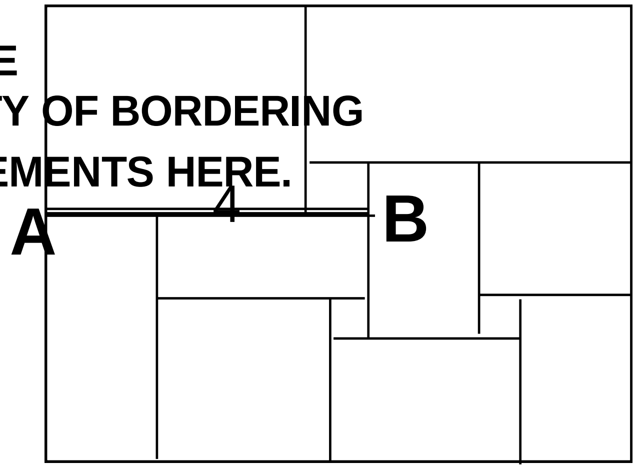
[12] 326 x 307



THE DEGREE IS ONE LESS THAN THE QTY OF BORDERING ELEMENTS HERE.

[9] 66x64 66 x3 + 64 x2= 326

[12] 335 x 303



[9] 75x55 75x3 + 55x2 = 335
9 = 12 - (4 -1)

NOTE: IN THE CASE OF 222x RECTANGLES THIS DOUBLE LINE COULD BE DRAWN FROM EITHER SIDE. TAKE THE LOWER NUMBER AND DEDUCT ONE.

G3.7. DEGREES FOR S2223+ SOLUTIONS

To BE DEALT WITH!!

G3.8. DEGREES FOR SIDES S3333 UPWARDS

Subject to checking these appear to be first degree dimensions for the Dominant dimension.
In the Above 223+3+ solutions there are 4 Elements bordering the double line.
Deduct this amount from the Order and add 1. This gives 9. The bottom line proves that these are 3rd degree dimensions.

G3.9. THE RECESSIVE DIMENSION

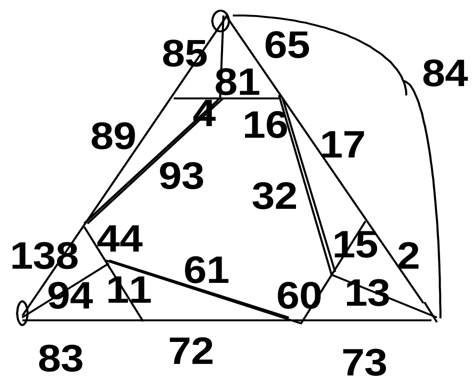
As already mentioned the Dominant dimension is usually the larger but not always.
Likewise the recessive one is often the smaller but not always.
Is there a similar relationship for the recessive dimension with smaller Order solutions?
Some larger recessive dimensions seem to obey the above rules -
Take [11] 185 x 183 2233 where there is a vertical divider of 5 squares. Now 11 - 4 is 7 and from [7] 24 x 21 24 x 5 + 21 x 3 = 183!
Some others also work likewise. The problem is that the range of sides for Order 11 Above ranged from 176 to 224 whereas the smaller dimensions for Order 11 are 112, 127, and 144 to 196. Actually 21 out of 33 are too small!

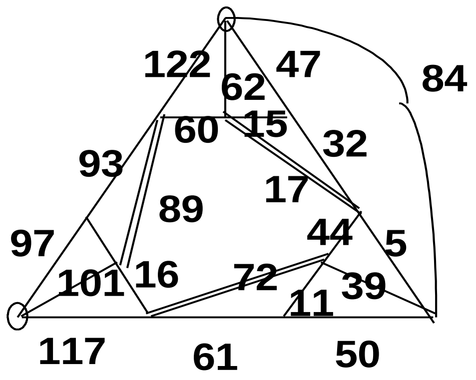
G4. TWINS AND ROTORS

G4.1. TWIN SOLUTIONS AND ROTOR STATORS (SEE ALSO SECTION H)

amendments needed below!

Although twin solutions can have any Reduction Index (or even different RI's between the two solutions as in [13] 112 x 75) some twin solutions which have regular Smith Diagrams have reduced Elements.
1. All solutions with identical Elements, as those shown have reduced Elements.
2. All solutions with Smith diagrams similar in style to below where there is a single rotor Element, have reduced dimensions. Some are considerably reduced. The Smith diagrams are shown later for all solutions to Order 22.





TWIN RECTANGLES WITH ROTOR ARMS
[22] 315 x 312 Perfect
Note extra Stator wire 84. This Wire has the same value in any pair.
Note the Rotor Arms 93 32 61 & 89 17 72

G5. USING REDUCED RECTANGLES

The term Reduction Index has been defined. This is the linear ratio between the full and reduced Elements (taken individually) or between the full and reduced dimensions.
Reduction Index = Larger full dimension ÷ by Larger reduced dimension.
An RI can be 1, 2, 3 or any positive integer, and may be a huge number in higher Order solutions.

G6. REDUCED RECTANGLES

1. What type or types of solution tend to reduce?
2. How can very reduced solutions be found readily?
3. How does the Reduction Index relate to other features of squared solutions?
These are not easy issues! It is relatively easy to make general statements and difficult to give specific ones. We begin with an elementary fact that the smaller the reduced Elements in a solution are found to be, the greater the probability that an Element is duplicated in size, and also that more Elements are duplicated.
From this statement we may expect Invalid solutions (always Imperfect) and Imperfect solutions to tend towards higher Reduction Indexes than Perfect solutions.
My list of solutions clearly shows this to be true i.e. in Order 13 solutions, Invalid solutions RI's 192 down to 5, Imperfect RI's 32 down to 4 and Perfect RI's 6 down to 1. Wording needs to be corrected*.**

G6.1. STATEMENT ON REDUCTIONS

The following statements do not apply to [7] 8 x 7 which has an Reduction Index of 3 and a full size of [7] 24 x 21. Nor is it true for Repeater solutions or Duds both of which are Compound by their construction. Otherwise it appears true for all normal Valid solutions -
1. It is impossible for any Imperfect or Invalid Simple Squared-Rectangle of Squared-square to have a Reduction Index of One Two or Three.

Put differently -
2. In any Rectangle or Square where the Reduction Index is found to be One Two or Three that Solution will automatically be Perfect - containing no duplicated Elements.
Again put differently -
3. Imperfect and Invalid Solutions always have a Reduction Index of 4 or more.
So far however, I know no way of proving the statements above.

G6.2. THE MOST FREQUENT REDUCTION INDEX

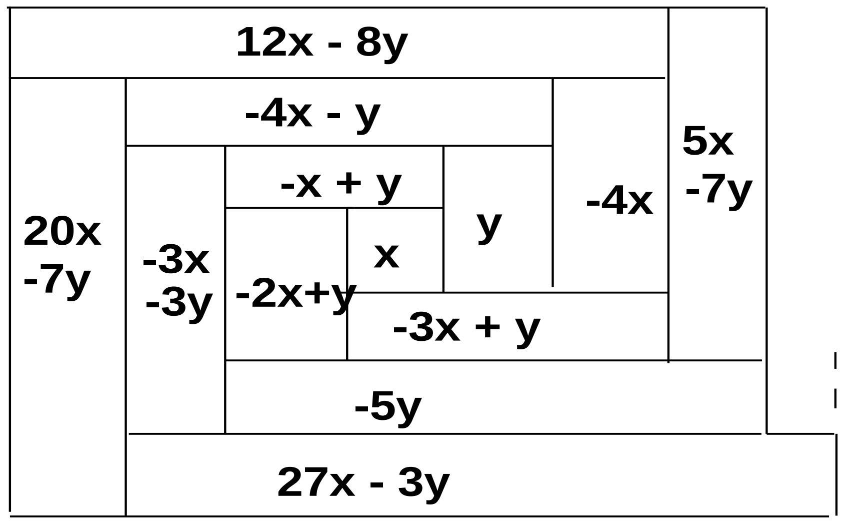
Without doubt for Rectangles, the most frequent Reduction Index encountered for any given Order is ONE, and all solutions for these are Perfect as already mentioned.
As it is possible for some Complex Compound solutions to also have an Index of ONE these may be also be automatically Perfect, but I need to check this out.

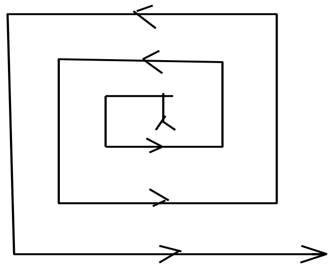
G7. SPIRAL CONSTRUCTION

F7.1. SPIRAL FORMS

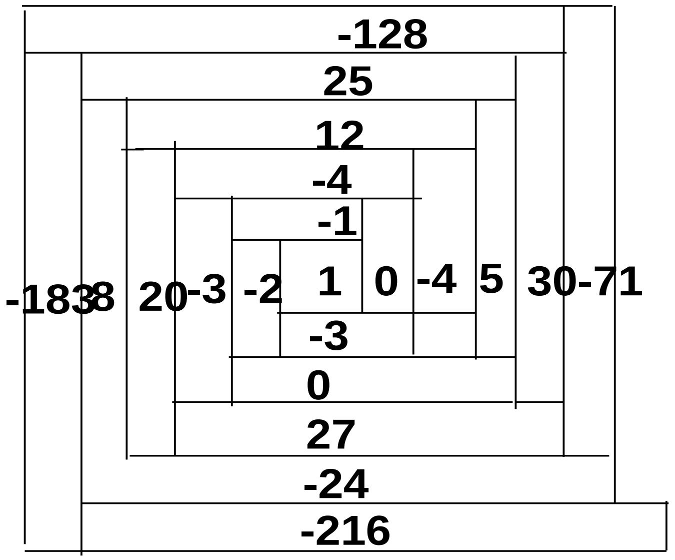
An example of this curious construction is shown below in algebraic format.
The inmost Element is called x, the next y, with Elements then drawn in spiral fashion for as long as we please. By chopping the last Element chosen e.g. at AB, a solution is formed which can be calculated. The fact that the actual solutions will look very different when drawn to scale in squares is not a problem. Also many Elements will show as negative, but this is not a problem either. To work out x and y coefficients for a greater amount of values, it is more convenient to consider each separately in two diagrams, leaving out the x's and y's. Below 1 and 2 show the coefficients of x and y respectively.

SPIRAL FORM WITH ALGEBRAIC WORKING.

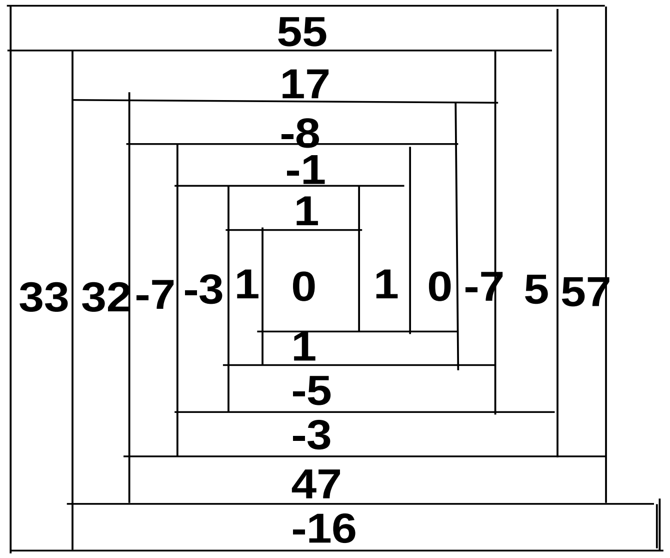




SPIRAL. (OTHER DIRECTION DOES NOT ALTER THE RESULTS).



COEFFICIENTS OF x



COEFFICIENTS OF y

x1	1	y1	0	x27	1,220	y27	-305
x2	0	y2	1	x28	1,525	y28	-11
x3	-1	y3	1	x29	1,536	y29	499
x4	-2	y4	1	x30	1,037	y30	1,209
x5	-3	y5	1	x31	-172	y31	2,024
x6	-4	y6	0	x32	-2,196	y32	2,745
x7	-4	y7	-1	x33	-4,941	y33	3,061
x8	-3	y8	-3	x34	-8,002	y34	2,573
x9	0	y9	-5	x35	-10,575	y35	865
x10	5	y1	-7	x36	-11,440	y36	-2,368
x11	12	y11	-8	x37	-9,072	y37	-7,137
x12	20	y12	-7	x38	-1,935	y38	-12,943
x13	27	y13	-3	x39	11,008	y39	-18,577
x14	30	y14	5	x40	29,585	y40	-22,015
x15	25	y15	17	x41	51,600	y41	-20,512
x16	8	y16	32	x42	72,112	y42	-11,007
x17	-24	y17	47	x43	83,119	y43	9,073
x18	-71	y18	57	x44	74,046	y44	40,593
x19	-128	y19	55	x45	33,453	y45	81,185
x20	-183	y20	33	x46	-47,732	y46	123,712
x21	-216	y21	-16	x47	-171,444	y47	155,231

x22	-200	y22	-95	x48	-326,675	y48	157,165
x23	-105	y23	-199	x49	-483,840	y49	107,499
x24	94	y24	-311	x50	-591,339	y50	-14,279
x25	405	y25	-399	x51	-577,060	y51	-219,176
x26	804	y26	-416				

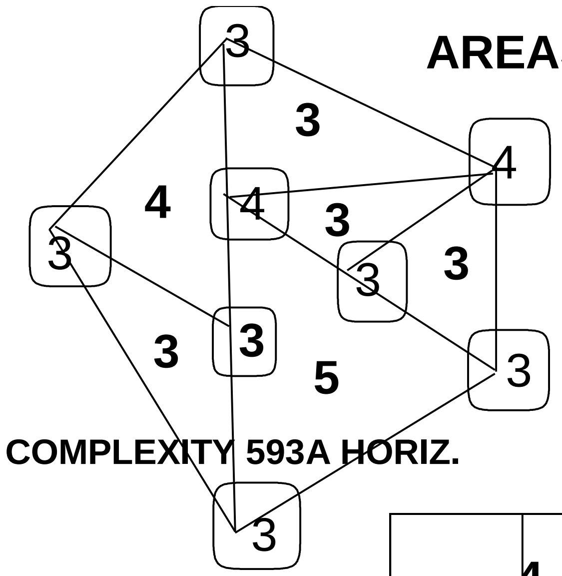
To be continued *****

Order	Dimensions & Reduction Index	Order	Dimensions
1	1 x 1 r1 inv square 2	20	27376 x 23984
2	2 x 1 r1 inv 3	21	47145 x 41305
3	3 x 2 r1 inv 5	22	81186 x 71129
4	5 x 3 r1 inv 8	23	139810 x 122491
5	8 x 8 r1 inv square 16	24	240765 x 210939
6	13 x 11 r1 inv 24	25	414616 x 363256
7	24 x 21 r3 inv 45	26	714005 x 625555 inv
8	40 x 35 r8 inv 75	27	1229576 x 1077267
9	69 x 61 r1 Perfect 130	28	2117432 x 1855131
10	120 x 104 r4 inv 224	29	3646397 x 3194693
11	205 x 181 r1 Perfect 386	30	6279400 x 5501528
12	354 x 309 r3 663	31	10813653 x 9474093 imp
13	610 x 535 r5 1145	32	18622018 x 16315181
14	1049 x 919 1968	33	32068674 x 28096111
15	1808 x 1584 3392	34	55225945 x 48383855
16	3113 x 2727 5840	35	95101984 x 83321056
17	5360 x 4697 10057	36	163773585 x 143485839
f18	9232 x 8087	37	282031840 x 247094641
19	15897 x 13929	38	485682464 x 425517679

G8. COMPLEXITIES

G8.1. A LOOK AT COMPLEXITIES (SEE SECTION B ALSO)

Look at the network below from which [13] 593 x 472, 473, 465, 480 and 510 are obtained as well as Singlend solutions of Order 12.



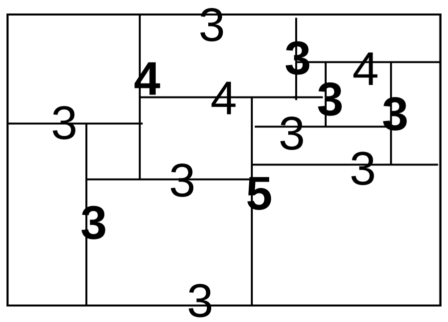
COMPLEXITY 593A HORIZ.

AREAS IN BOLD POINTS IN LIGHT

SHOWN ARE NUMBERS OF WIRES BORDERING THE AREAS (IN BLUE) & NUMBERS OF WIRES JOINING THE POINTS IN RED. THESE CORRESPOND TO THE VERTICAL & HORIZONTAL LINES IN ANY RECTANGLE OBTAINED FROM THE NETWORK, ONE OF WHICH IS . BELOW ([13] 593X480).

NOTE IN BOTH DIAGRAMS **3 3 3 3 4 5 &**

3 3 3 3 3 3 4 4 APPEAR!



BUT IT IS NECESSARY TO INCLUDE THE HORIZ TOP & BOTTOM FOR THE RED NUMBER. THE ABOVE SERIES IS CALLED V4110 & H6200 (IE SIX 3'S & TWO 4'S, NO 5'S OR 6'S.

-

It is evident that wherever Poles are chosen from any given network the "areas" contained in it are fixed. Since these represent the vertical internal lines in all solutions which can be drawn from it, the five related solutions all have code V4110 as explained above.
However the horizontals relate to the other complexities i.e. of 465 472 480 473 & 510. The light figures above marking the points are in fact the horizontals with code H6200.
Note however that two of these must include the two poles (wherever the poles happen to be).
This explains why the top and bottom border is involved in the above.
But in the vertical numbers the borders are excluded.
Observe that there are 5 Elements bordering the vertical sides which are the number of external wires.
But according to the choice of poles there could be 1 and 4, 2 and 3, 3 and 2 or 4 and 1 Elements at the left and right. All the Above comments are equally true for vertically arranged solutions.

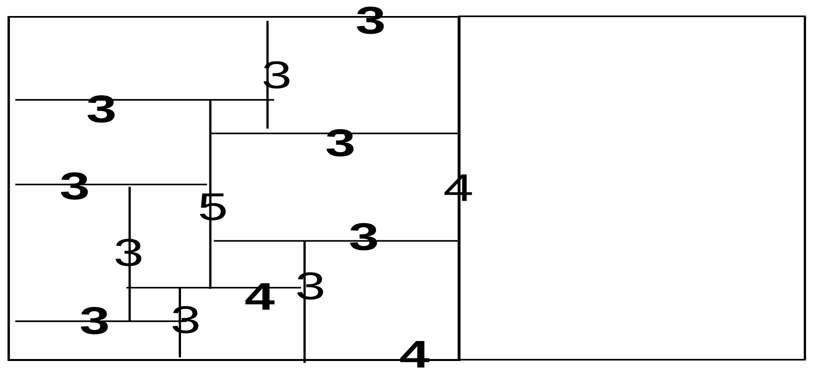
G8.2. SINGLEND SOLUTIONS OF THE ORDER BELOW

What happens when poles one wire apart are considered? A solution of the Order Below with a single end is produced with the single end removed, the Semi-perimeter is 593 as shown earlier.
Below is an example connected to that above and with the single Element has the same Complexity of 593A. As expected the two codes of V4110 and H6200 arise again, but note the involvement of some of the edges.

G8.3. FOUR DIFFERENT COMPLEXITIES OF THE SAME VALUE

It is easily observed that the single end can be placed at top left or right of the asymmetric solution Below with each having a Semi-perimeter of 593. But also observe that the codes will change - and this means four different complexities of 593. Sure enough four do exist for Order 13 (actually more).
It follows that there must be four complexities of same size from any Asymmetric solution and two from a symmetry 2 one. But in some instances the only available solutions are single ended, e.g. where there are only 2 Elements at one end and one at the other.
This effectively means that most Semi-perimeters will arise four times as different sides complexities for the Order above. This explains the puzzling repetition of sides which occurs when solutions are constructed in quantity - 4 out of 6 Order 10 solutions have sides of 209.

SINGLEND SOLUTIONS AT THE LEFT IS [12] 353 x 240¬



NOTE THE 33333344 H6200 CODE AGAIN!

NOTE THE 33345 V3110 CODE ALSO!

WITH THE SINGLE END, THE COMPLEXITY IS 593A. THIS IS CONNECTED TO THE LAST DIAGRAM

G9. UNIT AREAS

G9.1. UNIT AREAS

Codes used - LA17 means largest unit area for Order 17
SA14 means smallest unit area for Order 14
SA22? means smallest area so far found for Order 22... and so on ...
For any given Order there is a large, sometimes very large, range in unit size
Note that in this and f11 only full dimension solutions are relevant though Reduction indexes are shown in the tables to identify the solutions which can reduce, e.g. for Order 12 it varies from 73071 (in 353 x 207) to 112892 (in 338 x 334) the largest being 54% greater. By Order 18 the largest unit area is already over 3 times (200%) greater than the smallest.
Clearly the larger the Order the wider the range becomes.

G9.2. LARGEST UNIT AREAS FOUND BY COMPUTER "LA"

In the table below are a list of largest areas so far found. For Orders 16 onward there will almost certainly be larger examples than these shown.

Order Full Dimensions Index Unit Area Elongation Factor
7 24 x 21 INVALID 3 504 b 87% a/b
8 40 x 35 INVALID 15 1,440 a 87% 2.78
9 66 x 64 2 4,224 96% 3.02

10 114 x 110 2 12,540 96% 2.97
11 196 x 196 INVALID 28 37,248 100% 3.06
12 338 x 334 2 112,892 98% 2.94
13 595 x 581 7 345,695 97% 3.06
14 1031 x 1006 table 7 1,037,186 97% 3.00
15 1810 x 1794 Below 2 3,247,14 0 99% 3.13
16 3139 x 3117 1 9,784,263 99% 3.01
17 5904 x 5056 4 29,850,630 85% 3.05
18 10448 x 8728 1 91,190,144 83% 3.06
19 16801 x 16399 1 275,519,600 97% 3.02
20 29195 x 28566 1 833,984,400 97% 3.03
21 49773 x 48703 1 2,424,095,000 97% 2.91
22 90288 x 79443 3 7,172,750,000 87% 2.95

G9.3. OBSERVATIONS

1. The largest solution can be Invalid (or imperfect) but is frequently Perfect.
2. The high elongation - often over 95%. Note that Orders 9-16 all have factors 96-100%.
It is tempting to believe that all higher Orders may have factors in this range (when better solutions have been found), but this cannot be assumed.
For any given Semi-perimeter it is observed that the more square the solution is, the larger the area is (e.g. take 1000 700 x 300 gives 21000 whereas 600 x 400 gives 24000 and 500 x 500 gives 25000)
3. The factor which hovers around 3.000 these figures for Orders 14+ may be a little higher as larger solutions are discovered. Possibly, the factor is over 3 for all Orders apart from 10 and 12.
Most figures vary by under 5% which is significant.
4. In extending to Order 34, I discovered factors from 2.65 to 3.077. However as bigger solutions are to be found these lower factors must be greater, possibly at least 3.00 in each case.
5. There seems to be a tendency for the largest areas to be in xyz rather than xy solutions. In Order 11 where only one solution is xyz it happens to be the largest.

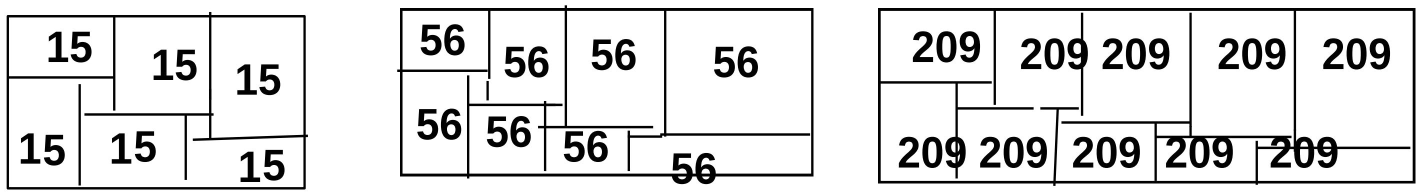
G9.4. SMALLEST UNIT AREAS PER ORDER “SA”

In Order to tackle the problem of establishing possibly smallest unit areas for full dimension solutions, I looked at various basic computer programs and also his register of all solutions to Order 13.
After a while the following became evident -

1. xy solutions *tend* to be smaller than xyz ones, and
2. Very elongated solutions *tend* to have much smaller unit areas than square-like ones.
This is not difficult to see, since if any particular Semi-perimeter is considered, the more elongated the solution, the smaller the unit area.
3. Often Invalid solutions are found to have the smallest (or particularly small unit areas).
Even smaller are Dud solutions, so it is important to look separately at each category and also at all non Dud categories.
4. Adding of triads is clearly the way to find very elongated and thus small unit areas.
Each additional triad involves only 3 Elements.
Most of the smallest unit area solutions have a number of triads at one or both ends.
5. Should we consider solutions with many triads at one end, or with several at each end?
There would have to be 6 or 7 Elements in the portion between the triads at both end, and as the construction is not particularly wide, the one end is better.
6. The most elongated solution is not always the one with the smallest unit area.

G9.5. THE INVALID SERIES "SA"

One group of smallest unit area solutions is now mentioned, but is rather dull.
xy Zero-one solutions containing just zeros and ones. Examples with full dimensions are below.
The series is for Orders 5, 8, 11, 14, 17, 20, 23, 26 ... only, and consists of Triads throughout.



...........& SO ON ORDERS 17,20,23,26 ....

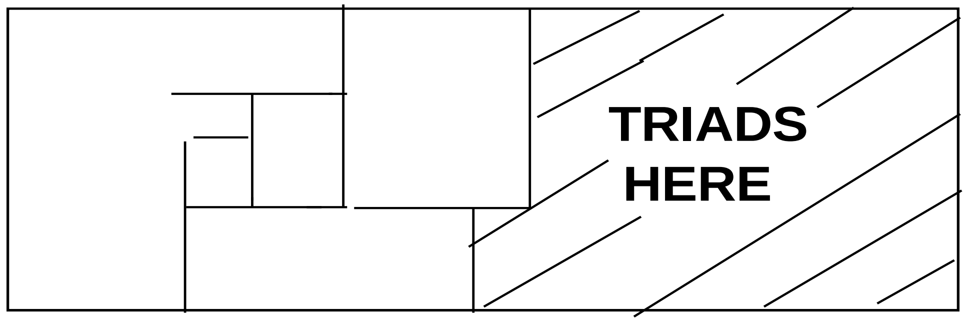
ORDER 8 45x30 UNIT AREA 1350 | ORDER 11 224x112 UNIT AREA 25088 | ORDER 14 1045x418 UNIT AREA 4368100

THE SMALLEST AREAS POSSIBLE APART FROM DUD SOLUTIONS

Since Triad constructions are being considered three series with Orders in progressions of three need to be discovered.

G9.6. THE SERIES FOR 5, 8, 11, 14… VALID "SA"

The following series commences with a fixed pattern of 8 Elements followed by a series of triads. Although there are some other patterns for 8 Elements this happens to have the smallest unit areas of the lot. The pattern is -



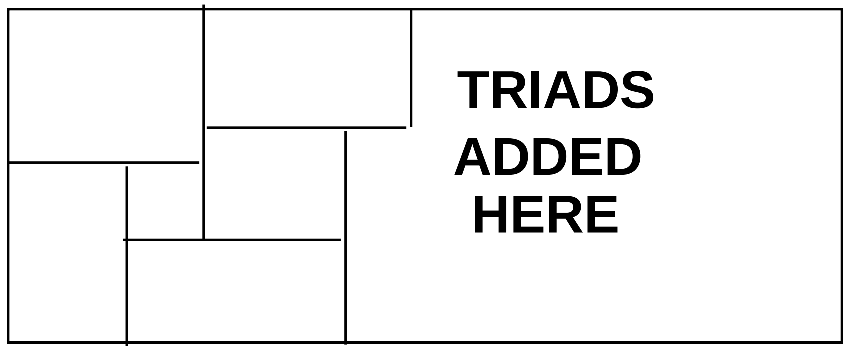
SMALLEST UNIT AREAS FOR VALID RECTANGLES OF ORDERS 11,14,17,20,23,26,29,32.......

The solutions found by computer are tabled below-

Order	Full Dimensions	Index	Triads	Unit Area and Elongation	
14	1015x473	r1	1	480,095 46%	
17	4670x1775	r5	2	8,242,550 37%	
20	20722x6587	r1	3	136,495,814 31%	
23	89627x24580	r1	4	2,203,300,541 27%	
26	76073x18349	r5	5	34,896,586,925 24%	
29	1590740x342397	r1	6	544,664,603,780 21%	
32	6575644x1277843	r1	7	8,402,640,655,892	19%
35	5385025x953795 x5	r5	8	128,405,247,996,875	17%

G9.7. THE SERIES 9, 12, 15, 18, 21 … VALID “SA”

The series below is not the most elongated one possible, but has the smallest unit areas

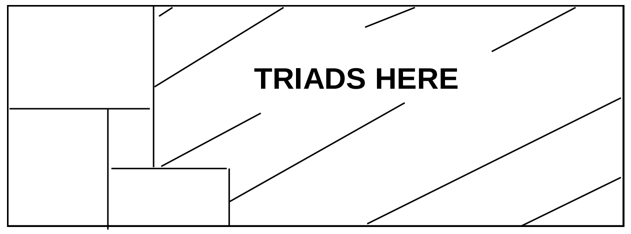


PATTERN FOR SMALLEST UNIT AREAS FOR ORDERS 9,12,15,18,21,24.......

Order	Full Dimensions	Index	Triads	Unit Area and Elongation	
9	75x55	5	1	4,125	

12	353x207	1	2	73,071	
15	1705x773	1	3	1,317,965	
18	7806x2885	1	4	22,520,310	
21	34576x10767	1	5	371,633,772	
24	13537x3653 times 11	11	6	5,983,529,981	
27	630711x149965	1	7	94,584,575,115	
30	2633684x559677	1	8	1,474,012,360,068	
33	10873414x2888743	1	9	22,711,767,378,602	
36	444777781x7795295	1	10		

G9.8. THE SERIES FOR 10,13,16,19....."SA"



PATTERNS FOR SMALLEST UNIT AREAS FOR ORDERS 7,10,13,16,19,22,25...............

The series is below. The Order 7 solution of 24 x 21 (8 x 7) is Invalid.

Order	Full Dimensions	Index	Triads	Unit Area & Elongation	%
7	24 x 21	r3	1	504	87%
10	130 x 79	r1	2	10,270	60%
13	633 x 295	r1	3	186,735	46%
16	2913 x 1101	r3	4	3,207,213	37%
19	12926 x 4109	r1	5	53,112,934	31%
22	55908 x 15335	r1	6	857,349,180	27%
25	79089 x 19077 x3	r3	7	13,579,027,677	24%
28	992287 x 213589	r1	8	211,941,588,043	21%
31	4101828 x 797125	r1	9	3,269,669,644,500	19%
34	5598562 x 991637	r3	10	49,965,671,033,946	17%

G10. ELONGATION

G10. "ELONGATION" AND "LENGTH RATIO"

How elongated a Squared-Rectangle is can be measured in two distinct ways.

1. Percentage of smaller side divided by the larger side, i.e. n / m 1% to 100%

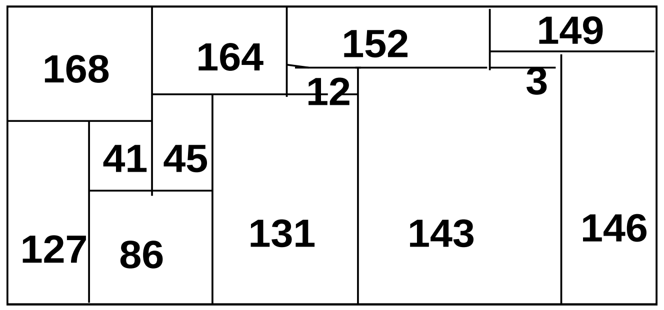
2. Numerical Ratio of Larger Side divided by smaller side. i.e. m / n
i.e. 1.00 (square) up to 2.00, 3.00 and so on ...
In the case of 1. above the term "<u>Elongation</u>" is used in this book and in this section, but in the case of 2. "<u>Length Ratio</u>"

Codes used - LN11 means largest elongation for Order 11 (most elongated)
SN22 means smallest elongation for Order 22 (least elongated or most square)

G10.1 ELONGATION VARIATION FOR AN ORDER

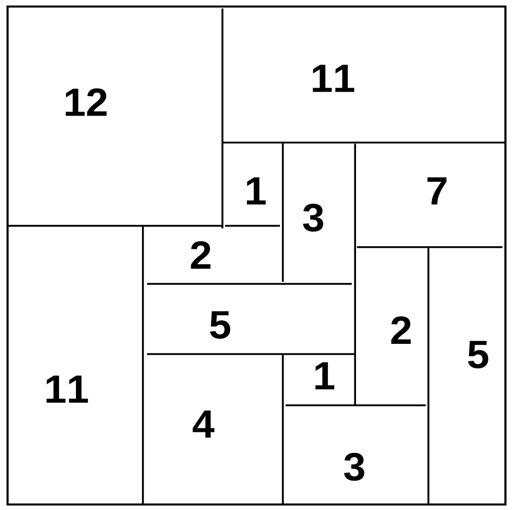
This is simply the lower dimension divided by the higher dimension of a solution and expressed as a percentage to 2 decimal places.
Whether full or reduced dimensions apply this does not affect the Elongation.
100% is least elongated and Square. As Orders increase so more and more Triads can be applied to one end, therefore the minimum elongation increases with the Order.
Obviously Singlend solutions have more elongated solutions still and duds the ultimate in elongation.
Below shows the most and least elongated solutions for Order 13.

MOST & LEAST ELONGATED - ORDER 13



[13] 633 x 295 46.60%

[13] 23 x 23 (529 x 529)   100.00%



G10.2. MOST ELONGATED INVALID SERIES "LN"

See F10.3 for Zero-one series.
The greatest maximum elongations for the Orders shown are as follows - These solutions are Invalid however.

[5] 2 x 2	100% Length Ratio	1.00
[8] 3 x 2	66.66%	1.50
[11] 4 x 2	50%	2.00
[14] 5 x 2	40%	2.50
[17] 6 x 2	33.33%	3.00
[20] 7 x 2	28.57%	3.50

G11. DIMENSION RECORDS

These are coded as follows -
LLD 12 means largest larger dimension for Order 12
LSD 13 means largest smaller dimension for Order 13
SLD 14 means smallest larger dimension for Order 14
SSD 14 means smallest smaller dimension for Order 14, and so on
LLD22? means largest larger dimension *so far found*, but may not be the absolute largest in fact. note the use of ?.
NB: The smallest and recessive dimension are often but not always the same.

G11.1. LARGEST LARGER DIMENSION "LLD"

The table below shows the Largest Dimensions found on the Computer Records to date. However the real maximums are likely to be somewhat larger for Orders beyond 17.

Ord.	RI	Full Dimensions	Type	Full Dimensions	Valid Type
7	3	24 x 21	INVALID	na	na
8	8	40 x 35	INVALID	na	na
9	1			75 x 55	SYMMETRIC
10	1			130 x 79..	PERFECT
11	16	224 x 144..	INVALID	209 x 129..	PERFECT
12		392	INVALID	386 x 277..	PERFECT
13	1			663 x 503..	PERFECT
14	1			1166 x 871..	PERFECT
15	1			2037 x 1544..	PERFECT
16	1			3581 x 2435..	PERFECT
17	1			6256 x 4261..	PERFECT
18	1			10925 x 8251	PERFECT
19	2			18456 x 9508	PERFECT
20	200			34000 x 22800	IMPERFECT
21	6			59316 x 33924	IMPERFECT
22	100			106000 x 64200	IMPERFECT
23	38			181260 x 124412	IMPERFECT
24	128			321536 x 190208	IMPERFECT
25	87			555060 x 328773	IMPERFECT
26	240			939600 x 558240	IMPERFECT

27 2304 1624320 x 1029888 IMPERFECT

G11.2. LARGEST SMALLER DIMENSION "LSD"

G11.3. SMALLEST LARGER DIMENSION "SLD"

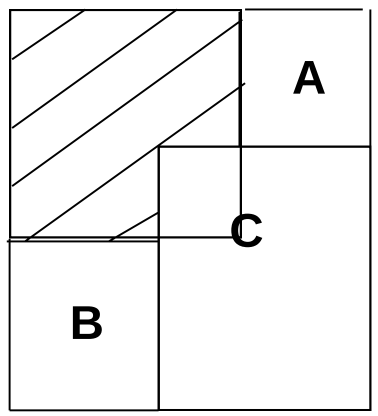
G11.4. SMALLEST SMALLER DIMENSION "SSD"

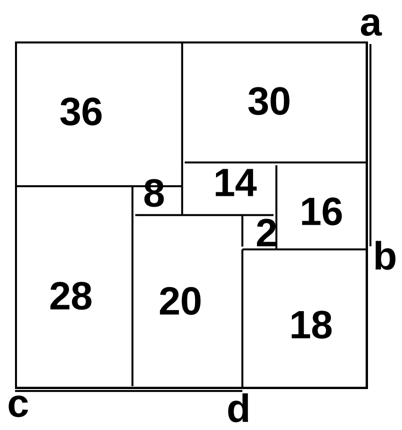
This is in for every Order simply "1" so is not interesting!

G12. ADD-ON SERIES

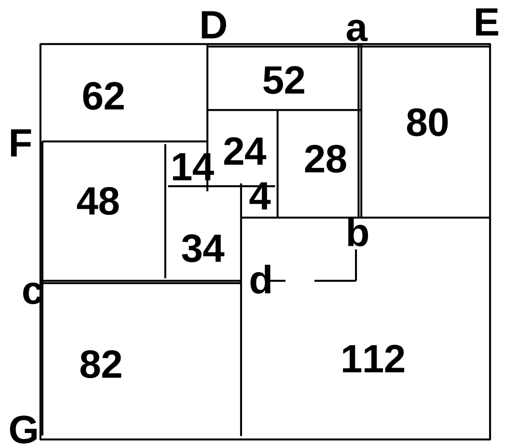
G12.1. S2233 SERIES

In the Squared-squares Section it is shown how easily S2233 solutions can be converted into many more by adding pairs of Elements as in A & B below where C is suitably increased in size to fill the Square.

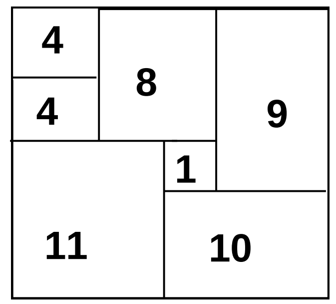




[9] 66 x 64 (33 x 32)



[11] 194 x 192 (97 x 96)



[7] 21 x 19 Invalid

But what happens when the S2233 solution is a rectangle and not a square? Two things are soon apparent - the Elements contained within the shaded area no longer stay the same and nor do they keep to the same proportions as is always true in the case of Squares. So the newly formed Rectangle has to be recalculated.

In Above 2 & 3 [9] 33 x 32 has been shown in full size 66 x 64 and Elements then added at ab and cd with the bottom right Element suitably increased in size. This gives [11] 97 x 96) as full size 194 x 192.

G12.2. THE DIFFERENCE IN THE DIMENSIONS IS FIXED IN THIS S2233 SERIES

The chart in F13.1 seems to provide little of interest - there seems to be no obvious links between the two Solutions, but by looking at several such pairs of Solutions an interesting feature is seen.

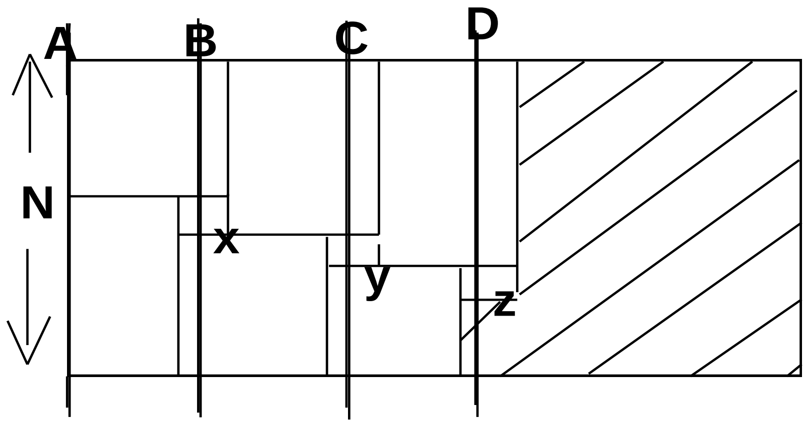
Looking at above 3 it can be seen that an Order 13 Solution could be drawn by adding Elements at DE and FG (and suitably increasing the top left Element). Then an Order 15 Solution could be drawn similarly from the bottom left as before. In fact the Series is unending with Orders 17 19 21 23 ...! If these were to be worked out we would find that the Larger (full) Element is always 2 more than the Smaller (full) Dimension!
From Solution [11] 185 x 183 with two added Elements [13] 530 x 528 is obtained - again the Dimensions vary by 2 in each case.
From [10] 105 x 104 the Solution [12] 297 x 296 is obtained. Here the Dimensions differ by one and this is true for the Order 14 16 18 20 ... Solutions in the Series. Also true for Order 12345678 if it was calculated! Also Solution [11] 177 x 176 (actually this has Sides 2234 which does not matter) with two added Elements gives [13] 552 x 551 (Sides 2233) - again both have a difference of 1 in the Dimensions!
The size of the differential is unimportant - it can be a wide range. For instance [12] 326 x 307 (difference 19) will give a S2233 series in which the Dimensions will always vary by 19 for the appropriate Oder 14 16 18 20 ... Solutions.

G12.3. CAN THE DIMENSIONS IN A SERIES BE ESTABLISHED BY A FORMULA?

It is very clear in the case of Squared-Squares that there is a Simple Formula from which the *Reduced Dimensions* are readily found.
See BRANCHLINK in M6. In the case of Squares [13] 23 x 23 [15] 41 x 41 [17] 70 x 70 [19] 117 x 117 [21] 193 x 193 and so on the series 23 41 70 117 193 ... Is easily extended by adding the Key Number 6 (in this instance) to the previous two values e.g. 6 + 193 + 117 = 316. (But what about the same series of *Full Dimensions* i.e. 529, 1681, 4900, 13689, 37249?) It would not be so easy to find 99856 our next number (316 x 316)!)
I cannot see at present any link between the Values!

G13. LENGTH RATIOS AND SYMMETRIC PATTERNS - TRIADS, PENTADS ETC.

Note: In this section Length Ratio is used rather than Elongation, meaning that it relates to the number obtained when the Horizontal (Greater) side is divided by the Vertical (Smaller) side.
A Squared-Rectangle may have parts which contain symmetry, leaving the rest of that Squared-Rectangle asymmetric. For example, a square with a Triad at left and a Pentad at the right, leaving an asymmetric section in the middle.
Surprisingly there is a definite relationship between the Symmetric parts and the *Length Ratio* of each such part.
For instance each Diad or Triad makes the solution Longer by 50% of the shorter side n. This was shown in

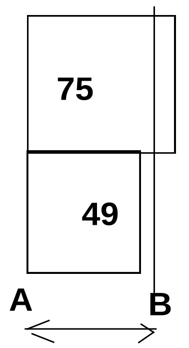


Lines B, C, D have been purposely drawn through the middle of Elements x y z.
Distances AB AC AD . . are then found to be half N, N, one & a half N . .
Also AB = BC = CD . .

D10.

G13.1. LINKING ELEMENTS INCLUDED OR EXCLUDED?

In Above three Linking Elements x y z are shown. If the pattern does not have the Linking Element "z" then the position of the vertical line at D does not change.



n here is 75 + 49 = 124

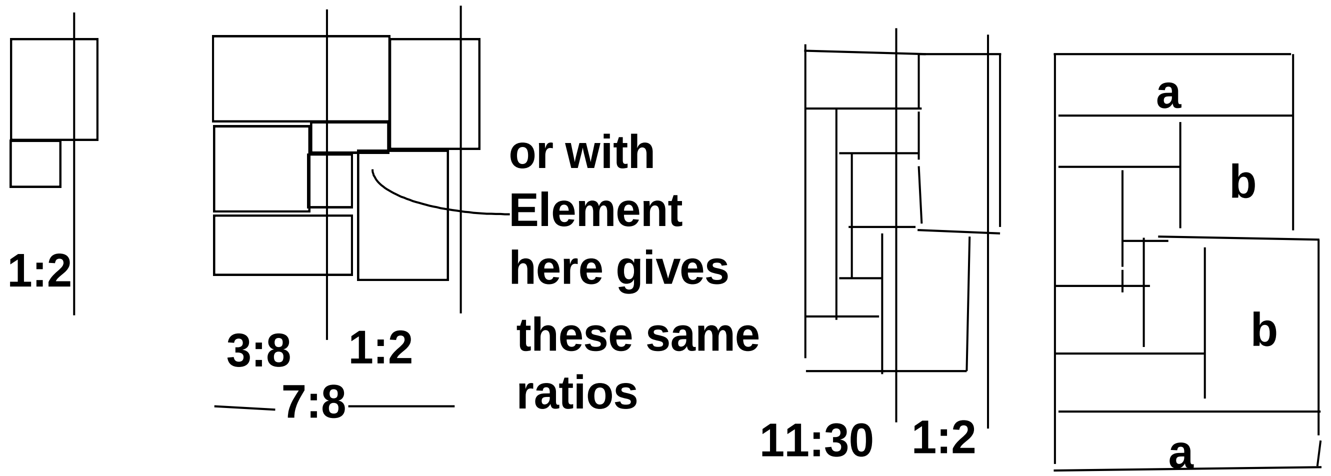
AB = 62 i.e. half of 124!

62 (i.e. average of 75 and 49)

Note interestingly that if a solution starts with a Diad, but *not* a Triad then these two Elements effectively also increase the *Length Ratio* by 50%, as shown above where two values have been chosen at Random.
<u>In fact whenever a Linking Elements is excluded instead of included it never affects the increase in *Length Ratio*</u>.

G13.2. ANY SYMMETRIC END OR COMBINATION OF THEM HAS A FIXED *LENGTH RATIO* ADDITION!

Upon testing a number of patterns with hypothetical values, the writer has discovered that each symmetric pattern has a fixed amount of additional Length Ratio. But for each part which is not regular, the additional Length Ratio is <u>never</u> fixed and varies with varying values of Elements. An attempt is made to show proofs by using random arithmetic - the reader can easily verify any of these if necessary!



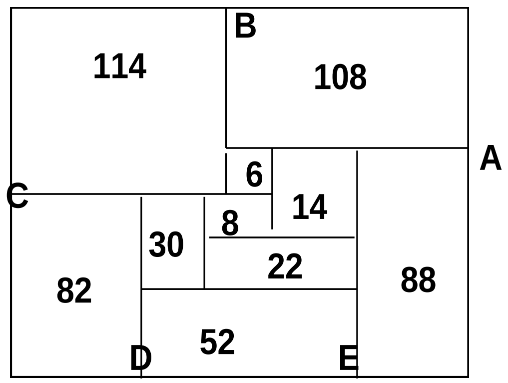
Without a & b
Ratio 1:1
Without a
Ratio 5:6
With a and b
Ratio 5:16

The patterns continue *ad infinitum* of course.

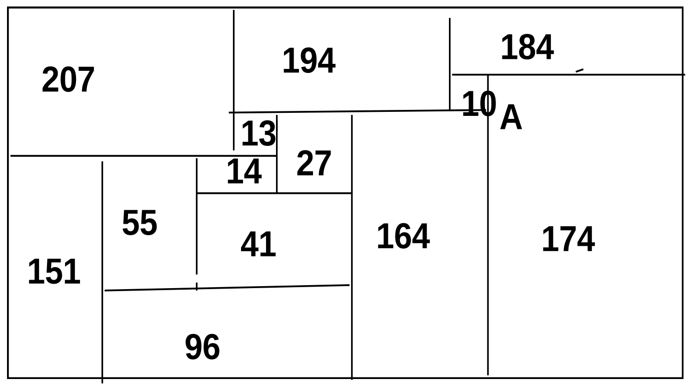
G14. CREATING SYMMETRIC SOLUTIONS

G14.1. SYMMETRIC SOLUTIONS FROM TWO KNOWN ORDINARY SOLUTIONS

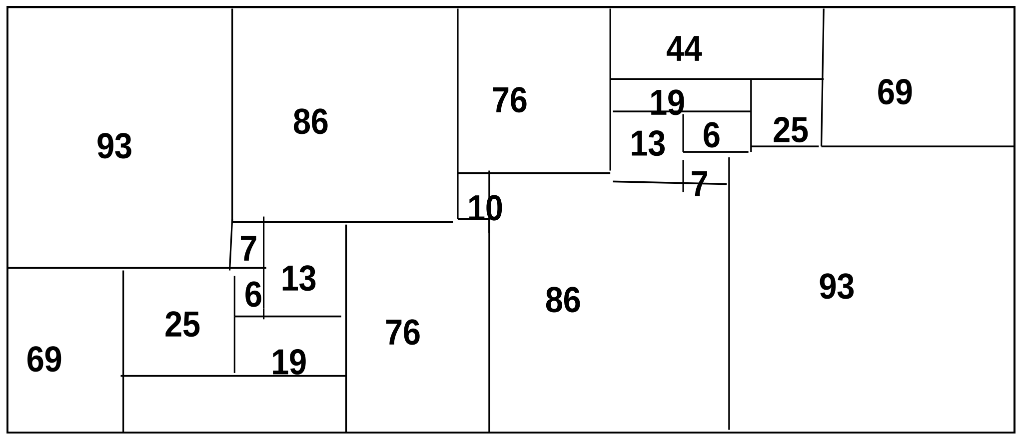
Consider any Squared-Rectangle e.g. [10] 111 x 98 and multiply each Element by two. Then calculate the same solution with a Triad added at some point on the edge - say point A as shown below.
For the new Solution to work it is necessary that both solutions are of full dimensions with the first doubled in size.



[10] 222 x 196 i.e. 111 x 98 times 2



[13] 585 x 358



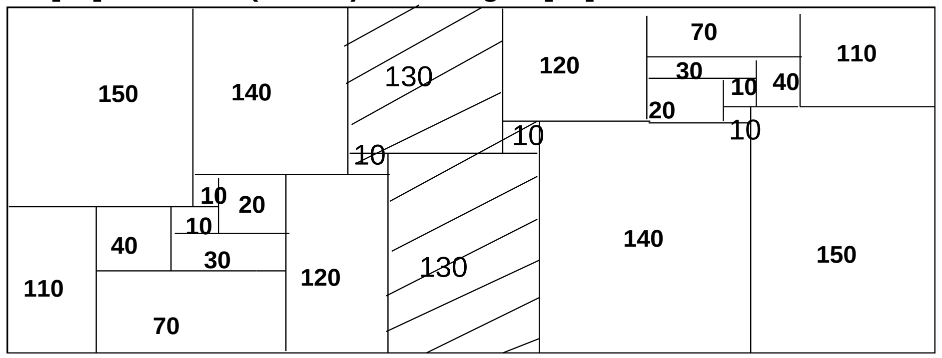
[21] 368 x 162
Symmetric

Now deduct the Elements in above 1 from the Elements in the corresponding places in Above 2 e.g. 207 - 114 = 93, and so on, ignoring the Triad part in Above 2. The first half of above 3 is found. By adding Element 10 (see above 2) we can then repeat each Element to make the Symmetric solution shown!
So from [10] 222 x 196 and [13] 585 x 358 we can obtain [21] 585 x 358, but note that four other solutions are possible with Triads added at B C D or E!
If B D or E is used the first Solution must be regarded as [10] 196 x 222 however.
This property is always found to apply.
If the first rectangle is coded [o] m x n, and the second [O] M x N the resulting Symmetric Solution is always <u>[2o + 1] (2M - 2m - N) x (N - n)</u>

G14.2. A SECOND SERIES OF SYMMETRIC SOLUTIONS

Interestingly if we subtract [10] 111 x 98 (rather than 222 x 196) from [13] 585 x 358 a different Symmetric solution is found as shown below.
The result here happens to be Invalid and reduces to [24] 72 x 26 but Valid Solutions are often found using this process. The Order becomes twice the original Order plus four (i.e. 10 x 2 + 4 in this case). The central Elements are always the same (130 & 130 here)

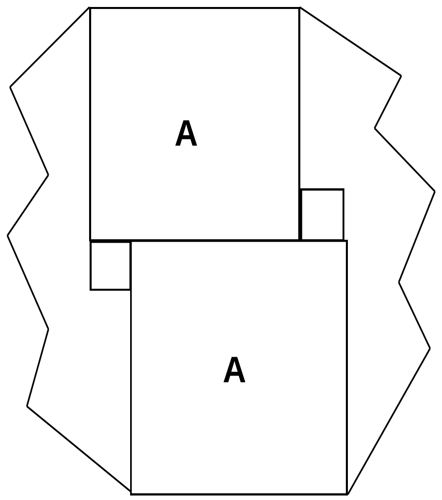
[24] 720 x 260 (Invalid) reducing to [24] 72 x 26

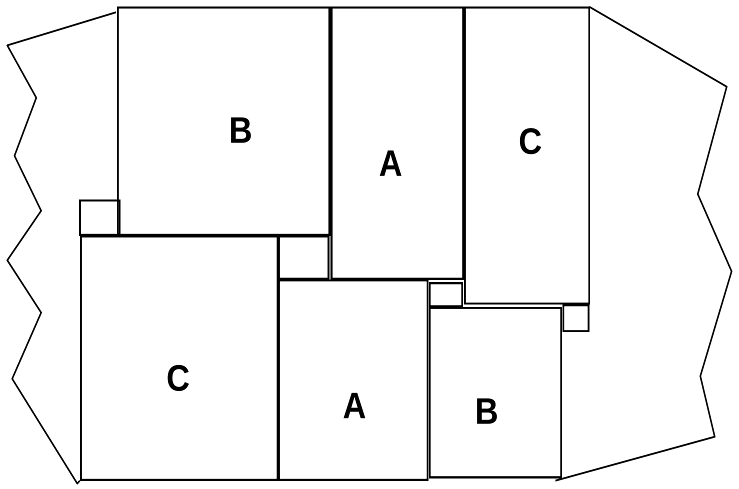


Again, if the original rectangle is coded [o] m x n and the second (one with Triad added) coded [O] M x N then the resulting Symmetric Solution is always
<u>[2o + 3] (2M - 2m - ½N - ½n) x (N - n)</u>

G14.3. FURTHER SERIES OF SYMMETRIC SOLUTIONS

We have seen the effect of deducting
1. The smaller x 1 from the larger rectangle (with Triad) x 1, and
2. The smaller x 2 from the larger rectangle (with Triad) x 1, but what about -
3. The smaller x 1 deducted from larger x 2 and
4. The smaller x 1 deducted from larger x 3 and
5. The smaller x 1 deducted from larger x 4
Yes further Symmetric Solutions are found! But in the case of No.5. it is not symmetric. It is in fact the solution for the larger rectangle with a Triad added (or the smaller rectangle with a double triad added! This has been mentioned elsewhere where the effect of adding repeated Triads was discussed.





Central format for 1. above like this. Central format for 3. above like this

In the case of 4. above there are five portions of numbers in the centre - not one or two as shown for type 1and type 3.

But if we continue the series (larger x 5 etc.) there are no new solutions found. However there is a Symmetric Solution for the larger x 1 + smaller x 1.

G15.1. HIGHEST COMMON DENOMINATOR (HCF) FOR REDUCED SIDES

Take [9] 33 x 32. Here the HCF of 33 and 32 is 1, and the Reduction is 2 (from 66 x 64).
In [9] 69 x 61 the HCF is again 1, with a Reduction of 1.
From the Writer's vast data on SR's the following have been found -
1. For any given Reduction from 1 upwards, Solutions are possible with a HCF of only 1.
2. Where the Reduction is 1, the HCF is always 1. Hence the Dimensions cannot be EVEN x EVEN and are always prime to each other.
3. Where the Reduction is 2, the HCF is either 1 or 2 with no obvious reason why some 1 and the rest 2.
If the Reduction is 3, the HCF is either 1 or 3 (2 not being a factor of 3).
4. The HCF can never exceed the Reduction, but on occasions is the same number.
5. The HCF is always a factor of the Reduction, e.g. for RI 12 the HCF will be one of the following 1 2 3 4 6 12.
6. It is not possible to determine what Reduction will be for any given HCF.
7. Imperfect Solutions always have a Reduction of at least 4, and usually higher. All Solutions with Reductions 1 2 or 3 are always Perfect.
8. It seems that for any given Reduction however factorial that Solutions can be found for every possible HCF factor from 1 to the Reduction itself.

G15.2. IMPERFECT SOLUTIONS WITH REDUCTION FOUR

Perfect Solutions with a Reduction of 4 are found to have HCFs of 1 or 2 or 4 as one might expect.
But Imperfect Solutions with a Reduction of 4 are found to have a fixed HCF 4 only!! So the Reduced Dimensions are always double-even times double-even, as in say 104 x 100 or 126 x 96. Note however that some solutions with such double-even dimensions are Perfect! (In 2500+ Imperfect Solution of RI 4, all had a HCF of 4 without exception).
The Writer has no explanation for this weird fact!

G15.3. TABLE OF HCF V. RI SOLUTIONS

From the Writer's large database the following numbers of Solutions (Order 16 and greater - rectangles only) have been found. The results are somewhat inconsistent but often the Quantity of Perfect Solutions decreases as the HCF Increases within a given Reduction. With Imperfect Solutions the statistics seem quite chaotic however with no solutions for found for 1:1, 2:1, 2:2, 3:1, 3:3, 4:1, 4:2, 6:3 and 9:3! For RI 12 out of 1950 Solutions there are just 4 for 12:6 and 60 for 12:3 but 203 for 12:12.
There seems to be relatively few Imperfect Solutions where both RI & HCF are divisible by 3.
For many RI's, the largest quantity of Imperfect Solutions arises where HCF is 1.

REDUCTION: HCF	IMPERFECT SOLUTIONS	PERFECT SOLUTIONS	REDUCTION: HCF	IMPERFECT SOLUTIONS	PERFECT SOLUTIONS
1:1	0	47,629	6:3	0	3,660
2:1	0	24,869	6:6	22	1,637
2:2	0	12,652	7:1	308	12,164
3:1	0	25,963	7:7	18	1,681
3:3	0	7,306	8:1	65	15,665
4:1	0	16,576	8:2	1,673	5,057
4:2	0	11,726	8:4	925	2,115
4:4	2,577	4,131	8:8	879	1,074
5:1	2,048	22,204	9:1	498	9,121
5:5	649	3,237	9:3	0	2,378
6:1	102	13,290	9:9	129	816
6:2	642	6,806			

H. TWIN RECTANGLES

H1. TWINS INTRODUCTION

This is a tricky subject to study and to date my research on it is poor and limited at best.

H1.1. DEFINITION

TWIN RECTANGLES are simply two or more different rectangles with the same dimensions

They have several meanings, but the Rectangles concerned always have the same Elongation.

1. Same size as regards full Dimension solutions.
2. Same size as regards reduced Dimension solutions.

It does not follow that if the reduced Dimensions are Twins that the full Dimensions are always Twins.

An example of this is [13] 112 x 75 (1) and (2) the full Dimensions being 672 x 450 and 560 x 375!

3. Same size after one or both solutions have been suitably up-rated e.g. 33 x 32 multiplied up by 3, and 99 x 96.

Twins are numbered (1), (2), (3)… where they exist for the same Order. But the Order used is according to 1. largest Element size downwards.

If this is insufficient then

4. Smallest Element size downwards.

It is often possible to have Twins with different Orders, with or without the need for up-rating, but not with both full size.

H1.2. TWINS FOUND IN ALL GROUPS AND TYPES

Twin solutions do not always have the same status, e.g. one may be Perfect and the other Imperfect.

They can be Valid and Invalid, Perfect and Imperfect, Simple and Compound, symmetric and asymmetric, and solutions may be Rectangles or Squares.

Some can be made Twins by suitably up-rating the Elements by a common factor in either one or both solutions.

H1.3. ALL COMPLEX SOLUTIONS HAVE RELATED TWIN SOLUTIONS

In the whole of Section G, the term “Twins” should be read as *multiple* Solutions rather than only two’s. Whatever type of cover surrounds the original solution, it is clear that the solution can be switched round to different positions. Thus

1. 4 Twins for each asymmetric original solution and
2. 2 Twins for each symmetric original solution.

H1.4. OCCURRENCE OF TWINS IN SOLUTIONS

Compared to all existing possible Solutions, twins appear uncommon particularly where low Orders are considered, and occurrences may seem to be somewhat coincidental.
But Orders 14 upwards have many examples, and in high Orders - say 40 upwards there are so many Twins possible that it may be coincidental when a Solution *does not* have Twins!.
For small Dimension ranges (say under 1000) 20% of Solutions or more in higher orders, may be twins. Every Compound solution has Twin Solutions (in fact most have at two or four).

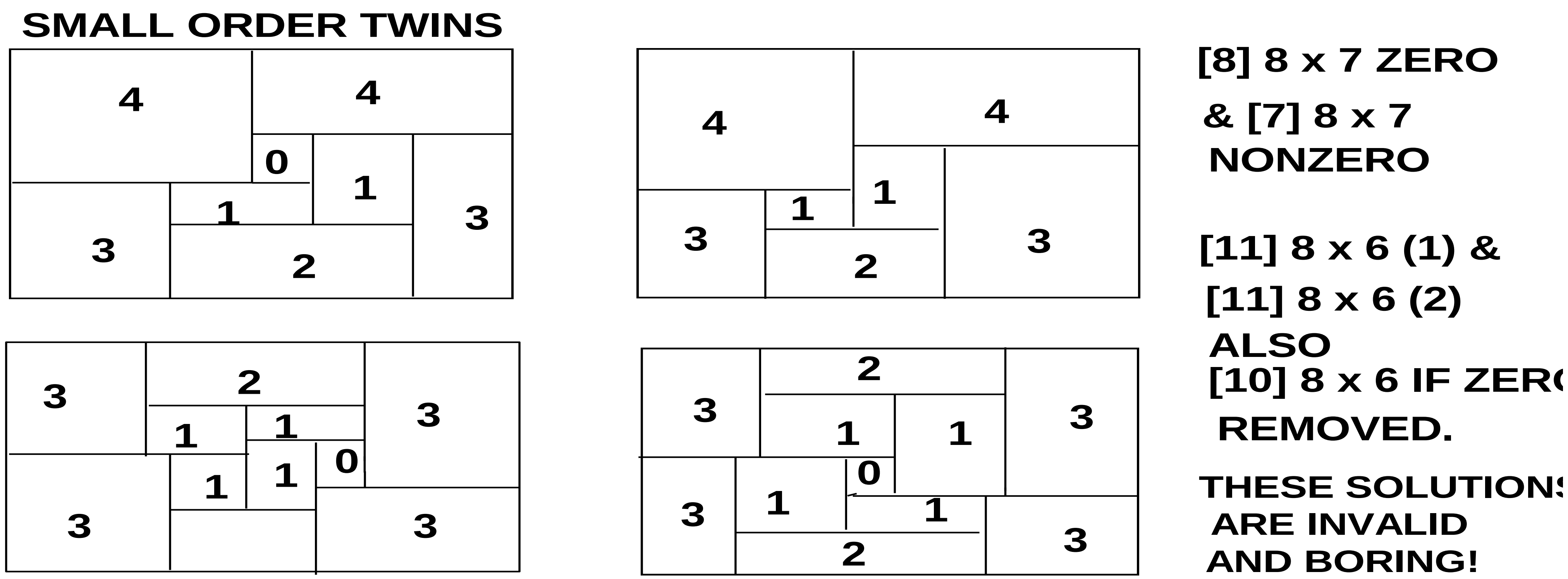
H1.5. USES OF TWINS IN THIS STUDY

The subject arises throughout this book, as Twins form a useful insight into the theory. Briefly it has two particular applications.
1. In study and formation of squared-squares section K. and L
2. In connection with highly reduced solutions. Neither of these is discussed here.

H2. TWINS - LISTING

H2.1. TWINS TO ORDER 12 - INVALID

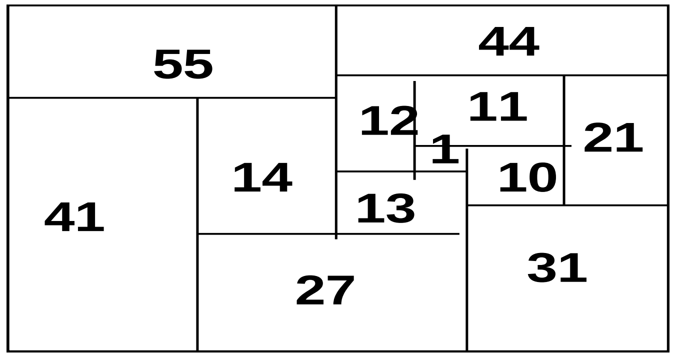
In the case of Zero solutions, if a zero Element can be removed without a dud resulting then twins result for two consecutive Orders.
The smallest is [7] 8 x 7 and [8] 8 x 7] where the reduced dimensions are the same but not the full.
Twins exist for all Zero solutions (but with different Orders). An example of 1. in G1.1.
Next is [10] 8 x 6 coupled with [11] 8 x 6(1) and (2). All are below.

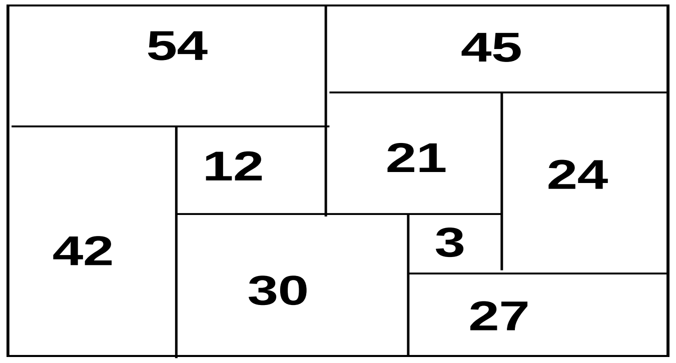


H2.2. TWINS TO ORDER 12 - VALID

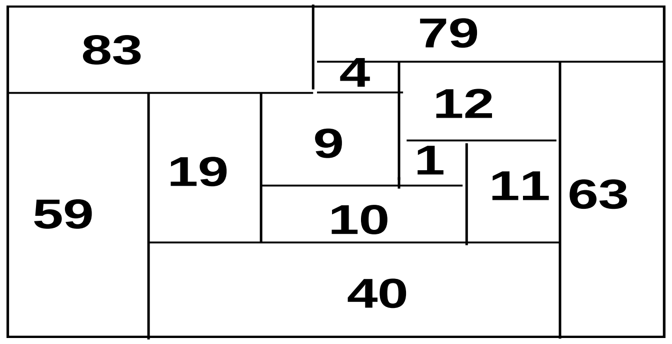
**The next example is [9] 33 x 32 up-rated 3 times with [12] 99 x 96, two Perfect squares.
An example of 3 in G1.1. Then comes [12] 162 x 142 (1) and (2) an example of 2
(and 1) in F1.1
Amazingly these are the only Twins to Order 12! See below.**

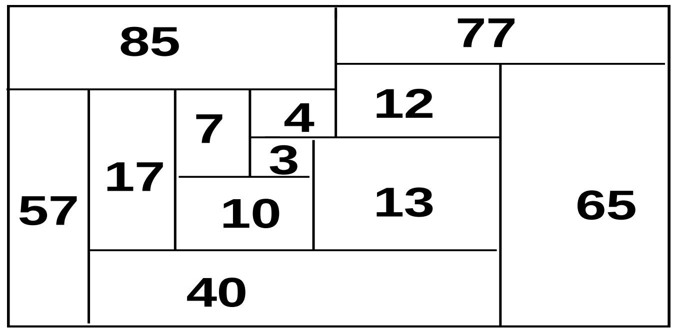
LOW ORDER TWINS - EACH PERFECT.





[9] 33 x 32
UPRATED TO
99 x 96 AND
[12] 99 x 96





[12] 162x142
(1) & (2)
NOTE DUP'D
EVEN NUMBERS
ONLY IN THIS
EXAMPLE.

H2.3. LIST OF VALID TWINS OF ORDER 13

No	Order and size	Elements Duplicated	Qty dup	Complex Same?	xyz	Sides
1	13 72x64(3)	(1 and 2) 3 5 16 17 35 (1 and 3) 3 5 6 16 35 (2 and 3) 3 5 16 27 35 37	5 5 6	different different 512 same	2 2 2 2 2 2	s2343 2234 s2343 2223 s2234 2223
2	13 18x16					
3	13 112x75	the lot!	13	different	2 3	s2334
4	13 112x93	9 11 17 36	4	different	2 2	s2224 2343
5	13 140x132	1 4 24 33 40 51	6	different	3 3	s3333
6	13 258x206	8 26 28 36 40	5	different	2 2	s2225
7	13 270x226	12 22 28 40 48	5	different	2 2	s2225
8	13 274x270 and 137x128	2 24 40 52 84 none with Above	5 0	540 same 548 same	2 2 3	s2233 s2234

9	13 545x528	58 64 74	3	different	2 2	s2234 2233
10	13 552x463	20 57 80 195	4	463 same	3 2	s2333 2224
11	13 577x511	1 4 57 132	4	different	2 2	s2223
12	13 578x537	25 30 100 132	4	different	2 3	s2234
13	13 593x422	none	0	different	2 2	s2324 2333
14	13 608x407	57	1	different	3 2	s2324
15	13 608x465	133	1	different	2 2	s2334 2323

H2.3.1. OBSERVATIONS ON ORDER 13 TWINS

1. The Above table shows that the complexities are usually different. 2. The Unknowns xy and xyz show no uniformity.
3. Many share the same sides Index, whilst others vary, some considerably.

H2.4. LIST OF TWINS WITH DIFFERENT ORDERS

In addition there are Twins for two different Orders up to Order 13 as follows:
1. [9] 33 x 32 times 3 and [12] 99 x 96. 2. [9] 33 x 32 times 4 and [13] 132 x 128 imp.
3. [10] 57 x 55 times 3 and [13] 171 x 165. 4. [10] 65 x 47 times 3 [13] 195 x 141.
5. [12] 106 x 99 and [13] 106 x 99.
Having the same dimensions for different Orders naturally means different Reduction indexes must apply.

H3. TWINS - WHY THEY OCCUR

H3. DO TWINS OCCUR BY COINCIDENCE OR DESIGN?

A study of this subject clearly shows definite relationships between some Twins, whereas others would appear to arise by coincidence. By general chance, one would expect small reduced solutions to be duplicated rather than large ones. Invalid solutions though fewer in amount seem to have a greater coincidence of Twins simply because the dimensions are so small.
By sheer chance one would expect Twins to occur less often than in fact they do!
This shows that they arise more often by design than coincidence.
In fact the Coincidence concept also works conversely, and for high Orders and highly reduced ranges of solutions it would be incredible coincidence if some were not frequently duplicated in size!

H4. TWINS - MAIN GROUPS

H4.1. TYPES OF ELEMENT DUPLICATION IN TWINS

It is useful to put Twins in groups. The following is just one possible division based on having all solutions to Order 13 available.

1. No duplication in any Elements for the twins. H5.
2. Even numbers of Elements duplicated (see above). H6.
3. Complete duplication of all Elements and H7.
4. All other duplications H8.

H5. TWINS - NO DUPLICATION OF ELEMENTS

H5.1. NO DUPLICATION IN TWINS

The only set to Order 13, [12] 593 x 422 shows no apparent relationships.
This type can be used to construct complex but Simple squared-squares.

H6. TWINS - EVEN ELEMENTS DUPLICATION

H6.1. EVEN NUMBER DUPLICATION IN TWINS

These form an interesting group when the Elements are compared in ascending Order of even followed by odd:-

4 10 12 40 1 9 11 19 59 63 79 83 for [12] 162 x 142 note differences of 2 in odds.
4 10 12 40 3 7 13 17 57 65 77 85
8 26 28 36 40 7 19 33 45 73 81 125 133 for [13] 258 x 206 differences of 2 in odds.
8 26 28 36 40 9 17 35 43 71 83 123 135
12 22 28 40 48 9 13 31 35 83 99 127 143 for [13] 270 x 226 differences of 6 in odds.
12 22 28 40 48 3 19 25 41 89 93 133 137
2 24 40 52 84 17 19 21 61 69 121 125 149 for [13] 274 x 270 differences of 6 in odds.
2 24 40 52 84 11 13 15 67 75 115 119 155

H6.2. OBSERVATIONS ON EVEN DUPLICATION

1. Differences in odd numbers of 2 2 6 and 6.
2. Highest numbers all odd - possibly a coincidence.
3. All contain Element 40 - possibly a coincidence.
4. All Reduction 2 cases.

H7. TWINS - TOTAL DUPLICATION OF ELEMENTS

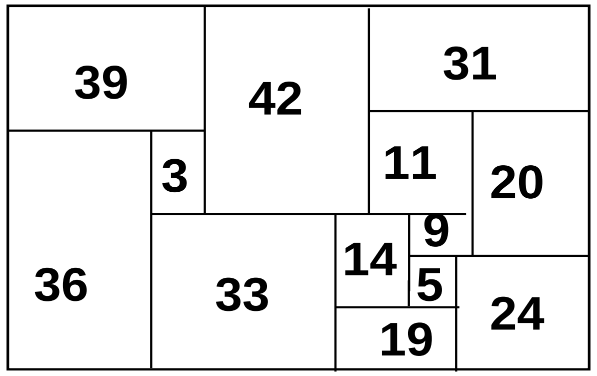
H7.1. HISTORY

Mr. Brooks who studied Squared-Rectangles with his friends in 1936 was so pleased with the solution [13] 112 x 75 that he cut one into pieces and asked his wife to reassemble, which she did. But her solution was different to his! The Twins with total duplication caused excitement and the team soon discovered why this happened by drawing both Smith Diagrams.

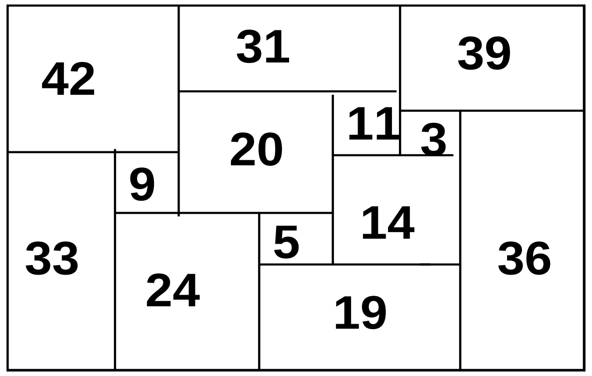
H7.2. EXAMPLE OF TOTAL DUPLICATION

These unusual solutions are shown below. Their full sizes are 560 x 375 and 672 x 450 and oddly the Reduction Indexes are different, 5 in the first and 6 in the second.

TOTAL DUPLICATION OF ELEMENTS



[13] 112 x 75 (1) R5

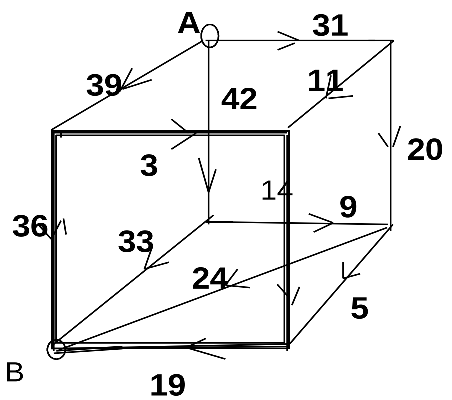


AND [13] 112 x 75 (2) R6

EACH RECTANGLE CONTAINS3,5,9,11,14. 19,20,24,31,33,36,39 & 41! THIS PROPERTY OCCURS IN RELATIVELY FEW RECTANGLES.

H7.3. THE CUBE NETWORK

Above 1 has an amazing smith diagram which can be arranged as a three dimensional cube!

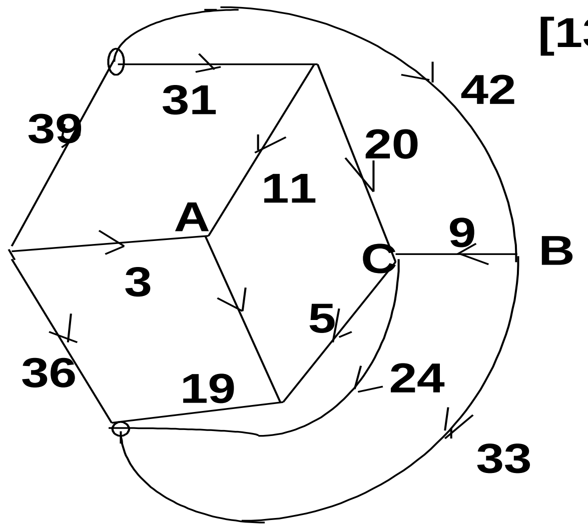


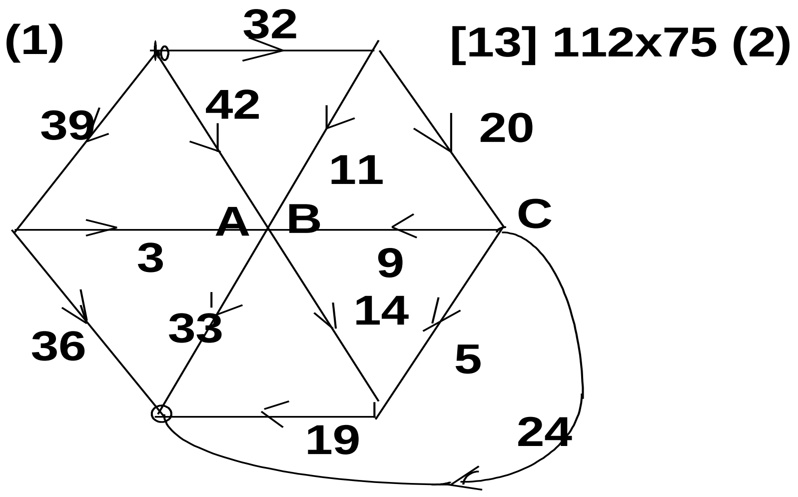
[13] 112x75 (1) EXPRESSED AS A THREE DIMENSIONAL CUBE WITH ONE DIAGONAL WIRE, AND WITH POLES FROM A TO B. THIS NETWORK CAN BE FLATTENED INTO A PLANAR DIAGRAM (SEE NEXT DRAWING).

H7.4. SMITH DIAGRAMS FOR TOTAL DUPLICATION

Below explains why the Elements are unchanged. In Below 2 three wires with Elements shown in bold are made to join at point A. The Smith diagram is made up of 3 parts -

1. A regular pattern (light numbers) called the Rotor.
2. A single wire (24 in above 2) called the Stator.
3. Three wires (dark numbers) which form part of the Stator in above 1. But form part of the Rotor in above 2.





[13] 112x75 (1) IS MADE INTO [13] 112x75 BY MOVING THE BOLD ELEMENTS. B IS SWUNG OVER TO MEET A IN 2ND DIAGRAM. THE EFFECT IS THAT NO ELEMENTS CHANGE VALUE IN THE MO

H7.5. LIST OF IDENTICAL TWINS TO ORDER 22 INCLUSIVE

Below is the listing of identical Element Twins up to and including Order 22. This list is probably not complete.

Amount	Order	Dimensions	Type of symmetry
1	13	112 x 75	3 triangular
2	17	259 x 181 Imperfect	4 square
3	19	1025 x 592	3 triangular
4	19	1025 x 592 different	3 triangular
5	21	5335 x 3750	5 pentagonal
6	22	153 x 120 Imperfect	3 triangular
7	22	2088 x 1155	3 triangular
8	22	2088 x 1155 different	3 triangular
9	22	3128 x 1375	3 triangular

H7.6. TABLE OF SYMMETRIES FOR IDENTICAL ELEMENT TWINS

Similar solutions are possible for Orders 25 26 28 31 33 34 36 37 40 41 43 45 46 49 50 . With 1 deducted from these numbers the result is a multiple of 3,4,5... None are possible for 1-12 14 15 16 18 20 23 23 27 30 32 35 38 39 42 44 47 48

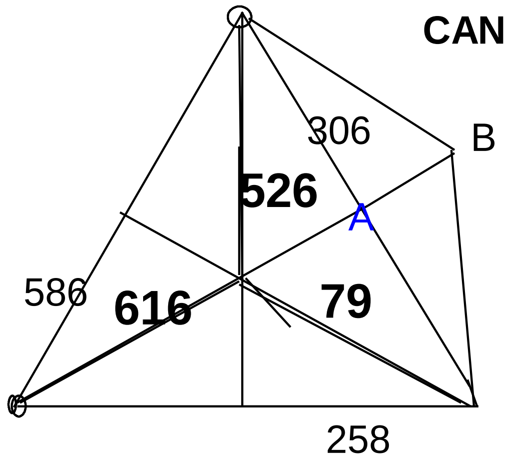
3 (=6=12) fold 13 16 22 25 28 31 34 37 40 43 46 49...
4 (=8=16) fold 17 21 25 29 33 37 41 45 49...........
5 (=10) fold 21 26 31 41 46......................

7 fold 29 36 43 50........................	
11 fold 45................................	
symmetry Orders possible	

H7.7. RULES CONCERNING IDENTICAL ELEMENT TWINS

Below is a pattern from which Twins might have been expected, but the results are
[15] 1639 x 1142 and [15] 2002 x 1395.

1. In the Rotor pattern outer Elements as those in light have to total zero for identicals to work but 586 - 306 - 258 does not.
2. Also the wires in bold also have to total zero but 616 - 526 - 79 does not either.
3. Also the Rotor pattern must stand free by itself, but AB Below joins it- hence the reason for 1 and 2 not working.



CAN THE BOLD WIRES BE DETACHED AND TAKEN EXTERNALLY TO PRODUCE AN IDENTICAL ELEMENTS PAIR? IT SEEMS SO - BUT NO! THE RED WIRES ARE NOT INDEPENDENT ie THEY TOTAL WRONGLY 526+79 DOES NOT EQUAL 616. OFTEN THE LIGHTER ELEMENTS WILL AGREE - BUT 306+ 258 DOES NOT EQUAL 586. SO WHAT IS WRONG? IT IS DUE TO EXTERNAL INTERFERENCE FROM WIRE AB.

H8. TWINS - OTHER TYPES OF ELEMENTS DUPLICATION

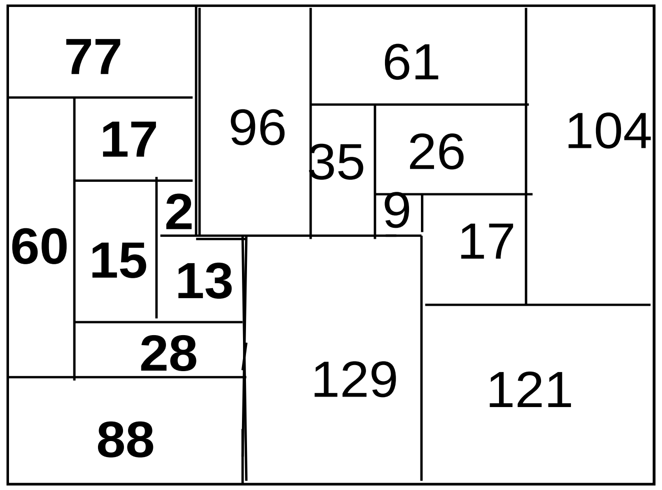
H8.1. OTHER TWINS DUPLICATION

In section E it was seen many solutions can be created from a given solution by applying various endings. Calling the original solution m x n it was seen that the dimensions of the resulting solutions could always be described in terms of m and n alone.
Despite the need to employ an Unknown Element a, the term a always vanished.
This means that the size of new solutions is not dictated by sizes of individual Elements.
Nor does the Order dictate the size. Now consider a solution with some ending common to both ends e.g. a Triad at each end. New solutions produced from the solution must be Twins!
To clarify this a solution with Triads at both ends m x n can be converted into Twin pairs of [o + 5] 15n x 15m - 2n, [o + 5] 8m + 8n x 7m + 5n [o + 5] 6m + 7n x 7m + 6n and sometimes [o + 4] 8m X 7m - 2n and [o + 4] 8m - n X 7m - n, regardless of the Elements sizes or shapes of the Triads.
Now the Twins property is not restricted to Triads - it is also true for any symmetric ending.
Where there are Diads at each end (which note must never be part of a Triad), Twins are again found, but are both symmetric.

In section E it was shown that Octads can replace Triads in solutions. If therefore a Triad at each end solution is converted into two Octad solutions the result is Twins. See Below
A quick test shows that any Triad solution can be converted to an Octad solution, although multiplying up the Elements by 3, 5 or 15 is often necessary.
In fact the Octad can be applied to the left or right end to get Twins.

H8.2. TWINS FROM OCTAD ENDINGS

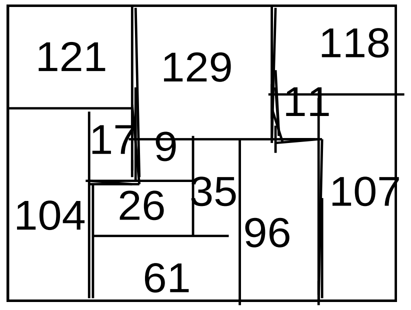
Look below. The middle rectangle has a Triad at each end from which result two Twins [17] 338 x 225 in Below 1 and 3! The Octad addition is a means of creating reduced size solutions where the triad end is divisible by 15. In such cases the Octad solution Order rises by 5, but the result slightly smaller. The status of the original and resulting solutions can vary, Perfect to Imperfect, Imperfect to Invalid etc. In the case Below Octads can be added to both ends at once but the result always Imperfect for reasons already given.

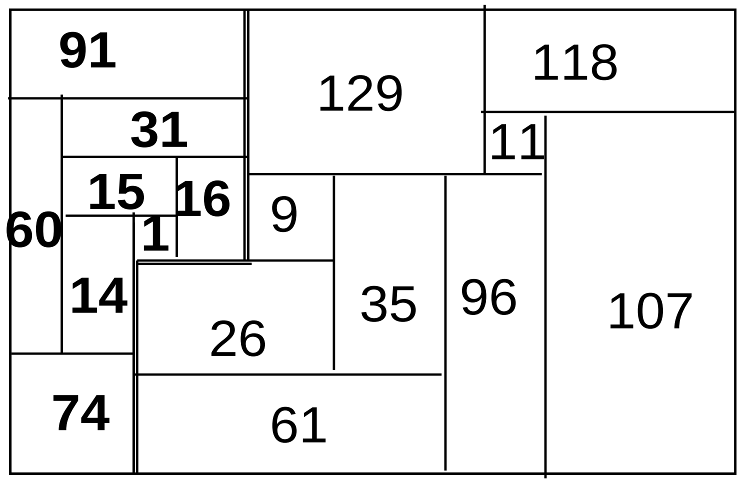


[17] 338 x 225 (1)

[17] 338x225

[12] 368x223





[17] 338 x 225 (2)

H8.3. TWINS FROM TRIAD SOLUTIONS

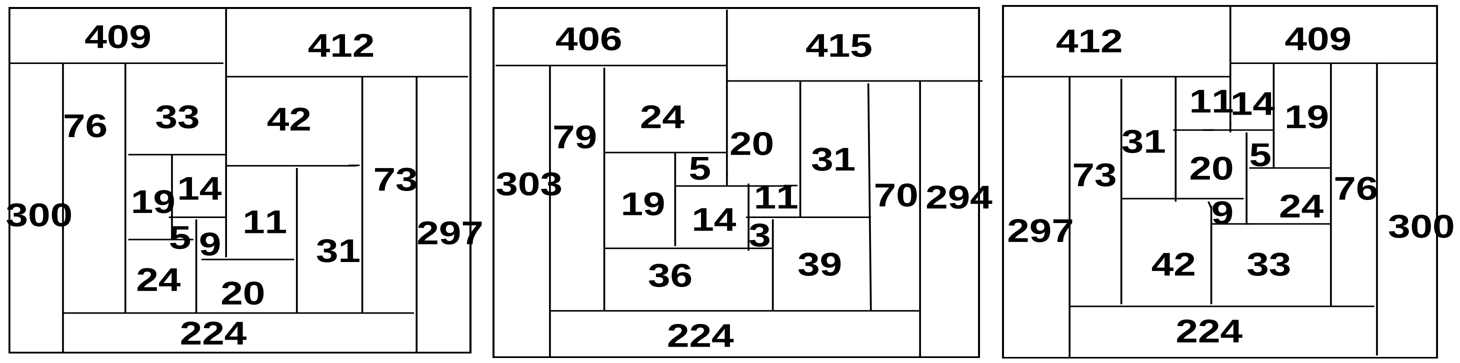
For the background to this subject refer to E3.
1. Any solution which has two Triads can be converted into two Twins.
2. Twin solutions which both contain a Triad can be converted into Twins.
3. Any pair of same size solutions of different Orders which both contain a Triad can be converted into twins.
4. Any pair of solutions which contain Triads and which can be made the same size by up-rating one or both can be converted into Twins.

G8.4. EXAMPLES OF TWINS

The figures below have been constructed from Twins [13] 112 x 75, one from one, two from the other. Note the common Element of 224 in each.

Each is Perfect.

[17] 821 x 709 THREE TWINS

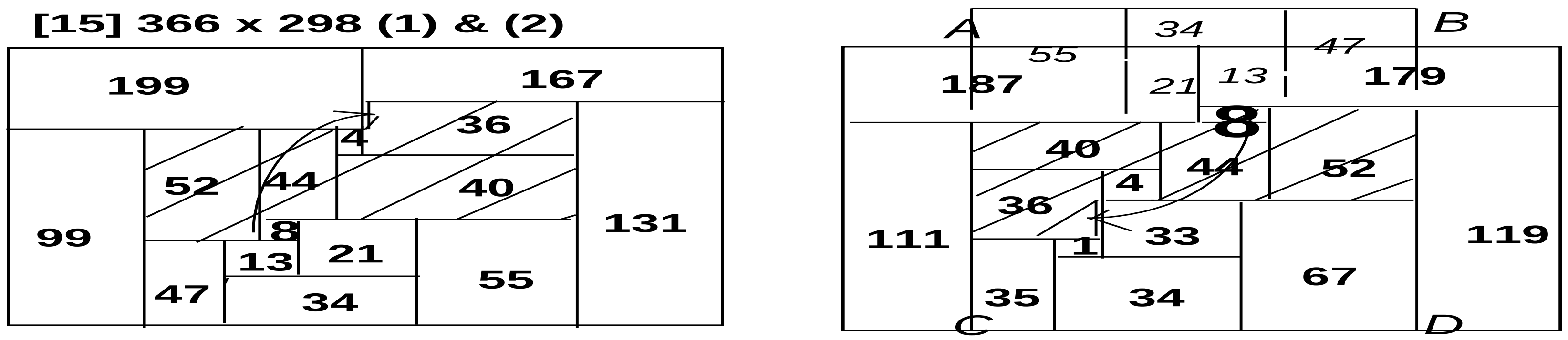


FROM [13] 112 x 75 (1) TWO SOLUTIONS FROM [13] 112 x 75 (2)

H8.5. PENTAD BETWEEN ARCH TWINS

Below is an interesting pair of twins. Note that each comprises an Arch and a Pentad positioned at the foot of it.
The Shaded portion in the first is turned up-side-down in the second.
It can be proved by algebra that whatever Elements comprise this shaded portion that a Twin is always possible from it. However there is an easier way of showing why this relationship works. Simply that an Arch can always be replaced by a Pentad - see G4.1 - look at the solution ABCD in Below 2 in which the pentad in Above 1 has been added. This can be readily converted into both solutions.

[15] 366 x 298 (1) & (2)



H8.6. LIST OF TWINS FOUND FROM TWO-TRIAD SOLUTIONS

Using all Two-Triad solutions up to Order 12 the Twins found have the Orders and dimensions shown. ABCDE denote the type of convention used (D and E are one Order lower and not always possible).
This lists all Compound Twins possible to up to Order 16 and some Order 17 (Other Order 17 are found from Order 13 solutions).
Obviously, this method only produces a small fraction of Twins which actually occur to Order 17.

1. [10] 130 x 79 to 15a 118 x 1792 15b 1672 x 1305 15c 1593 x 1384 14d 260 x 108 14e 961 x 831
2. [11] 112 x 81 to 16a 506 x 405 16b 1544 x 1189 16c 1463 x 1270 15d 448 x 311 15e 815 x 703
3. [11] 209 x 144 to 16a 949 x 720 16b 2824 x 2183 16c 2680 x 2327 16d 1672 x 1175 15e 1528 x 1319
4. [12] 193 x 126 to 17a 881 x 630 17b 2552 x 1981 17c 2420 x 2107 16d 1544 x 1099 16e 1418 x 1225
5. [12] 303 x 257 to 17a 4031 x 3855 17b 2240 x 1703 17c 4223 x 3663 -------

6. [12] 307 x 278 to 17a 4170 x 4049 17b 4680 x 3539 17c 4402 x 3817 -------
7. [12] 336 x 257 to 17a 4526 x 3855 17b 4744 x 3637 17c 4487 x 3894 16d 1344 x 919 16e 2431 x 2095
8. [12] 353 x 207 to 17a 1627 x 1035 17b 2240 x 1753 17c 4273 x 3713 16d 2824 x 2057 16e 2617 x 2264
9. [12] 368 x 225 to 17a 338 x 225 17b 4744 x 3701 17c 4519 x 3926 16d 1472 x 1063 16e 2719 x 2351
10.[12]386 x 207 to 17a 1792 x 1035 17b 4744 x 3737 17c 4537 x 3944 16d 772 x 572 16e 2881 x 2495

H8.7. TWINS FOUND FROM TWO-DIAD SOLUTIONS

These are all symmetric Twins of higher Orders than in the last Section. They are numerous, and not a especially interesting group.

H9. SYMMETRIC TWINS

H9.1. SYMMETRIC TWINS FROM S2223+ SOLUTIONS (REFER SECTION E)

Already dealt with in E14.13.1.

This property works with any S2223 to S2229 solution provided there are two or more Elements Bordering both AB and CD. Otherwise one or both Twins will be Compound.

The original solution may be a Triad-arch or Diad-arch. It does not matter.

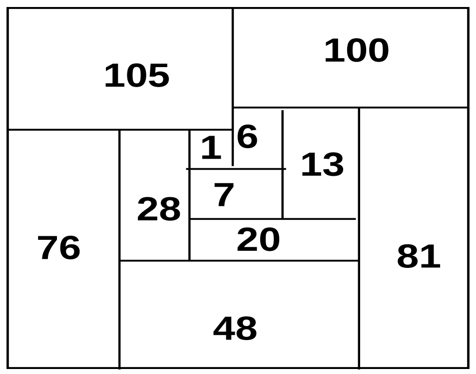
If the original solution is [o] m X n the resulting Twins are found to be [2o - 3] 2m - n X n.

Both Twins are symmetric, and no solutions of this type exist under Order 17.

In fact the only Order 17 solution is from [10] 111 x 98 from which [17] 124 x 98 (1) and (2) are found - formulas |57 26 41:11 15:7 4:3:10:3 7:41 15 4:11:26: and :44 26 54:11 15:7 4:3 16:54:3 7 44:15 4:11:26:

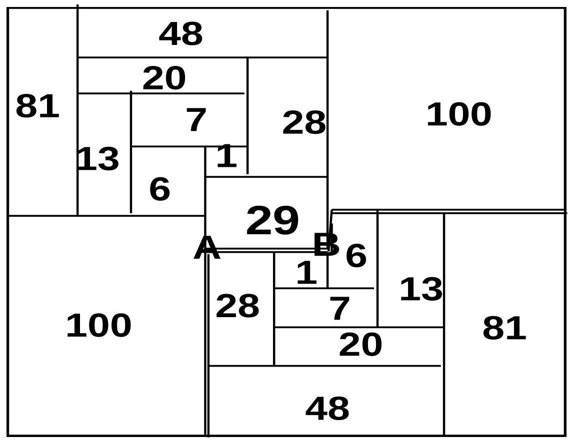
Although there are many suitable 222x solutions, the Twins resulting are of fairly high Orders.

Twins of lower Orders are few.

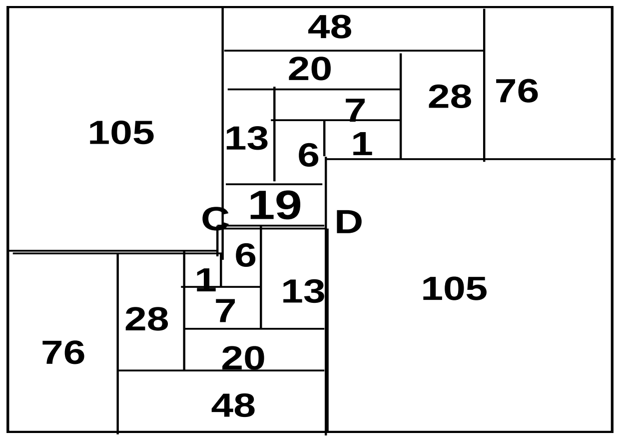


[11] 205 x 181

TO-



[19] 229 x 181 (1)



[19] 229 x 181 (2)

H10. OBTAINING TWINS FROM TWINS

H10.1. TWINS TO MORE TWINS BY JOINS

-------do a section on this----------

H12. EXAMINATION OF UNUSUAL OCCURRENCE OF TWINS

H11.1

A computer program has produced 51 solutions of Order 20 each with the upper dimension of 7076. These are listed below as no less than 34 of them form twins (indicated by the number of Twins [2] [3] [4] and [5]) .

20	7076	x	3874	:	4	26	2019	1	0	4	2424	0-5-	+2019-1131-1976+1950:596 535:225 310:292 304:140 85:26+1924:499-1898:+1855 456:444:-1399	54
20	7076	x	3925	:	4	67	2208	1	0	4	2425	2-0-	+2208-1587-1319+1962:676 643:1112 475:642+1963:67 609:542:+1717 491:472-1321:-1226 377:-849	55
20	7076	x	3934	:	4	246	2213	1	0	4	2425	2-2-	+2213-1195-1512+2156:878 317:807 1022:632 246:378+1778:+1721 492:386 667:-1400:-1229 281:-948	55
20	7076	[3]	4333	a	4	49	2344	1	0	4	2424	0-0-	+2344-1506-1183+2043:323 860:679 1150:613+2290:208 471:+1989 306 49:257:563:557-1677:-1120	61
20	7076	[3]	4333	b	4	79	2344	1	0	4	2424	2-1-	+2344-1506-1099+2127:407 692:1193 720:435 257:178 79:+2206:473 860:+1989 355:-1634 387:-1247	61
20	7076	[3]	4333	c	4	79	2344	1	0	4	2424	2-0-	+2344-1506-1183+2043:323 860:1193 636:178 435+2290:557 79:257:+1989 355:692:-1634 471:-1163	61
20	7076	[4]	4522	a	4	21	2604	1	0	4	2424	2-0-	+2604-1074-1053+2345:21 1032:1095:84 688 260:575 604:428+2177:+1918 686:546 29:-1749:-1232	63
20	7076	[4]	4522	b	4	84	2604	1	0	4	2424	2-0-	+2604-1074-1397+2001:751 323:428 688 604:919 260:84+2521:1032:+1918 686:546 373:-1405:-1232	63
20	7076	[4]	4522	c	4	84	2632	1	0	4	2325	0-2-	+2632-1825+2619:891 934:716+1903:+1890 658 84:574 401:358 576:541 218:-1232:323-1187:-864	63
20	7076	[4]	4522	d	4	84	2647	1	0	4	2325	0-2-	+2647-1825+2604:906 919:686+1918:+1875 688 84:604 386:373 546:218 541:-1232:-1187 323:-864	63
20	7076	x	4538	:	4	56	2524	1	0	4	2424	0-1-	+2524-1187-1124+2241:63 1061:1250:1005 56:+2297:87 684 479:+2014 597:205 274:-1417 69:-1348	64
20	7076	x	4565	:	4	25	2647	1	0	3	2335	2---	+2647-2227+2202:25 1042-1135:689 918 645:+1918 729:273 372:460 229:-1420:-1321 93:+1228:-1189	64
20	7076	[2]	4730	a	4	80	2734	1	0	4	2324	2-0-	+2734-1877+2465:621 668 588:80 708+2265:454 167:120 628:287:+1996 738:520 221:-1557:-1258	66
20	7076	[2]	4730	b	4	80	2734	1	0	3	2335	2---	+2734-2211+2131:80 668-1383:741 962 588:+1996 738:374 882:520 221:-1557:-1258:167+1216:-1049	66
20	7076	[2]	4802	a	4	81	2764	1	0	4	2324	0-1-	+2764-1774+2538:1071 703:368 335:274+2264:+2038 645 81:609:564 380 576:184 196:-1393:-1381	67
20	7076	[2]	4802	b	4	81	2787	1	0	4	2324	0-1-	+2787-1751+2538:1048 703:345 358:274+2264:+2015 576 196:184 564 645:632:380:-1439 81:-1358	67
20	7076	[4]	4870	a	4	57	2690	1	0	4	2324	0-1-	+2690-1807+2579:940 867:288+2291:674 193:+2180 453 57:396 601:481:644 205:439-1522:-1083	68
20	7076	[4]	4870	b	4	89	2738	1	0	4	2324	0-0-	+2738-1923+2415:815 616 492:452+2455:288 328:+2132 866 466 89:377:780:400 443:-1266:-1223	68
20	7076	[4]	4870	c	4	108	2758	1	0	4	2324	2-0-	+2758-1813+2505:905 508 400:108 292:616:432+2365:686 219:+2112 646:467 800:-1466 333:-1133	68
20	7076	[4]	4870	d	4	213	2738	1	0	4	2324	0-1-	+2738-1759+2579:979 780:452 328:288+2291:+2132 866 466 253:616:213 492:400 279:-1387:-1266	68
20	7076	[3]	4874	a	4	33	2697	1	0	4	2324	0-1-	+2697-1793+2586:1072 721:351 370:298+2288:+2177 352 168:184 720 687:668:536:33-1322:-1289	68
20	7076	[3]	4874	b	4	70	2764	1	0	4	2434	0---	+2764-1358-1569+1385:1147 211:184-1201:947 1017:259 888:877 70:+2288:+2110 913:284-1481:-1197	68
20	7076	[3]	4874	c	4	33	2764	1	0	4	2324	0-1-	+2764-1726+2586:1005 721:284 437:298+2288:33 720 536:+2110 687:735:184 352:-1423 168:-1255	68
20	7076	[5]	4890	a	4	104	2602	1	0	4	2324	1-0-	+2602-1945+2529:609 856 480:376 104:272+2361:362 247:+2288 314:791 960:676:-1298 169:-1129	69
20	7076	[5]	4890	b	4	104	2669	1	0	4	2324	0-1-	+2669-1781+2626:888 893:362+2264:+2221 856 480:579 314:676:376 104:272 411:-1365 139:-1226	69
20	7076	[5]	4890	c	4	19	2717	1	0	4	2324	0-1-	+2717-1757+2602:960 797:435 362:314+2288:+2173 856 376 272:676:104 603:480:-1317 19:-1298	69
20	7076	[5]	4890	d	4	36	2810	1	0	4	2324	0-0-	+2810-1739+2527:1107 632:476 156:320+2363:+2080 694 36:658 485:796:173 312:-1386 139:-1247	69
20	7076	[5]	4890	e	4	36	2810	1	0	4	2324	0-1-	+2810-1977+2289:869 796 312:+2601:320 476:+2080 694 36:658 247:411 156:632:-1386 377:-1009	69
20	7076	x	4898	:	4	9	2922	1	0	4	2324	0-1-	+2922-1673+2481:801 872:730 71:64+2417:1007:+1976 639 307:25 357 348:332:9-1346:-1337	69
20	7076	[5]	4918	a	4	24	2786	1	0	4	2324	0-1-	+2786-1753+2537:1009 744:548 196:156+2381:352:24 702 283:+2132 678:419 764:-1454 345:-1109	69
20	7076	[5]	4918	b	4	24	2814	1	0	3	2324	2-0-	+2814-1779+2483:1075 704:752+2435:+2104 710:670 405:24 188 540:265 164:352:-1394 251:-1143	69
20	7076	[5]	4918	c	4	24	2814	1	0	4	2324	2-0-	+2814-1873+2389:981 540 352:188 164:24+2529:752:+2104 710:670 311:359 704:-1394 345:-1049	69
20	7076	[5]	4918	d	4	24	2822	1	0	4	2324	0-0-	+2822-1775+2479:1071 704:744+2439:+2096 702 24:678 417:196 548:261 156:352:-1394 247:-1147	69
20	7076	[5]	4918	e	4	24	2822	1	0	4	2324	0-1-	+2822-1873+2381:973 548 352:196 156:+2537:744:+2096 702 24:678 319:359 704:-1394 345:-1049	69
20	7076	x	4970	:	4	9	2998	1	0	4	2324	0-0-	+2998-1683+2395:971 712:532+2575:698 273:805:+1972 681 345:336 9:327 380:-1291 53:-1238	70
20	7076	x	5238	:	4	82	2703	1	0	4	2334	0---	+2703-1423-1516+1434:1002 421:82-1352:328 1270:749:278 724:556 193:+2535 446:363+2452:-2089	74
20	7076	x	5339	:	4	8	2946	1	0	3	2323	0-0-	+2946-1689+2441:937 752:295+2898:320 507 110:405:+2393 608 265:78 421 8:413:343:-1785	75
20	7076	[2]	5350	a	4	114	2743	1	0	4	2334	0---	+2743-1271-1588+1474:954 317:114-1360:773 1246:518 436:300 473:736:+2607 654:563+2516:-1953	75
20	7076	[2]	5350	b	4	99	2825	1	0	4	2334	1---	+2825-1735+2516:954 781:691 1246-1360:436 518:+2525 300:736:654 555:99-1588 114:-1489:+1474	75
20	7076	[3]	5455	a	4	111	2851	1	0	4	2334	0---	+2851-1733+2492:974 759:510 1315-1426:144 535 295:+2604 391:805:926:361-1648 111:+1537:-1287	77
20	7076	[3]	5455	b	4	111	2875	1	0	4	2334	1---	+2875-1709+2492:926 783:534 1315-1426:535 391:+2580 295:144 781:974:337-1648 111:+1537:-1311	77
20	7076	[3]	5455	c	4	73	2963	1	0	3	2323	0-0-	+2963-1509+2604:805 704:101 212 391:361 434 111:323:144+2851:288 73:974:+2492 759:-1733	77
20	7076	x	5583	:	4	19	2889	1	0	4	2334	1---	+2889-1944+2243:1140 505 299:206 917-1419:711:+2694 195:1259 76:1202 502:+1921:-1240 19:-1221	78
20	7076	x	5614	:	4	8	2968	1	0	4	2344	2---	+2968-1969-1029+1110:948 81:-1191:1321 640 8:632 324:308-1207:681 899:+2646 322:-2324:+2106	79
20	7076	x	5638	:	4	103	2943	1	0	4	2334	0---	+2943-1463-1106+1564:648 458:1172 291:190-1832:1129:308 864:616 513:+2695 556:103+2242:-2139	79
20	7076	x	5818	:	4	132	3059	1	0	3	2333	0---	+3059-1713+2304:1122 591:681 782-1432:224 748 150:598 233:+2759 524:132 650:365:-2235:+2082	82
20	7076	[2]	6250	a	4	149	3293	1	0	3	2343	0---	+3293-1701+2082:1320 381:1107-1356:272 880 168:712 563:+2957 608:314-1042:149 728:-2349:+1770	88
20	7076	[2]	6250	b	4	59	3397	1	0	4	2343	0---	+3397-1909+1770:728-1042:1320 589:1003 314:-1356:168 880 272:+2853 712:608 667:-2141 59:+2082	88
20	7076	[2]	6370	a	4	68	3488	1	0	4	2343	0---	+3488-1958+1630:328-1302:1109 1177:203-1099:421 620 68:552 896:+2882 1027:828 344:+2339:-1855	90
20	7076	[2]	6370	b	4	68	3488	1	0	4	2343	0---	+3488-1958+1630:328-1302:1385 901:828 73:-1375:145 896 344:+2882 751:552 620:-2131 68:+2063	90
20	7076	x	6410	:	4	62	3808	1	0	4	2234	00--	+3808+3268:835 897-1536:+2602 911 295:616 452 62:390 569:164 678:-1691:499 70:+1606:-1177	90

51 solutions 34 forming twins! All RI 4 and perfect No pentads seen

H11.2. OBSERVATIONS FROM THE TABLE

There are many instances found where selected upper dimensions for a given order tend to have an unusual occurrence of Twins, but the above is particularly spectacular! Possibly some of the singles shown might form part of Twins in this incomplete list. 7076 is 2 x 2 x 1769 (prime) - nothing significant.
Of the lower dimensions of 4333 4522 4730 4802 4870 4874 4890 4918 5350 5455 6250 and 6370 only two - 4333 and 5455 are odd and the rest are an odd number times 2. The author has noticed with this phenomenon generally that both higher and lower dimensions *tend* to be even rather than odd.
From the Elements shown in Red above it is observed that sometimes both largest and smallest Elements have the same numbers whilst both are different in others or just one dimension the same.
In some groups the Side Indexes are the same. In others, different.
Possibly all these statements have little significance, but mentioned in passing.

H11.3. UNANSWERED QUESTIONS

1. Can Twins be classified in what can be described in conveniently matching pairs?
2. Where 3 twins exist, could it be one is merely a coincidence and the other a ‘matching pair’?
3. Can definite rules linking the Dimension sizes be established?
4. Why are Higher and Lower Dimensions the same in some pairs and different in others?

H11.4. ANOTHER EXAMPLE FOR COMPARISON

Order 19 Dimension 536 26 solutions of which 19 are part of Twins, but owing to the Reduction Indexes varying, some Twins are Reduced Twins and not True Twins (unlike the 7076 example)! Also the tendency for even lower Dimensions is untrue in this case. Notice how few highest and lowest Elements pair up in this case. Side Indexes often vary.
There is an odd mixture of Perfect and Imperfect.

19	536	[2]	399	a	20	2	209	1	0	3	2323	0A0- 50 24 26 2 28 22 8 14 36 30 140 206 16 190 75 59 134 209 193	74
19	536	[2]	399	b	24	2	220	1	1	3	2323	0-0- 19 36 19 36 17 53 70 123 17 193 2 15 55 13 220 206 28 151 179	74
19	536	[2]	409	a	24	1	205	1	1	3	2323	0-0- 29 4 13 20 7 49 78 127 6 23 1 204 27 205 50 205 77 127 204	76
19	536	[2]	409	b	24	4	247	1	1	3	2336	4--- +247-141+148:134 7:-155:+162 85:65 69:49+106:-77 8:-69 4:-65 8:-57	76
19	536	x	435	:	23	3	283	1	0	2	2224	002- +283+253:71+182:+152 26 17 25 22 41:9 8:3 19:36 35:20-111:-91	81
19	536	[3]	445	a	24	10	250	1	0	4	2344	1--- +250-149+137:12-125:161:+195 55:48-77:40 15:10 35-135 29:25:+106:-100	83
19	536	[3]	445	b	24	12	253	1	1	3	2334	0A-- 12 40 28 52 16 48 32 80 112 100 32 132 164 192 119 15 134 253 149	83
19	536	[3]	445	c	24	7	270	1	0	3	2236	40-- +270+266:73 100-93:+175 95:41 32:7+86:-80 15:28-79:9 23:-65:-51	83

19 536 [3] 460 a 24 2 239 1 0 3 2343 1--- +239-161+136:25-111:70 116:24 46:+221 18:30-81:2 22:20:-183 51:+132
85
19 536 [3] 460 b 24 5 240 1 1 3 2343 0--- 70 65 5 75 145 60 59 20 240 40 220 180 44 136 92 11 70 81 151 85
19 536 [3] 460 c 24 2 245 1 0 4 2343 2--- +245-159+132:27-105:116 70:46 24:18-87:+215 30:22 2:20:-185 49:+136
85
19 536 x 463 : 24 5 240 1 1 3 2343 1--- +240-129+167:91 38:112-93:32 59:+223 17:5 27:22:19-74:-184 55:+129 86
19 536 x 473 : 25 1 273 1 1 2 2223 000- +273+263:5 4 12 32+210:1 3 4 2:5:1 3+200 74:20:52:-126 88
19 536 [2] 476 a 24 2 271 1 0 4 2234 01-- +271+265:6 78 40 42-99:+205 72:38 2:44:-133 47 8:39 13:+112:-86 88
19 536 [2] 476 b 24 5 272 1 1 3 2245 01-- +272+264:8 84 71-101:+204 76:13 58:-128 45:48-53:-83 20:-63 5:+58 88
19 536 [2] 481 a 21 7 293 1 1 2 2246 40-- +293+243:138-105:+188 105:33-72:39-100 32:-83 22:25 7:-61 18+61:-43 89
19 536 [2] 481 b 26 4 249 1 1 4 2334 0A-- +249-134+153:115 19:-172:+232 76 56:20 36:80 16 16+156:68:-76 4:-72 89
19 536 x 489 : 21 7 272 1 1 2 2223 001- +272+264:39+225:+217 24 31:17 7:10 37 30:27:7 23:55 16:39:-94 91
19 536 x 493 : 21 4 310 1 0 2 2234 01-- +310+226:84-142:+183 63 34 39 75:29 5:24 20:17+125 4 16:-120:-108 91
19 536 x 497 : 24 4 250 1 0 3 2344 0A--
68 10 78 88 58 40 18 22 4 26 176 140 110 250 113 21 134 247 155 92
19 536 x 513 : 24 2 204 1 0 3 3333 ----
31 24 7 38 69 17 163 28 135 204 11 174 185 8 177 169 2 165 167 95
19 536 [2] 520 a 24 8 292 1 1 4 2234 01-- +292+244:48 61-135:+228 99 13:74:20 48+141:55 36 8:28:19-93:-74
97
19 536 [2] 520 b 24 5 292 1 1 4 2234 01-- +292+244:48 75 63-58:+228 85 27:5-53:20 48:58 36 8:28:+165:-143 97
19 536 [3] 532 a 24 1 272 1 1 3 2234 10-- +272+264:20 52 61 65-66:+260 12:32:75 9:66 4:68 1:-67:-141:+135 99
19 536 [3] 532 b 24 14 279 1 1 4 2343 1--- +279-121+136:106 15:-151:46 60:32 14:+253 26:18 94-113:76:-151 19:+132 99
19 536 [3] 532 c 24 13 285 1 1 3 2343 0--- +285-119+132:106 13:-145:60 46:14 32:26-119:+247 94 18:76:-153 17:+136 99

H11.5. FURTHER OBSERVATIONS

In examining a number of pairs of Twins having the same Side Indexes (e.g. S2223 and S2223, S2234 and S2234) it appears that the values of one set of Elements differ from the values of the other set by multiples of a given number. For example, [14] 238 x 225 (1) and (2) are both 2234 and the Elements -

122 116 103 59 50 44 41 40 36 32 25 19 9 4

133 105 92 59 61 44 41 51 36 21 25 19 2 15

The positive differences (ignoring signs -/+) being -

11 11 11 0 11 0 0 11 0 11 0 0 ** 11

Subject to verification it appears that there is always a duplication of at least one pair of Elements which means a difference of zero.

In some Pairs one Solution converts to the other by a Certain Element being 'screwed' the corner or to the other side.
This Element has the same value in both rectangles and if this is always the case then the statement in the last previous sentence will be true.
**** It is annoying to discover that the rule of Differences above often *seem*s to have exceptions as in ** above where 9 - 2 = 7 is clearly NOT divisible by 11! However 9 + 2 *is divisible by 11.* More logically, it is the difference between 9 and -2 which is divisible by 11.**
The reason is that there are often similarities in structures between two Twins, but the two need to have like-for-like format and to get identical format it is sometimes necessary to 'push an Element round a corner' thus effectively making it negative! In [15] 814 x 784 the Elements are -
324 460 354 145 167 234 231 81 43 38 140 32 119 17 123
303 481 333 124 132 199 217 158 13 52 84 32 119 17 116
The positive differences (ignoring signs -/+) are -
21 21 21 21 35 35 14 77 ** 14 56 0 0 0 7 all divisible by 7. If 13 is rewritten as -13 the difference 43 - -13 is 56 and also divisible by 7!

H12.1. OTHER SOLUTIONS FROM BLOCKS

By 'messing about' with any given pair of Blocks, many other solutions might be found.
In fact, infinite BLOCKS exist if no restriction is applied to Order size.
Such solutions include Squared-squares and Symmetric, some Twins and some not.
RELATIONSHIPS BETWEEN GENERAL FEATURES
a. Type ab ac ad ae af ag ah aj ak
b. Dimensions bc bd be bf bg bh bj bk
c. Semiperimeters cd ce cf cg ch cj ck
d. Sides Index de df dg dh dj dk
e. Reduction ef e.g. eh ej ek
f. Elongation fg fh fj fk fl
g. Unknowns gh gj gk
h. Twins hj hk
j. End-types jk
k. largest Element

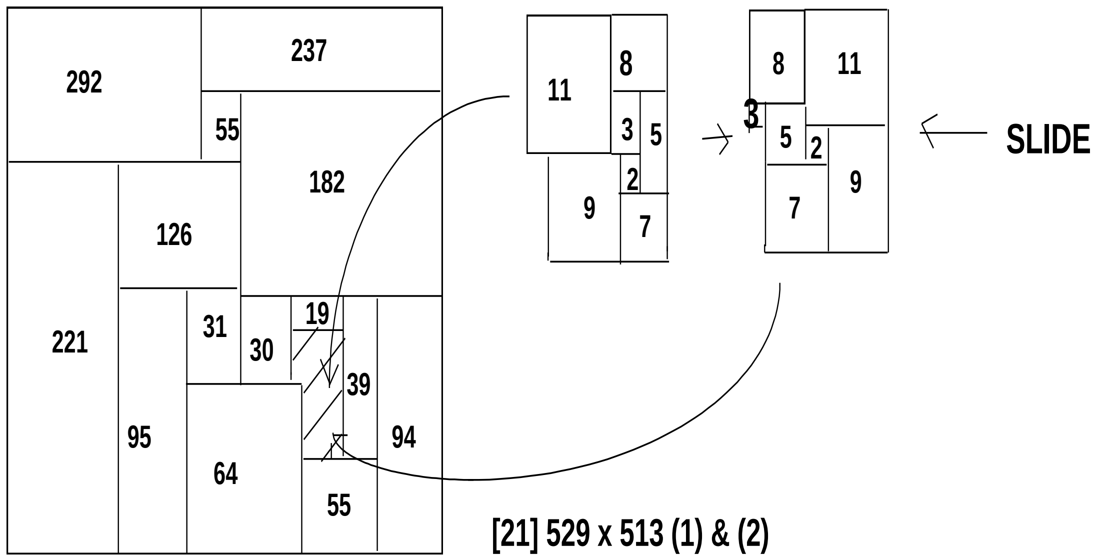
Type	xxxx	e	e	e	e	e	d	f	d	e
Dimensions	e	xxxx	e	e	e	e	d	f	d	e
Semi-per	e		xxxx	e	e	e	d	f	d	e
Sides	e			xxxx	e	e	d	f	d	e
Reduction	g				xxxx	e	d	f	d	e
Elongation	e					xxxx	d	f	d	e
Unknowns	d						xxxx	f	d	e

Twins	f							xxxx	d	e
End-types	d								xxxx	e
Largest Elem	e	e	e	e	G	e	d	f	d	xxxx
SUBJECT	Type	Dims	SP	Sides	Redn	Elon	Unk	Twin	Ends	Larg

H13. TWIN VARIBLOCKS

H13.1. VARIBLOCKS

An example is below -



[21] 529 x 513 (1) & (2)

In above, either of the patterns on the right will fit into the gap shown forming twins [21] 529 x 513.
These are termed Twin Variblocks and it is clear that for <u>any</u> solution (not necessarily above) containing one of these, another solution containing the other is possible providing adjacent Elements do not arise.
Although there are endless examples of Blocks, they are only found from a large catalogue of Solutions.
Also they usually contain many more Elements that the 7 in 2 3 5 7 8 9 11.
This is the smallest both in quantity of Elements and actual size 20 by 18.
So with any Block, Twins are possible, and in classifying TWINS can be regarded as Block Type or Non-Block Type.
This could be a useful general division.

H13.2. VARIBLOCKS WITH IDENTICAL ELEMENTS

In the above Variblock, the Elements are Identical (2 3 5 7 8 9 11), a feature that occurs in many Variblocks.
Others have a few duplications of Elements, whilst some have totally different Elements throughout.

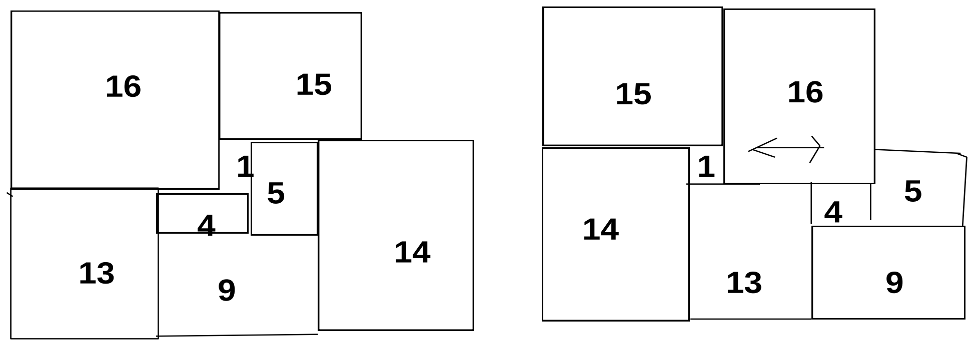
H13.3 VARIBLOCKS WITH SLIDES

Some Identical Elements Variblocks exist by reason of a SLIDE, above 3 being an example where the line under 3 & 11 is on the same horizontal. See how it is easily converted into above 2 by sliding the Elements!

H13.4 LIST OF VARIBLOCKS WITH 10 ELEMENTS OR LESS

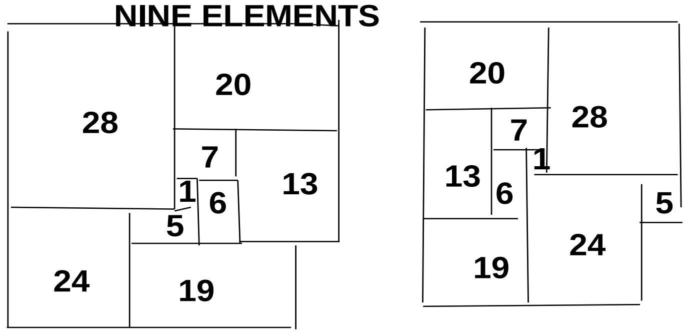
The Above Variblock can be conveniently Coded as "V07B11"
V for Variblock 7 is No. of Elements B11 means two Blocks both with maximum Elements 11.
Another is V10A29A17 meaning 10 Elements with One Block containing 29 and the other 17 (Maximum Element sizes).
If the two blocks differ in numbers of Elements then the format V(Order1)A(Max.Value1)-V(Order2)A(Max.Value2) is used.

EIGHT ELEMENTS



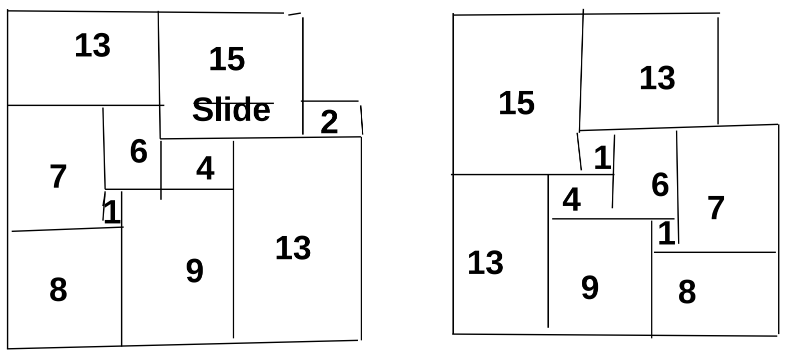
V08B16 VARIBLOCK (Identical Elements - with Slide)

NINE ELEMENTS



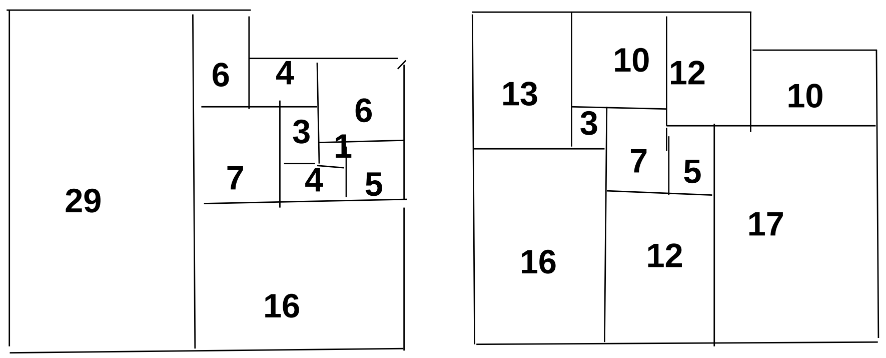
V09B28 VARIBLOCK (Identical Elements - no Slide)

TEN ELEMENTS



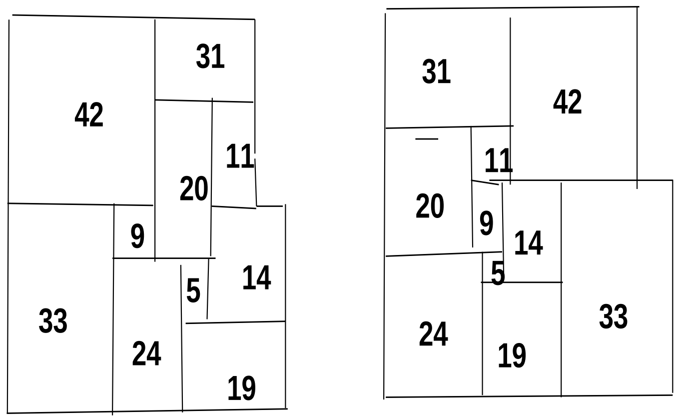
V10B15 (Identical Elements - with Slide)

TEN ELEMENTS



V10A29A17 (Some Elements duplicated)

TEN ELEMENTS



V10B42 (Identical Elements - with Slide)

Slides only exist where the Elements are Identical throughout. Note how each Element stays at the same horizontal level. Theoretically some Variblocks might exist with have no relating Solutions, but I know no such cases, and think this unlikely in practice.
[21] 668 x 556, [21] 764 x 636, [23] 428 x 384, [26] 456 x 456 (1)(2), [25] 556 x 556 (1) & (2) and [25] 540 x 540 (1) & (2) all include the V08B16 Variblock, and no doubt there are many more - possibly an unlimited number of cases!
Note that Twins are not guaranteed as some Solutions form adjacent Elements.
The V10B42 Variblock is contained in the [13] 112 x 75 twin Solutions.

J. REDUCTION AND REDUCTION INDEX

J1. COMPLEXITIES

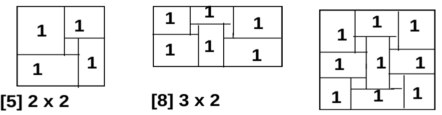
J1.1. COMPLEXITIES AND REDUCTION INDEX

There is a loose relationship between Complexities and Reduction Indexes. Clearly, if the Complexity is a prime number then the Reduction Index must be 1. Where a Reduction Index is >1 it must obviously be a factor of the Complexity.
In Section B, Complexity 593A for Order 13 is shown. 593 being prime, it is possible to state in advance that the Reduction Index for all solutions, no matter which pole positions are chosen, will be 1.

J2. ZERO-ONE

J2.1. ZERO-ONE SOLUTIONS

EXAMPLES OF ZERO-ONE RECTANGLES



[5] 2 x 2 [8] 3 x 2 [13] 3 x 3

	2	3	4	5	6	7	8	9	10
2	5	8	11	14	17	20	23	26	29
3	8	13	18	23	26	33	38	43	48
4	11	18	25	32	39	46	53	60	67
5	14	23	32	41	50	59	68	77	86

ORDERS FOR RECTANGLES OF VARIOUS DIMENSIONS

ZERO-ONE describes the most reduced solutions of all - those containing 0's and 1's only. These obviously Invalid forms are created from boxes of 1's with 0's at each intersection. In a rectangle of x Elements by y Elements deep, the Order for the rectangle is $xy + (x - 1)(y - 1) = 2xy - x - y + 1$ Above 4 shows a table of Orders.

J2.2. REDUCTIONS IN ZERO-ONE SOLUTIONS

By looking at a few examples it is possible to establish the relationship between the Reduction Index and these solutions.
*** to be finished ***

J3. REDUCED RECTANGLES

J3.1. REDUCED RECTANGLES (SEE SECTION H ALSO)

In section A the term Reduction Index was defined. This is the linear ratio between the full and reduced Elements (taken individually) or between the full and reduced dimensions.
RI (Reduction Index) = Larger Full dimension divided by Larger Reduced dimension.
An RI can be 1,2,3 or any positive integer, and may be a huge number in higher Order solutions.
This section is concerned with the following
1. What type or types of rectangle tend to reduce?

2. How can very reduced solutions be found readily?
3. How does the Reduction Index relate to other features of Squared-Rectangles?
These are not straightforward problems! It is relatively easy to generalise and difficult to be specific.

J3.2. STARTING OFF

We begin with an elementary fact - The smaller the reduced Elements in a solution are found to be, the greater the probability that an Element is duplicated in size, and also that more Elements are duplicated.
From this statement we may expect Invalid solutions, which are always Imperfect, and Imperfect solutions to tend to have higher Reduction indexes than Perfect solutions. A list of Squared-Rectangles clearly shows this to be true. i.e. in Order 13 solutions Invalid solutions RI's 192 down to 5, Imperfect R1's 32 down to 4 and Perfect RI's 6 down to 1.

J3.3. SOME FACTS ABOUT THE REDUCTION INDEX

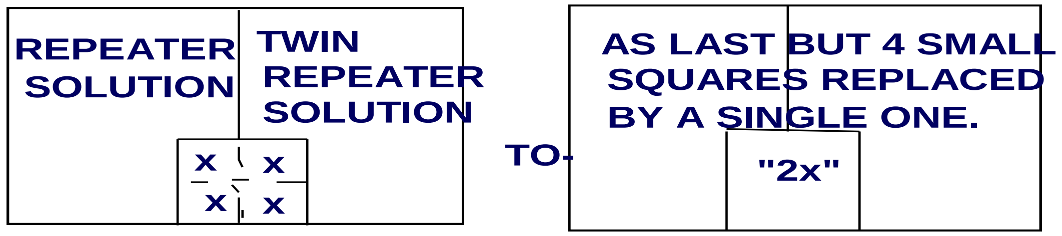
1. It is possible to find an Invalid solution at even Order 13 which is less reduced than a Perfect one of the same Order.
2. It is impossible to have an Imperfect solutions with Reductions of 1, 2 or 3. Only exception is [8] 8 x 7 invalid.
3. All solutions with Reduction indexes of 1, 2 or 3 are Perfect.
4. Many Simple Perfect solutions have a Reduction of 1, and a considerable proportion of those left, have a Reduction of 2.
5. It is possible to have reductions of 1, 2 or 3 in the case of duds.
6. I have never discovered why Simple solutions of Reductions 1, 2 or 3 always have different Elements.

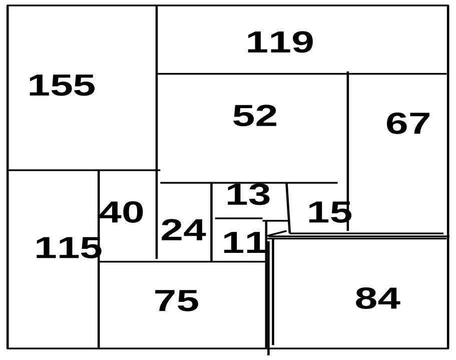
J4. CONSTRUCTING VERY REDUCED RECTANGLES

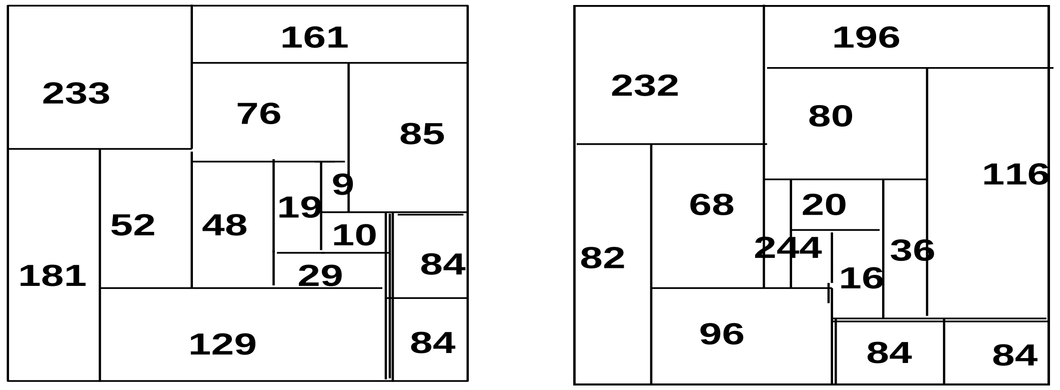
J4.1. USING RECTANGLES TO CONSTRUCT VERY REDUCED RECTANGLES

It is sometimes possible to alter the form of a pair of twin solutions in such a way that they will fit together to form an extremely reduced solution. The Twin solution chosen must have a corner Element which is common to both, although either rectangle may be increased if necessary in size in Order to produce this state of affairs. All the solutions mentioned above in No.2 come within this category, but up to Order 13 solutions provide only the following pairs with an identical corner Element:- [13] 70 x 66 (increased to 140 x 132) and 140 x 132A.
[13] 274 x 270A and B. [13] 552 x 463 A and B [13] 72 x 64.
Taking a Twin pair, the pattern of each is drawn and the corner square divided into two either vertically or horizontally (making the Order 14).
The solutions for the higher Order are calculated by algebra e.g. by the Add and Deduct Rule the full Element in the single and double corners are the same, e.g. 84.
But it does not always happen that the resultant solutions are the same size, as in the case of [13] 72 x 64(1) and (2) where horizontal double corner solutions of Order 14 turned out to be [14] 944 x 703 and [14] 962 x 719 and ones such as these cannot be fitted together to form a highly reduced solution. However double cornered solutions from [13]140 x 132, [13] 274 x 270 and [13] 552 x 463 do give rise to Twin solutions both in the horizontal and vertical double cornered Elements Above 4 & 5.

FITTING REPEATERS TOGETHER



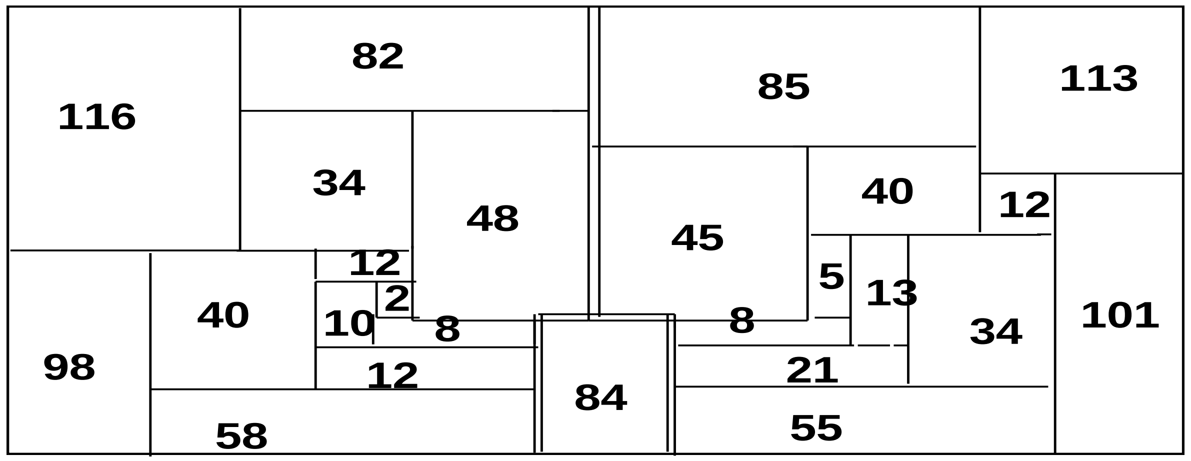




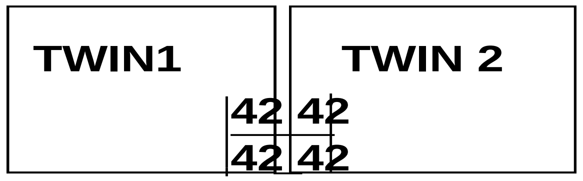
[13] 274X270(2) ANDTHE SOLUTIONS PRODUCED BY DOUBLE CORNERS VERTICALLY (LEFT) & HORIZONTALLY (RIGHT)

These give rise to 6 reduced solutions for Order 25 by the construction of above 1 & 2.
It is observed that the four Elements created by the two 'double corners' are replaced by one Element of twice the dimension. The six solutions are 1. An Invalid Solution. 2. 392 x 214 from [13] 274 x 270's 3. 392 x 220 from [13] 140 x 132.
4. 788 x 414 from [13] 274 x 270's 5. 1614 x 608 from [13] 552 x 463's. 6. 698 x 781 from [13] 552 x 463's.
All these are Imperfect unfortunately. Just one is shown below. Refer to the cards for the other solutions.

ADDITION OF TWO DOUBLE CORNERED TWINS



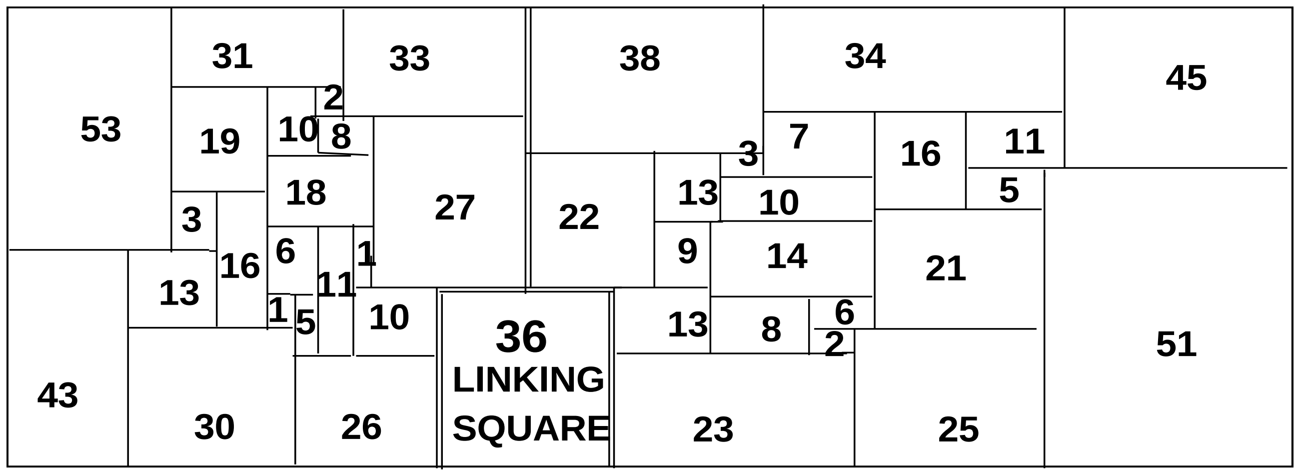
[25] 796 x 214 imperfect from [13] 274 x 270 (1) & (2)



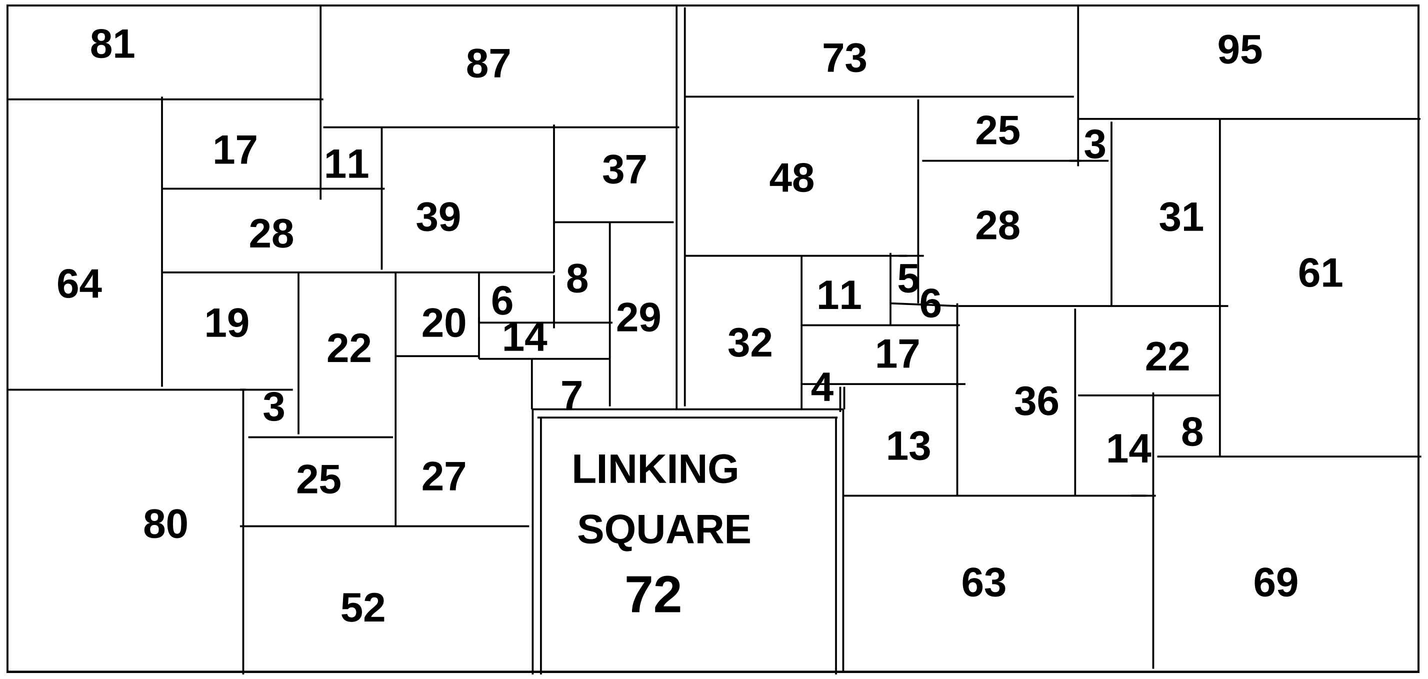
FOUR 42 ELEMENTS REPLACED BY 84

Where there is duplication of Elements between one original Twin solution and another, there is often found duplication between the respectively placed Elements in the resultant Twin solutions (those with the double cornered Elements).
Now we shall consider the Twin solution of 3-fold symmetry - Orders 16, 19, and 22 which have a single Stator corner Element which is duplicated. From these, highly reduced solutions of Orders 31, 37, 43 respectively may be found.

If the original solutions of Twin rectangles have dimensions m x n then the two double repeater solutions have dimensions of (m-s) x (n/2 + s)… (1) Smaller m > n and (m + 2s) x (2 (n - s) …(2) larger. s is the stator. Providing that s is divisible by 2. s is the stator value. m is the horizontal or larger dimension.
From the above it is evident that if we consider a given Twin pair (with a single Stator), we can easily calculate the dimensions of the large reduced solutions obtained from it, e.g. 16 - 671 x 504 which has a Stator of 154. m=671 n=504 s=154. Now (671 - 154) x (504 / 2 + 154) = 517 x 406 - Order 31 solution is 1034 x 406 and (671 + 2 x 154) x (2 x 504 - 2 x 154) = 979 x 700 - Order 31 solution is 979 x 1400. In the formula (1) and (2) Above, it must be stated that (1) relates to the Twin solutions when regarded horizontally with the double corner Elements the vertically.
In (2) relates to vertically regarded Twin solutions, again with the corner Elements vertically.
When the two Twin parts are combined one of the dimensions clearly doubles in size. Which?
Observation shows that (m - s) in the first, and (2n - 2s) in the second formula are the dimensions which double.
In the case of 2(2n - 2s) this exceeds thus the formulae for the reduced solutions are:-
2(m - s) x (n / 2 + s) horizontal form and smaller solution and (m + 2s) x 4(n - s) vertical form and larger rectangle. Linking square = s and 2s respectively.
If s is odd, then the resulting rectangle may have Elements with halves occurring, and may have to be doubled to make them integers. If n is odd, then n / 2 becomes a fraction, and the resultant solution is twice the size stated.
Now the object of the exercise is to find particularly reduced solutions with high Orders.
Therefore it is reasonable to consider the smallest sized solutions for twin solutions of this size, namely
1. [16] 561 x 464 Stator 136. Duplicated Elements.
2. [16] 571 x 504 Stator 154. Duplicated Elements.
3. [19] 1025 x 592 Stator 200 Duplicated Elements.
4. [19] 1175 x 913 Stator 275 Odds duplicated.
5. [22] 153 x 120 Imp. Stator 36. Duplicated Elements.
6. [22] 315 x 312 Stator 84 Duplicated Elements.
7. [22] 637 x 483 Stator 147 Odds duplicated.
8. [22] 663 x 520 Stator 156 Duplicated Elements.
9. [22] 1104 x 891 Stator 264 Duplicated Elements.
10. [22] 1144 x 845 Stator 260 Duplicated Elements.
For the smallest rectangle possible from the above list, we use [22] 153 x 120. Fortunately 120 and 36 (Stator) are even numbers, and so the two rectangles possible are [43] 234 x 96 and [43] 336 x 225 using the above formulae. See Below.



[43] 234 x 96 IMPERFECT.



[43] 336 x 225 IMPERFECT.

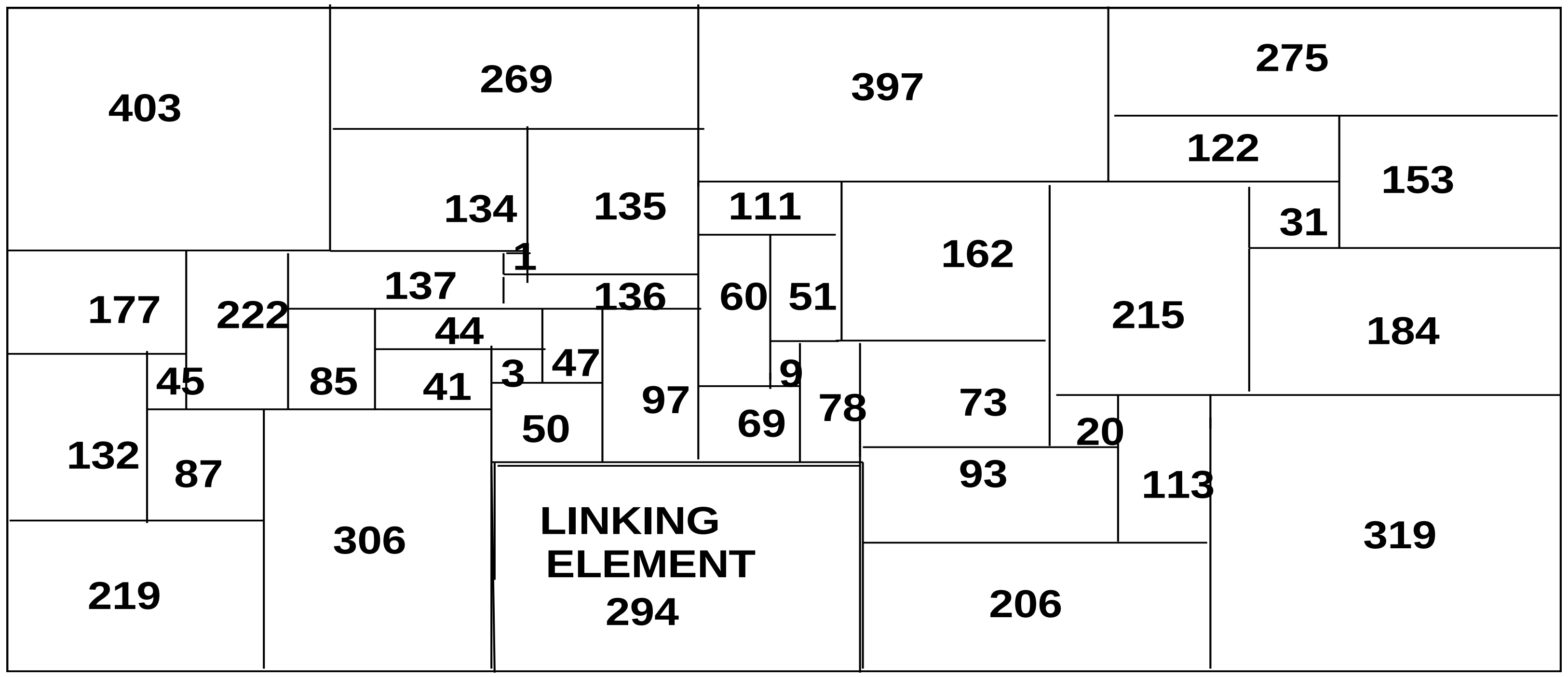
It is found generally that where there are duplications in the Rotor arms or in sets of three wires symmetrically placed with regard to one another in the original Twin solutions - duplications similarly occur in the resultant Twin solutions - those with double corners. This rules out numbers 1-3, 5-6 and 9 as far as the possibility of creating Perfect solutions is concerned. Number 4 has no duplication at all, but the dimensions are not particularly small at 1175 x 913.

Numbers 7,8 and 10 have what are termed coincidental duplications only, and Perfect solutions may be possible, although there is no guarantee since coincidental duplications may arise in the final solution. The smallest of these three (7, 8 and 10) is 637 x 483, from which the formulae gives reduced Rectangles of 980 x 408½ and 1344 x 931.

The fraction in 408½ occurs as s is odd (147), and these dimensions must be doubled to 1960 x 817 to make sense, which is unfortunately larger than 1344 x 931. However [43] 1344 x 931 is Perfect. See Below.

The next smallest pair is 663 x 520 with s = 156 an even number.

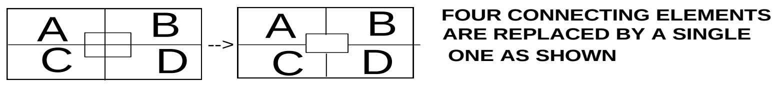
This gives [43] rectangles of 1014 x 416 (imperfect) and 1456 x 975. The pair [22] 1144 x 845 give solution [43] 3536 x 1365 and [43] 2340 x 1664, and the pair [19] 1175 x 913 gives rectangles [37] 3600 x 1463 and [37] 2552 x 1725.



[43] 1344 x 931 PERFECT.

J4.2. USING FOUR TWIN RECTANGLES

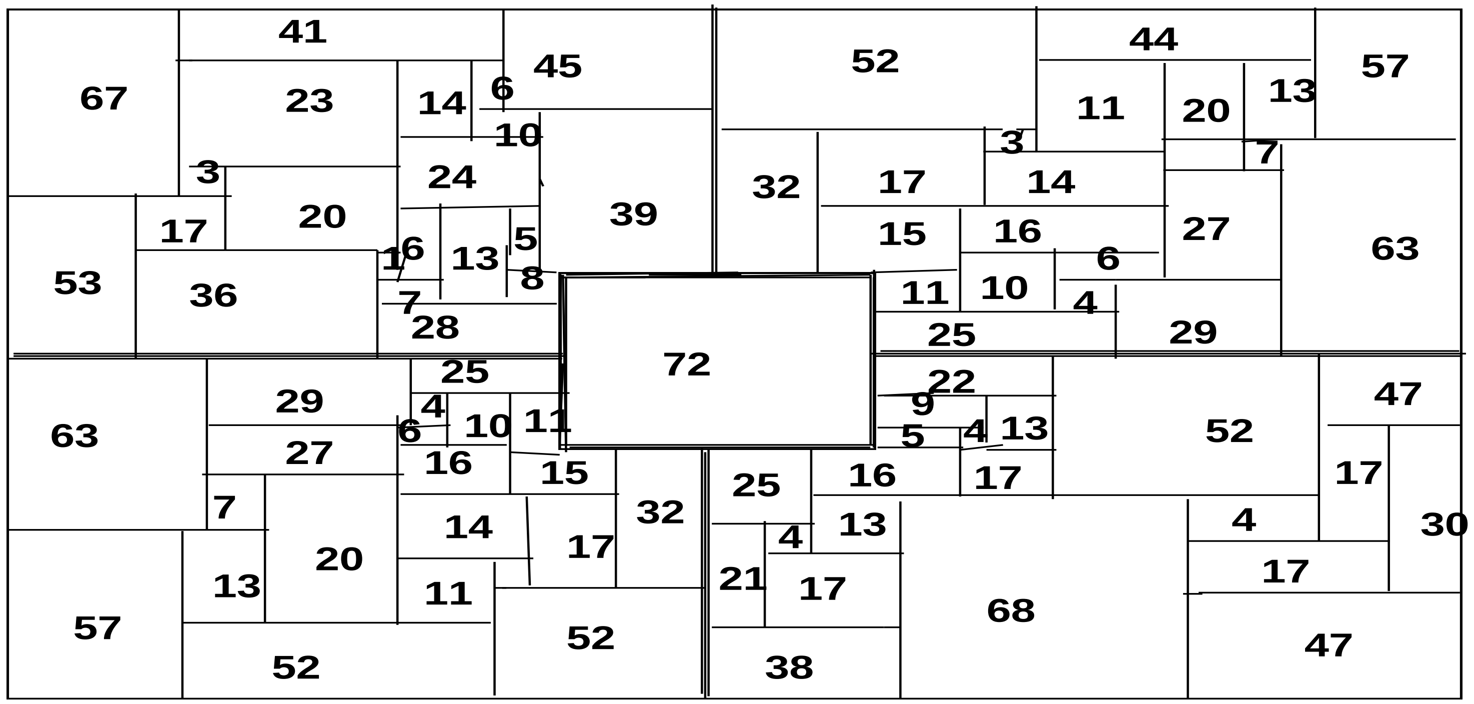
Another method of producing very reduced solutions of high Orders is shown below.



Four Twin solutions are required each with a common corner Element which can be put together, and the four inner Elements replaced by one larger one. Naturally solutions are extremely unlikely to turn out Perfect.
Although 4 Twins are needed one of these can be duplicated and an example of Order 85 which is not symmetric and is Simple is shown.
Assuming Imperfect solutions are accepted only 2 Twins may be used thus AB (calling rectangles “A” and “B”) ...
BA in the case of 2 to 4 Twins ABC and D the following combinations are possible-
AB AB AB AC AB AD AC AB AC AD 1 for A and B only 2 for ABC
CD BC BA BC CA BC CB DC DB CB 6 for ABCD obviously there are other useless combinations which give Invalid solutions. As similar constructions it is not possible to find the Reduction Index. However it is huge!
The only solutions from solutions known are

1. [9] 132 x 128(2) and [13] 132 x 128(1) giving [41] 552 x 463(1) and (2)
2. [13] 552 x 463(1) and (2) giving [49] 1104 x 926 Imp and sym.
3. [13] 274 x 270(1) and (2) giving [49] 548 x 520 (1) and (2)
4. [13] 140 x 132 and [13] 140 x 132(2) giving [49] 280 x 264 Imp and sym.
5. [13] 72 x 64(4) and (1) giving [49] 144 x 128 Imp and sym.

Apart from Twins known to me for Orders 16, 19 and 22 all of which also give Imperfect and symmetric results (apart from the three [22] 150 x 120) the most reduced are these:
[61] 1122 x 928, 1342 x 1008. [73] 2050 x 1184, 2350 x 1826... [85] 630 x 624, 1274 x 966...



ORDER 85

306 X 240

IMPERFECT

(NOTE THE FOUR SEGMENTS AND THE CENTRAL ELEMENT

J5. CONSECUTIVE NUMBERS

J5.1. CONSECUTIVE NUMBER THEORY

Is it possible for a rectangle of Order n to be constructed with one and only one of each Element 1 2 3 4...n?
This is a real problem as the scope and amount of Orders is infinite. Obviously if such solutions existed they are hugely reduced! I am tempted to believe that "Consecutive" rectangles cannot exist in view of the enormous restrictions which apply. But if 1 in 100 million did exist, a possibility of one existing with an Order like 25 or 26 with Elements up to say 90 x 70 might exist. <u>See below.</u>

J5.2. TENTATIVE INVESTIGATIONS

1. There is a maximum Elongation that a possible solution would have. Take Order 26 where Elements 1 to 26 total 6201. The square root of this is about 79. Somewhere 26 would have to appear. The minimum width

ORDER 2	SIZE	Unit gap 1. top 2 1
3	3 x 2	1 top 3 1 bottom 2
4	7 x 5	5 top 4 3 bottom 2 inside 1
5	9 x 7	8 top 5 4 bottom 2 3 inside 1
6	11 x 9	8 top 6 5 bottom 3 4 inside 2 1
7	14 x 11	14 top 6 7 bottom 5 4 3 2 inside 1

8	15 x 14	6 top 8 7 bottom 6 5 4 inside 3 2 1
9	18 x 17	21 top 9 8 bottom 7 6 5 inside 4 3 2 1
10	24 x 17	23 top 9 5 10 bottom 8 6 3 7 inside 4 1 2
11	31 x 17	22 top 11 10 9 bottom 6 5 4 7 8 inside 2 3 1
12	29 x 23	17 top 12 10 7 bottom 11 8 9 inside 2 3 5 4 1 6
13	35 x 24	21 top 13 12 10 bottom 11 9 7 8 inside 2 3 5 4 1 6
14	42 x 25	35 top 13 14 8 6 bottom 12 11 10 9 others inside
15	47 x 27	29 top 15 8 11 13 bottom 12 10 9 14 others inside

If a supposed rectangle would have be at least half as much again say 40. This restricts the range from c. 40 x 155 to 79 x 79 but (Section J5.2)
2. The sum of 1 to n has to be a useful factorial number. It happens that all such numbers are factorial.
But some Orders are clearly impossible for Consecutive rectangles e.g. 46 which would need a unit area of 23 x 41 x 31 and even 47 x 713 cannot possibly include Elements over 40.
3. If the rule that the solution must be Perfect is relaxed then beyond doubt Imperfect solutions exist containing 1, 2, 3 … n. a square with 1, 2, 3 and 4 exists for Order 17.
4. One way to assess better the possibility is to attempt to create small solutions containing all consecutive numbers with as little gaps remaining. Obviously which shape they are is irrelevant.
The best I could find to Order 14 without bothering to illustrate them is as follows.

J5.3. UNIT AREAS IN CONSECUTIVE RECTANGLES

The summation of 1 to n must be a useful factorial number as already said, so the unit areas for n=7 to 30 are tabled below to show theoretical dimensions that the solutions would need to have. The [24] 70 x 70 would be most interesting, but a solution is definitely not possible. The Writer now has information that apart from [1] 1 x 1 only Order 24 where the natural numbers total 4900 is a squared number. NO CONSECUTIVE SOLUTIONS EXIST AT ALL.

Order	Unit Area	Dimensions would need to be -	Order	Unit Area	Theoretic dimensions
7	140	14 x 10	19	2,470	65 x 38
8	204	17 x 12	20	2,870	70 x 41 or 82 x 35
9	285	19 x 15	21	3,311	77 x 43
10	385	35 x 11	22	3,795	69 x 55
11	506	23 x 22	23	4,324	92 x 47 or 94 x 46
12	650	26 x 25	24	4,900	100 x 49, 70 x 70
13	819		25	5,525	85 x 65
14	1,015	35x29	26	6,201	104 x 53 or 106 x 52
15	1,240	40x31	27	6,930	90 x 77 or 105 x 66

16	1,496	44x36	28	7,714	133 x 58
17	1,785	51x35	29	8,555	145 x 59
18	2,109	57x37	30	9,455	155 x 61

J6. PUZZLE CONSTRUCTION

J6.1. PUZZLES FROM SQUARED-RECTANGLES
This is an investigation into the possibility of constructing jigsaw puzzles from Squared-Rectangles.

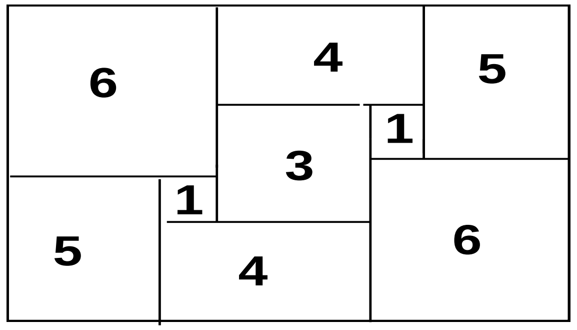
J6.2. IDEAL REQUIREMENTS FOR SUCH PUZZLES
Only a very small proportion of Squared-Rectangle Solutions are actually really suitable, as ideally the solutions should -
1. Have very reduced Elements.
2. Have a small or very small High / Low Element Ratio.
3. Contain no Elements of 1 2 3 or even 4 5 or 6. Smallest Element at least 6 mm.
4. Not contain 1 2 or 3 Elements much larger than all others.
5. Look pleasing to the eye when solved.
6. Be possibly Perfect rather than Imperfect - though Imperfect ones will naturally be easier to solve.
7. Be possibly Squares rather than Rectangles.
8. Be reasonably small - say less than 13” across!

J6.3. PROBLEMS IN FINDING IDEAL REQUIREMENTS
Finding Solutions which fulfill all the above conditions may not be possible!

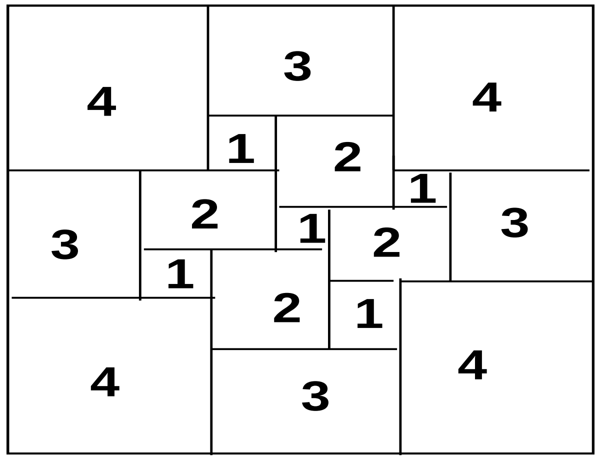
J6.4. SIZE PROBLEM
On looking at Computer produced solutions I found a quantity of Squared-Rectangles.
These have low Ratios when the largest and smallest Elements are compared:
25 found with Elements ranging from 8 to 120. 11 with Elements 8 to 99. 23 with 6 to 99. 15 with 7 to 99.
15 with 6 to 90. 27 with 5 to 90. 25 with 4 to 80. 33 with 3 to 70. 31 with 3 to 60. 31 with 3 to 50.
Order [9] 69 x 61 has Elements ranging from 2 to 36 which with Elements of say a quarter inch to 4.5 inches makes a Rectangle 8.6 x 7.6 inches - not particularly small, but the piece quarter inch square certainly is, making the puzzle unsuitable.

J6.5. LOOKING AT TWO SOLUTIONS WITH PUZZLES IN MIND
Order [9] 15 x 11 is one obvious choice for smaller children to solve since it contains a 3 and two of 1 3 4 5 and 6 only.
The ratio is 6 divided by 1 is just 6 and is very small considering the smallest possible Ratio is only 4.0.

But it is too easy for bigger children.
Order [17] 11 x 11 square (see L8.8. is another possibility) Only 1 2 3 4 are involved and if 1 cm represents “1” the Solution will fit into 5.5 cm square. But it is repetitive and not that exciting or difficult.



[9] 15 x 11



Possible Easy Dissection Puzzles

[17] 11 x 11

J6.6. FURTHER POSSIBLE SOLUTIONS

Of the many solutions found by Computer these had ranges of Elements between 3 and 60 -
ORDER 10 65 X 47 105 X 104 111 X 98 115 X 94 130 X 79
ORDER 11 98 X 95
ORDER 12 106 X 99 106 X 105 120 X 91 120 X 101
ORDER 13 109 X 96 112 X 75A 112 X 75B 112 X 93B 120 X 103 120 X 109
ORDER 14 106 X 75 145 X 111 168 X 109
ORDER 15 118 X 69 208 X 103
ORDER 16 153 X 87
ORDER 17 136 X 97 144 X 93
ORDER 18 104 X 99

J6.7. MRS. BROOK’S SOLUTION [13] 112 x 75

Referred to elsewhere, this is the one with two solutions with Identical Elements which are Twins, and contains the Elements 3 5 9 11 14 19 20 24 31 33 36 39 and 42 with a high / low ratio of 14.
With Element 3 measuring 6 mm the Solution measures 224 x 150 mm roughly 9 by 6 inches - see H7.2 for diagrams.

J6.8. SQUARE SOLUTION [15] 39 x 39

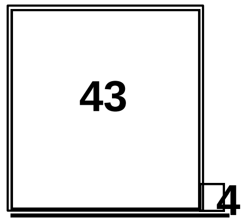
See L1.1 for illustration. The Elements range 2 to 20 with a High / Low ratio of 10.
Again with Element 2 at 6 mm square, the solution is 117 mm square, about 5” square, so is ideal.

J6.9. WAYS IN PRESENTING DISSECTION PUZZLES - THEIR MERITS & DEMERITS

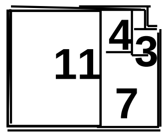
1. Have wood or cardboard pieces which do not show any Numbers!

This solves the problem of squeezing tiny numbers inside tiny pieces, but may not be suitable where two elements are proportionately very similar in size, like 55 and 56.

2. Have certain tiny Elements joined to a large one, or even have a 'bunch' containing all the really small ones -



Bunching in



Puzzles

This solves the tiny Element problem, but does not appeal that much to me somehow.

3. Having a Dimensions Box - or not

4. Having the Dimensions previously quoted or not.

These are two related ideas. Having a box and mentioning that the solution has sides of say, 69 and 61 will make the puzzle rather easier than not indicating the shape at all. Also having a box means the Puzzle can be left in a much tidier and permanent way. Having a set of boxes of different sizes for different solutions is another idea.

5. Following on with no. 4 what about a set of Boxes, say 5 with different dimensions, with one group of wooden pieces which can be fitted into any and each box using different combinations. This would present a challenge to the Puzzler who would not initially know 1 which pieces are needed for any box and even 2. how many pieces.

The Puzzler could be asked to draw all the results to prove he / she has solved them!

SQUARED OBLONGS

K1. DEFINITION OF OBLONGS

A SQUARED-OBLONG is a Squared-Rectangle where the horizontal side is an exact amount of times larger than the vertical side. Thus, a Squared-Rectangle of 400 x 200 is termed “2 by 1” and one of 600x 200 a “3 by 1” Rectangle etc. This Section has so far received little attention and needs much more work done.

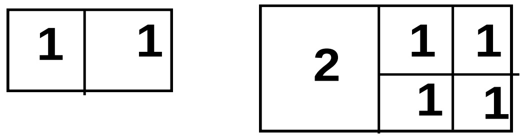
K2. PROBLEMS IN FINDING OBLONGS

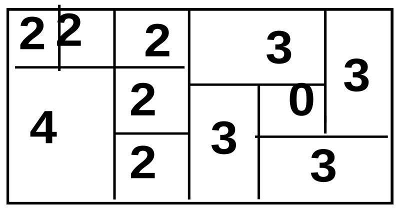
The comments made concerning Squared-squares equally apply to Squared-oblongs. Trial and error diagrams very rarely results in such Rectangles, and low Orders have few rectangles of 50% or less elongation.
There are no solutions of Orders 7 to 13 not surprisingly.
W.F.TUTTE in “More Mathematical Puzzles and Diversions” mentions if anyone wants to expand the research,
Here are two unsolved problems:
1. To establish the smallest Order for a Perfect Square, and
2. To construct a Simple, Perfect rectangle with the horizontal side exactly twice the vertical.
It is now known that such solutions do exist and all solutions to and including Order 26 have been found (and no doubt some of higher Orders).

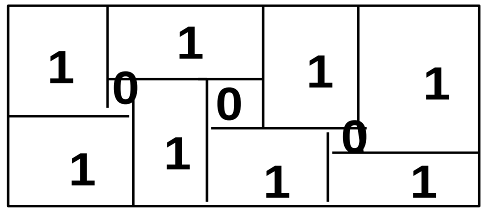
K3. INFERIOR SQUARED-OBLONGS

K3.1. INFERIOR TYPES OF “2 BY 1” OBLONGS

These may be readily found and shown in passing as they are of little interest.
Below 4 shows [11] 4 x 2 Invalid which may be regarded as the smallest and lowest Order Oblong with a Reduction of 56.

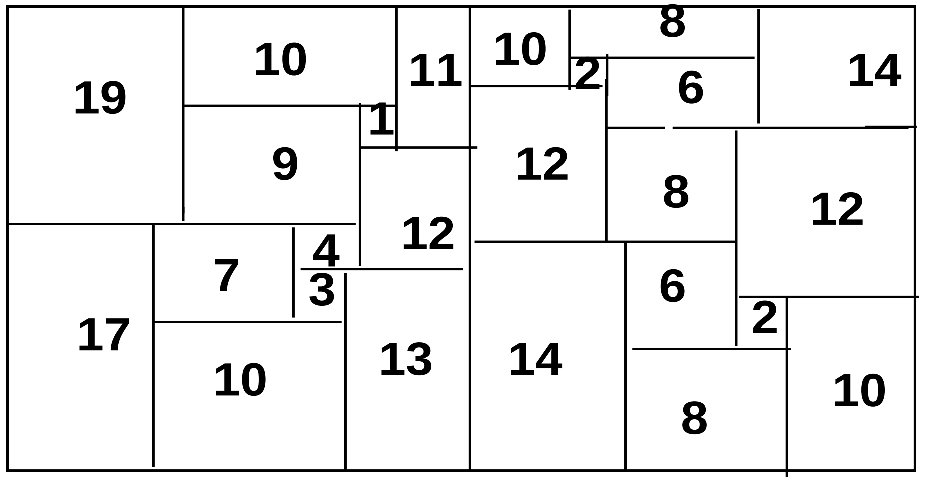


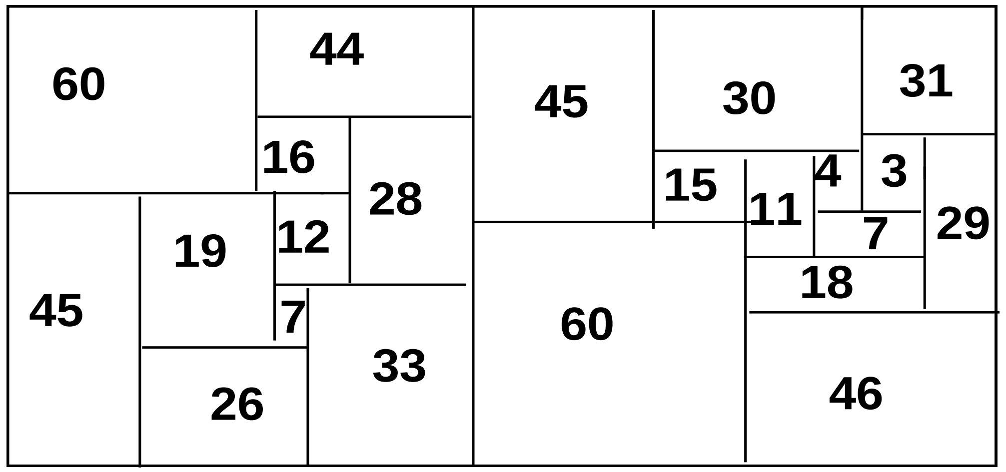




TWO BY ONE

RECTANGLES OF LITTLE INTEREST (DUDS & ZERO TYPE)





[25] 72 x 36 IMPERFECT 2-BY-1 [22] 210 x 105 IMPERFECT 2-BY-1

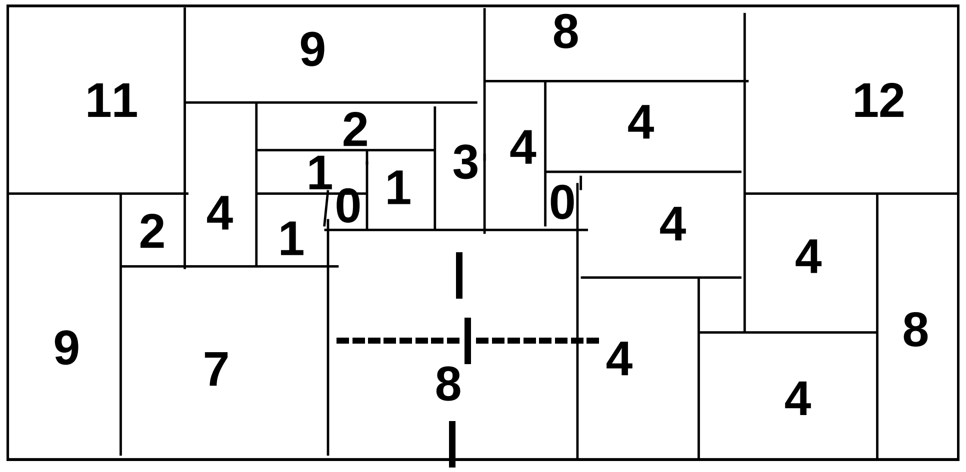
K3.2. ZERO SOLUTIONS

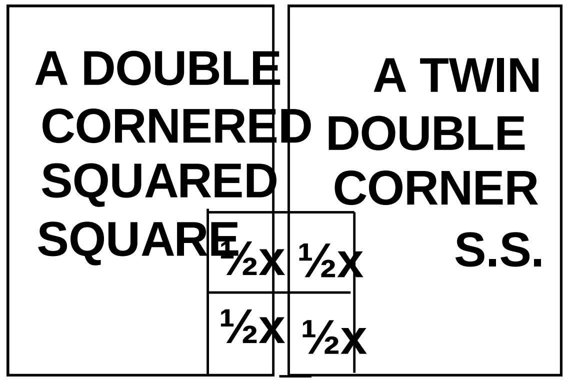
Further Compound solutions are possible by taking a known Squared-square, and adding a large Element along any one side. This is also of little interest, apart from indicating that Valid solutions which are Compound exist.

K3.3. COMPOUND SOLUTIONS

More interesting, but far from the best result are solutions formed by adding two suitable smaller solutions.
To do this, we require two solutions having one side identical and where the other sides total twice the identical side.
There are surprisingly few solutions possible involving solutions with Orders to 13. They are of course Non- Simple, but also unfortunately all Imperfect.
The smallest of these [25] 72x36 and the lowest Order [22] 210x105 are above. Such solutions are found by process of elimination after recording all solutions to Order 13 according to sides, as I have done.
Clearly Perfect Compound solutions of the type above are possible, but what about Simple solutions?
An example is below 1 and is [25] 40 x 20 but is Invalid.
Theoretically, if two Repeater Twin Squared-squares can be found with double corners (the double corner Elements being identical in each square) with no duplication of Elements between the two squares, then a “2 by 1” Perfect Simple solution is possible.

[25] 40X20 TAKEN FROM ADJUSTED SOLUTIONS



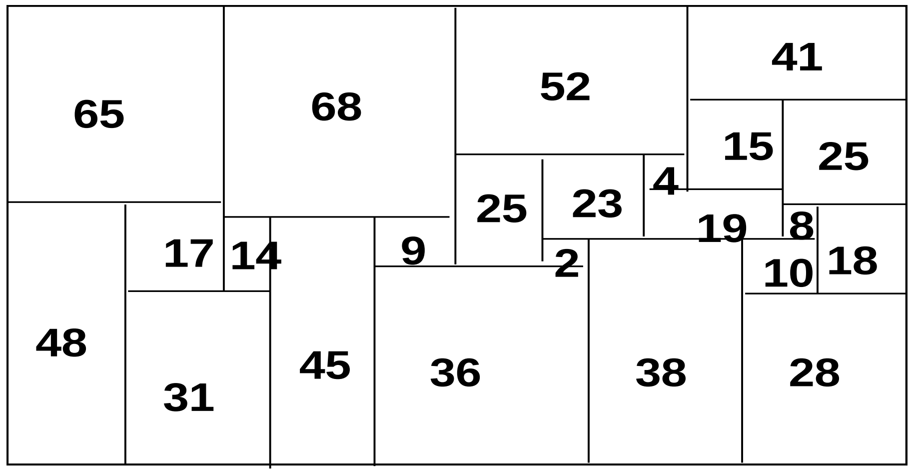


FORM OF CONSTRUCTION USED.

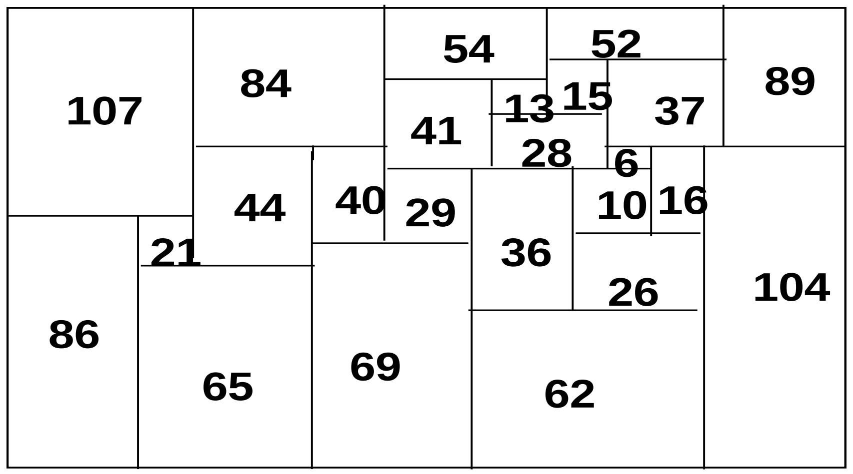
[13] 70 x 66 IMP & [13] 140 x 132(1) (rhs)

K4. PERFECT TWO-BY-ONE RECTANGLES

The two solutions of Orders 23 and 24 which follow were not discovered by me but taken from the Internet.



[23] 226 x 113 R452 PERFECT



[24] 386 x 193 RI 386 PERFECT

These are far more spectacular than the Reader might appreciate, and were the only two of their kind known to me before 2003, who has never found a Perfect Solution yet after calculating well over 620,000 solutions by computer.

The relationship between the Reduction and the sides is similar to Squared-squares, 452 being 113 x 4 and 386 being 193 x 2. As in Squares, Imperfect solutions are considerably more numerous but nevertheless rare against the millions of solutions obtainable for these Orders. The chances of calculating one by luck are virtually nil.

K5. ORDER 26 SOLUTIONS

In 2003 a list of 2 by 1 Perfect Rectangles of Order 26 was produced on the Internet with each drawn out.

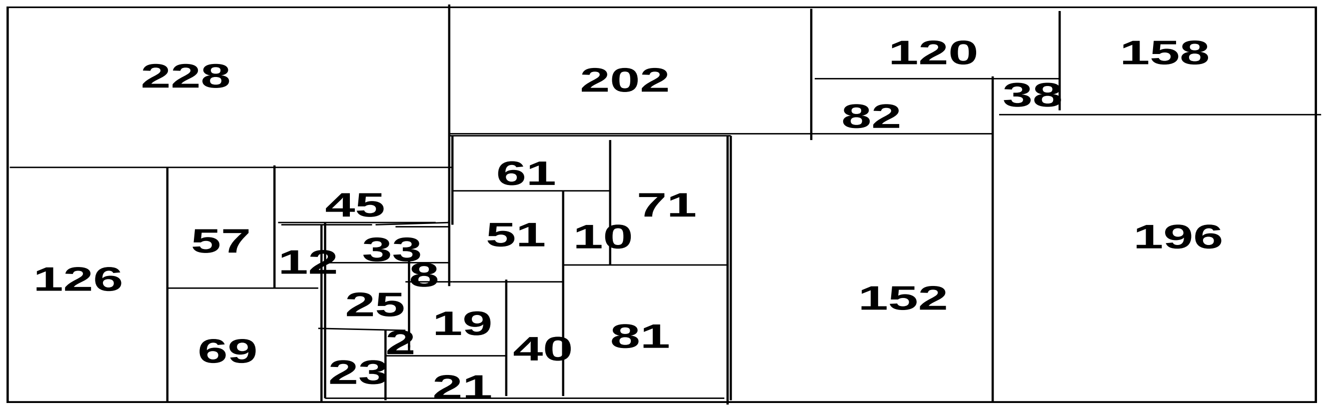
The Larger Dimensions of these are - the other Dimension being half the size.

308 312 315 322 326 328 352 368 376 384 428 444 452 456 458 464 468 480 488 500(1) and (2) 536 540 544 560 564 568 572 578 584 610 612 628 630 650 654(1) and (2) 662 664 666 668 670 672 688 698 704(1) and (2) 708(1) and (2) 712 714 740 746 748 750 756 758 766 768 792(1) (2) (3) and

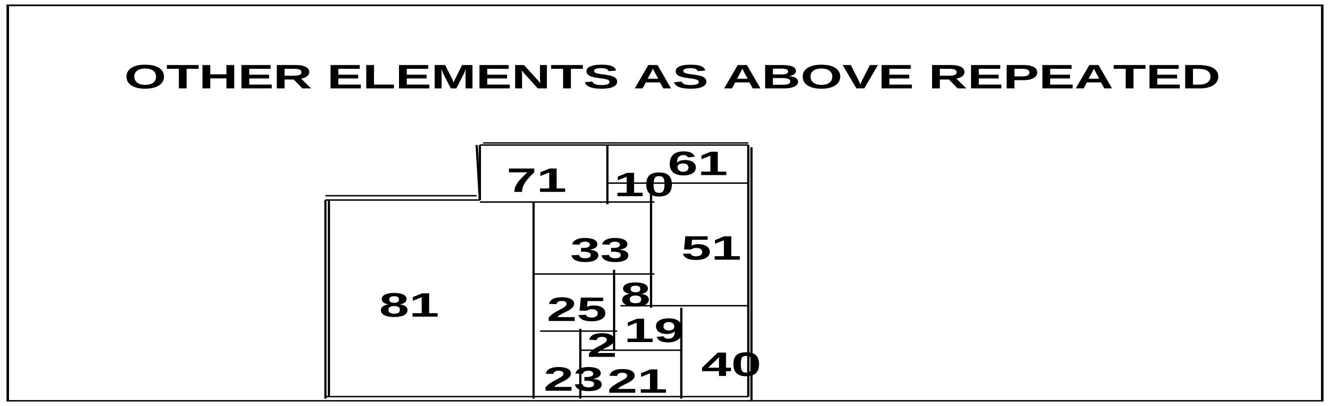
(4) 798 810(1) and (2) 818 820 822(1) and (2) 824 840 844 858 862 864(1) and (2) 876 1016 1034 1040 1048 1058 1064 1070 1090 1104 1106 1110 1116 1122(1) (2) and (3) 1126 1134 1142 1144 1146 1154 1180 1184.
The Number of Solutions is 101.

K6. EXISTENCE OF IDENTICAL PAIRS

The following Twin Pair was found on the Internet List - one incredibly similar to the other.
The Elements are the same for both but the pattern in the lower middle changes!



ORDER 26 TWO BY (
PERFECT RECTAN(
708 x 354 (1)

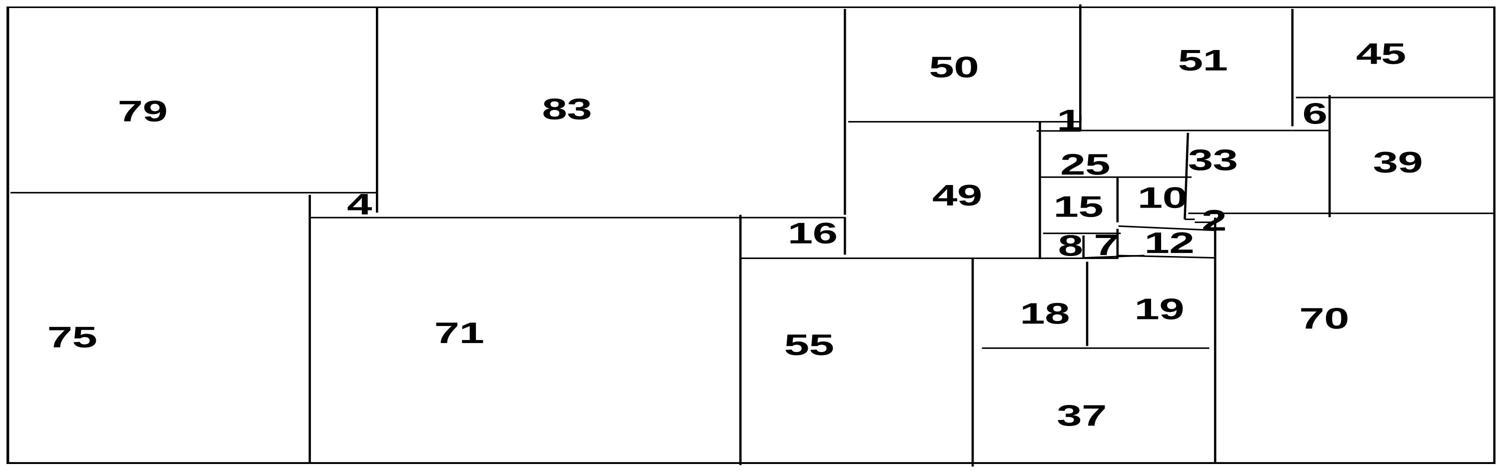


708 x 354 (2)

NOTE THE IDENTI(
ELEMENTS

K7. SMALLEST TWO BY ONE FOR ORDER 26

This is [26] 308 x 154 and is shown below-



[26] 308 x 15(

K8. SIDES INDEXES FOR ORDER 26 TWO BY ONE RECTANGLES

Clearly there must be at least two Elements at each end. Because of the 50% Elongation it appears that three need to be at least four Elements at Top or at the Bottom. However in the 101 Solutions the Smallest Sides Index found is 2-4-2-5, not 2-4-2-4 - it is just possible that the latter never exists.

The Following were found: - 2425 2426 2427 2428 2525 2526 2535 2536 2537 2436 2437 2445 2545 2547 2635 2536 2734

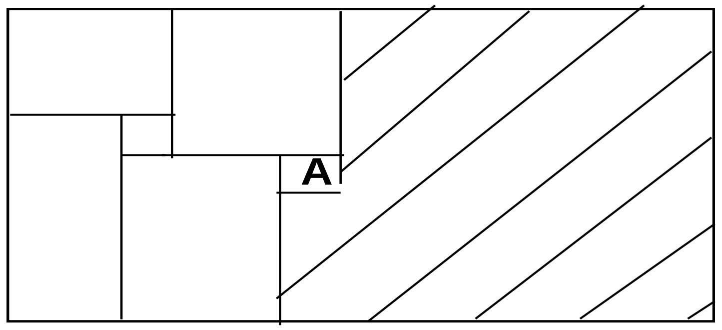
In the case of 2-4-2-5 only nine Elements are external and 17 internal. In 2-5-4-7 16 Elements are external and only 10 internal.

K9. A CURIOUS LINK RELATIONSHIP

K9.

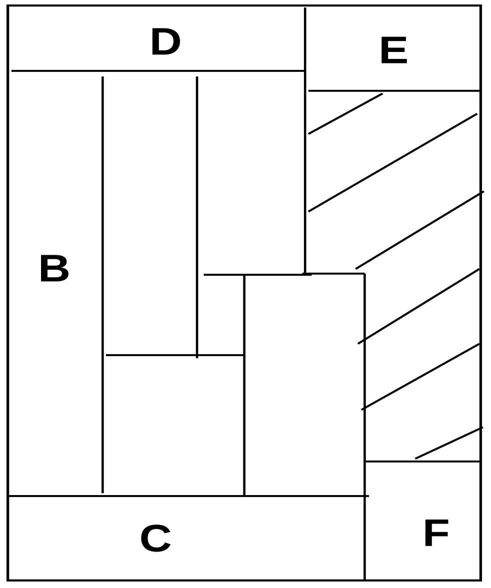
Refer to the Above Solution and observe the six Elements to the left (79 75 4 83 71 and 16).

It was shown earlier in the Links Section that where this ending exists so does another Solution from it -



[o] m x n=half m

RATIO 2:1



[o+4]

9m-7n x 10m-5n

RATIO 11:15

In Above 2 Element A has been omitted but B to F added. If Above 1 is a 2 by 1 Rectangle this means n is half of m and substituting this in the formula 8m - 7n x 10m - 5n the Reader may easily work out that Above 2 must have a Ratio of 11 to 15 or 15:11 if the rectangle is shown horizontally.

So if any Sides 2 3 2 4+ Rectangle of this type has the exact Ratio of 11:15 or 0.073 (recurring) exactly it automatically means that <u>a 2 by 1 Rectangle is always obtainable from it!</u>

If the original is Perfect - it is very likely that the resultant 2 by 1 Solution will be also.

L. COMPOUND SQUARED-SQUARES

L1. DEFINITION AND STATUS OF COMPOUND SQUARED-SQUARES

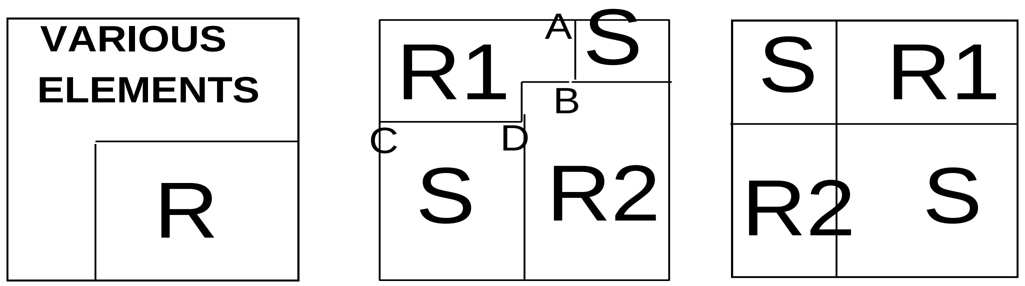
L1.1. DEFINITION

COMPOUND SQUARED-SQUARES are squares divided into individual squares, but also contain one or more Squared-Rectangles within. They can also be termed Complex Squared-squares.
See Section M for the more desirable Simple solutions.

L1.2. STATUS OF COMPOUND SQUARE SQUARES

Complex solutions have a desirability status as follows:-
1. Perfect (best grouping)
2. Imperfect - containing one rectangle as below 1.
3. Imperfect - containing two same solutions, inset, no Crossover, as below 2.
4. As last but different (Twin) solutions. as below 2.
5. As last but with crossover (i.e. corner Element common to both same solutions). As below 2.
6. As last but different (Twin) solutions. as below 2.
7. Imperfect with two complete Twin same solutions. As below 3.
8. As last but different (Twin) solutions. As below 3.
9. Non-zero - possible splits as above.
10. Zero - possible splits as above.

THREE TYPES OF COMPLEX SQUARES. S=SINGLE SQUARES



R1&R2 TWO RECTS. OF SAME SIZE.

R= ANY VALID RECTANGLE.

1. BEST TYPE. 2. 3. WORSE TYPE.

Squares may also be graded according to 1. No duplication (Imperfect)
2. Slight Duplication (Imperfect) in degrees to
3. Very Duplicated (highly Imperfect).

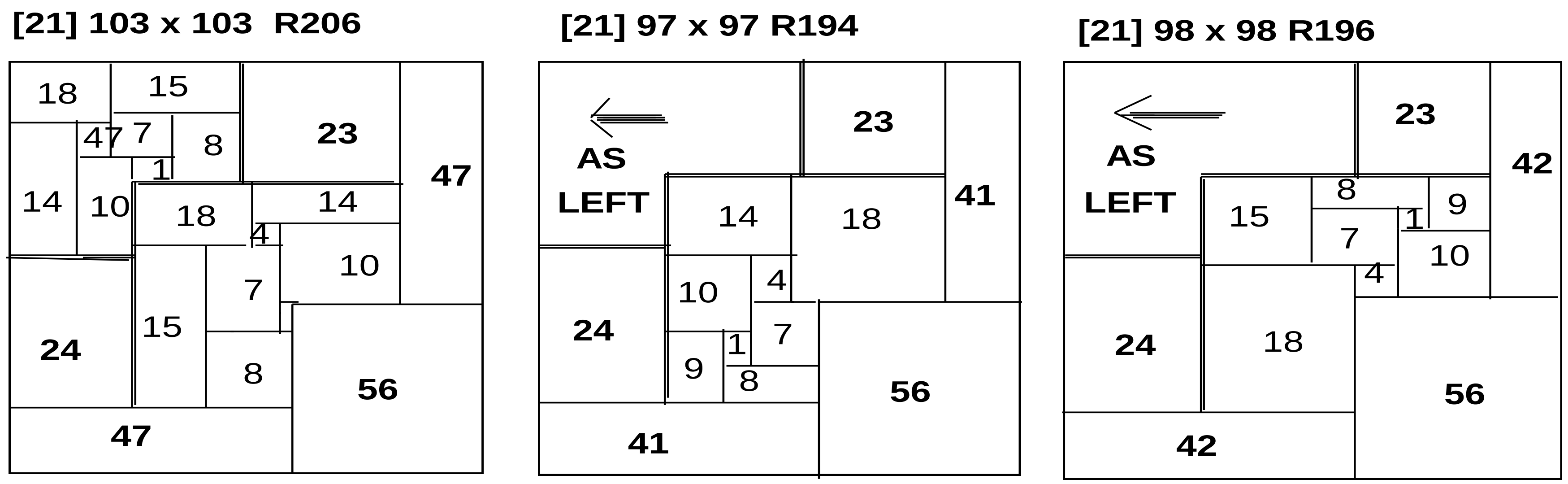
L2. CONSTRUCTION

L2.1. COMPOUND SQUARES USING TWINS - TYPE 3 ABOVE

**By adding two component Elements to a pair of solutions a Compound Squared-square is obtained.
If a corner Element is common to both twin.**

L2.2. COMPOUND SQUARES TYPE 2 ABOVE

**A square can be constructed provided there are at least two Elements between A and B and C and D, without which the solution is Invalid.
Where the corner Elements differ in size one such Element may be removed from one rectangle, and a square constructed as in ******put in*****



IMPERFECT SOLUTIONS OBTAINED FROM COMPLEX SQUARES & [9] 33X32

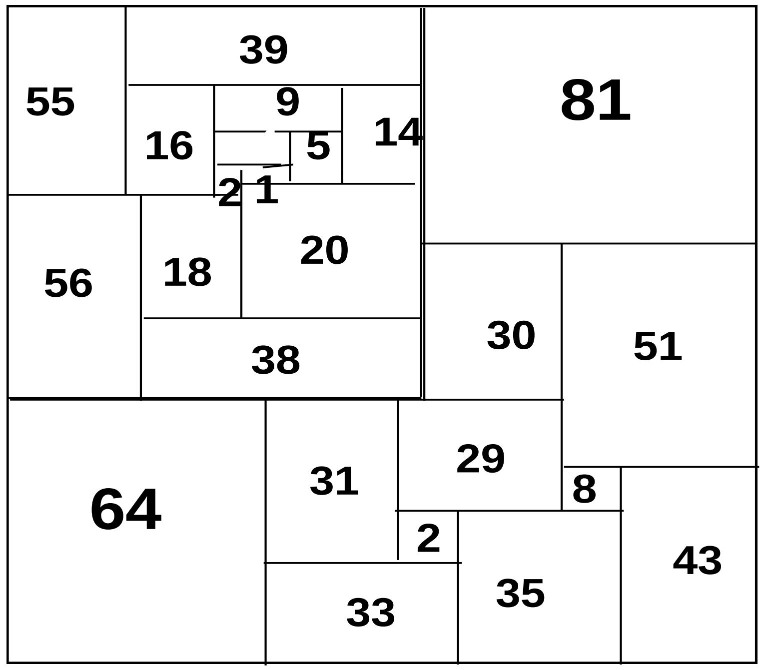
L2.3. COMPOUND SQUARES TYPE 1 ABOVE

**Again there must be at least two Elements as construction Type A is more Compound than Type B; Type B more Compound than Type C. Thus Type C is superior to Type B, and Type B to Type A.
It is better to construct Type C Squares. Clearly too, it is more satisfactory to find Perfect rather than Imperfect solutions. Usually it is necessary for the Twin solutions to be Perfect (1) themselves and (2) with one another.
However, it is sometimes possible to dispose of one unwanted corner Element which happens to be duplicated when employing constructions Type B or C.**

L.2.4. SQUARES FROM USING TWINS

We shall now experiment with the Twin solutions listed on 98 and 99 in Order to discover the lowest possible Order Squared-square possible from them. Obviously we choose the twin pair with the smallest Orders, this being set 15 namely [9] 33 x 32 (x3) and [12] 99 x 96. 9 + 12 Elements + 2 components gives Order 23 and a square of 195 x 195 by construction Type A. However, employing construction Type C and losing a corner Element in one of the Twins, two Order 22 solutions are obtained. There are two duplicated pairs, 12 and 21, and the solutions are of necessity, Imperfect.

The only Elements which can be lost is either 27 or 31 (if anything else is removed Invalid solutions result).
Perfect solutions from twin solutions - from the list to Order 13 the only set where all the Elements in one rectangle differ entirely from the other is [13] 593 x 422 (1 and 2). There is a duplication of 57 in [13] 608 x 407 (A and B) and 133 in
[13] 608 x 465(A and B) and 7 in [13] 106 x 99, but these duplications do not appear as corner Elements and so cannot be removed. Looking then at [13] 593 x 422 (A and B) this may be made into a Squared-square 1015 x 1015 by construction Type A.
This is an Order 28 square, and is illustrated ****



A PERFECT BUT COMPLEX SQUARED-SQUARE
[24] 175X175

THE TOP LEFT ELEMENTS DENOTE
[13] 111X94

THE BLACK ELEMENTS FORM A RECTANGLE WITH AN ELEMENT OMITTED.

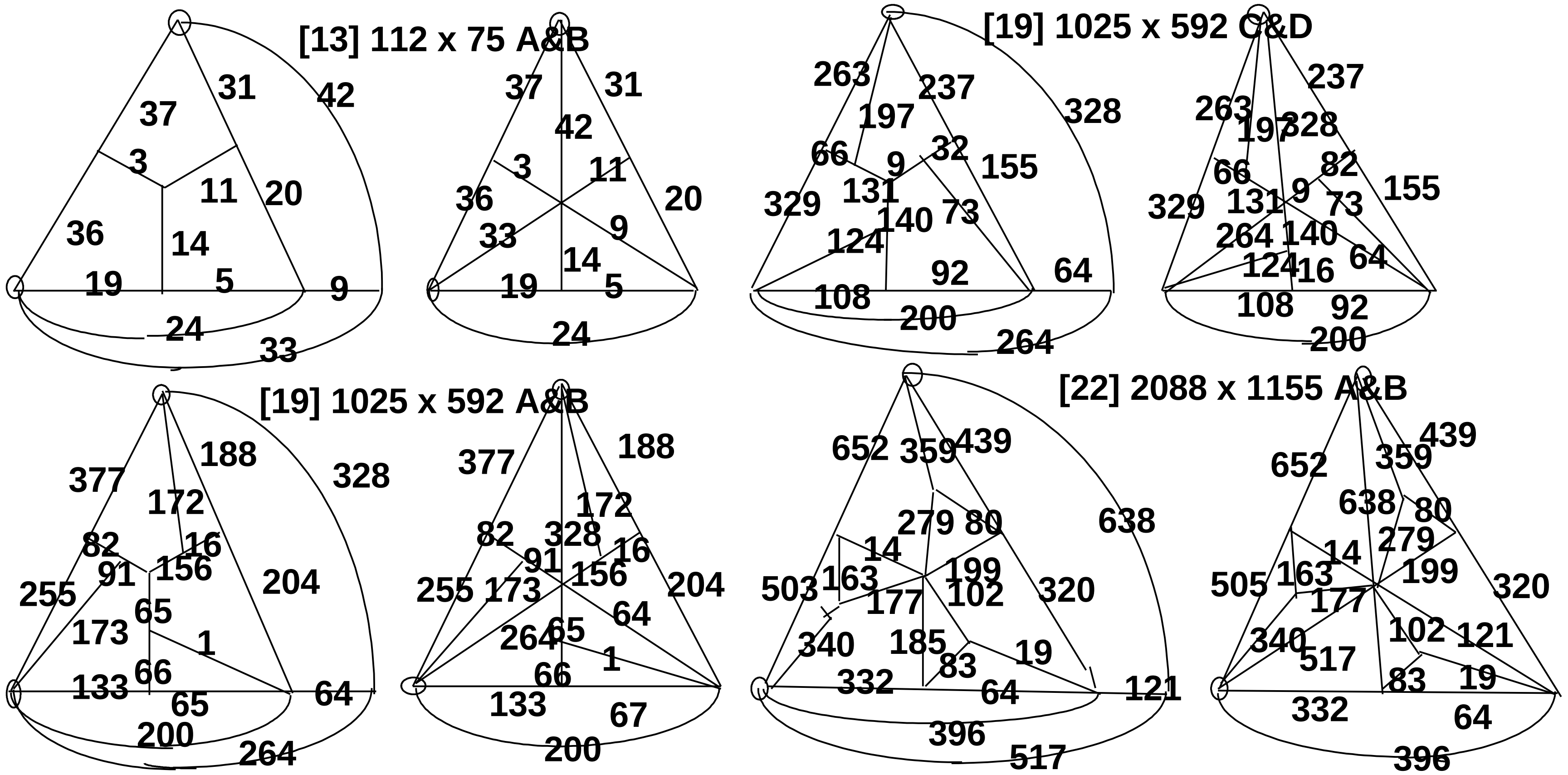
NOTE THAT THE RECTANGLE CAN BE SWITCHED LEFT TO RIGHT &/OR TOP TO BOTTOM TO CREATE 4 TWINS, 175X175

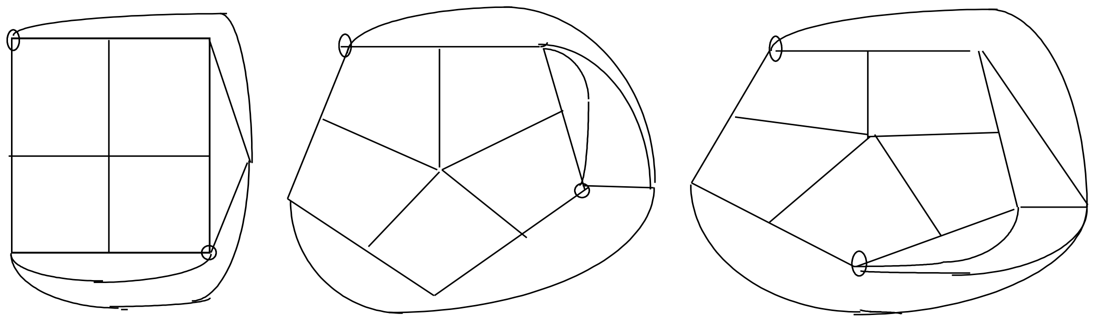
L2.5. SQUARES USING SINGLENDS

Another method of attempting to obtain Compound Squared-squares is to construct rectangles of the type shown in figure 120b, in the hope that some are of identical size to other known solutions.
Although this type of Squared-Rectangle is Invalid, it is Valid once one of the duplicated corner Elements is removed.
The solution is then added to the twin as in construction type c, with two component Elements added, to obtain a Squared-square. If there is only one Element between A and B and C and D, the resulting Squared-square is Invalid. Why employ this unusual type of construction? Well, mainly because of the restricted amount of solutions possible for low Orders, in the usual construction. There are in fact 100 constructions of this Type to Order 11 (inclusive) and mostly Invalid. A list of the specially constructed solutions has been calculated up to, and including Order 12.
Upon checking the dimensions against those of normal construction to Order 13, only one case, unfortunately, gives rise to a Twin pair, and this is illustrated - fig 120b. 111 x 94. With [13] 111 x 94, the Element 30 omitted, and two further Elements added, the magnificent solution (figure 121a) is obtained.
This is a really good example, as although it is Compound it is reduced 875 times, and the Order is low at 24.
Furthermore, it is a Perfect solution, something not guaranteed by this method.

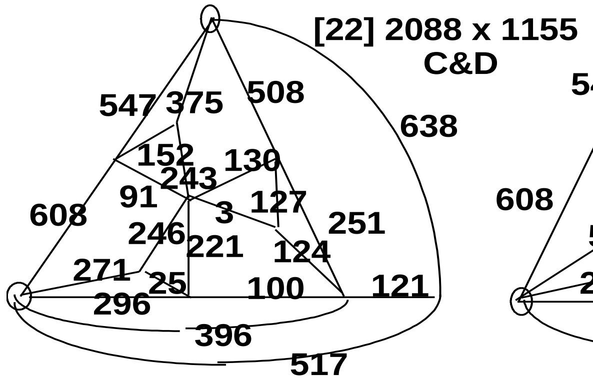
L2.6. SQUARES USING ROTOR ARMS

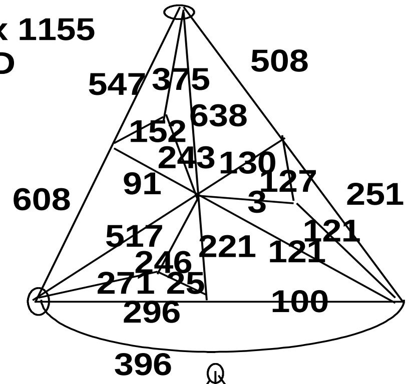
Symmetric Smith Diagrams with Rotor arms included are a certain means of producing Compound Squared-squares.
They are not necessarily Perfect however for the individual solutions may be Imperfect, or more common there may be duplication of Elements between the two solutions. The fact that a single Stator wire is always a duplicated value presents no difficulty in producing Perfect squares, as the following construction is used.
All the possible constructions of Smith Diagrams of this Type to Order 22, containing one Stator only are below.

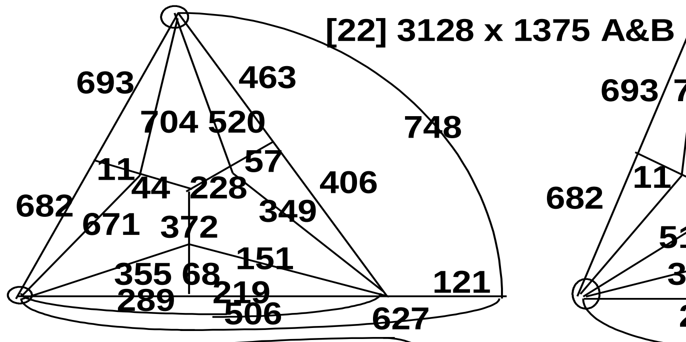


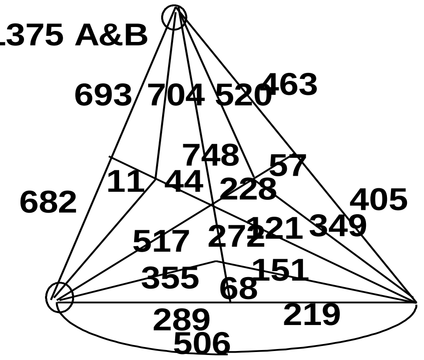


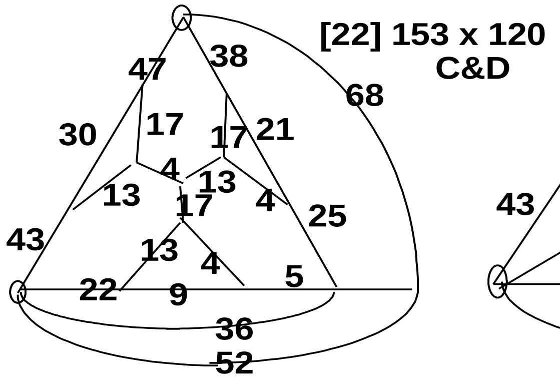
These fail to give proper solutions.

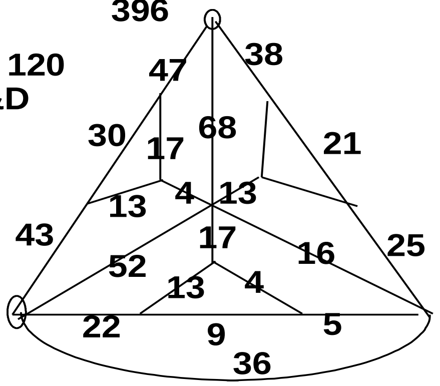


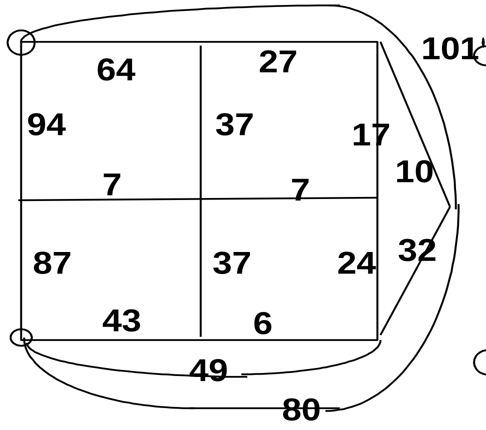


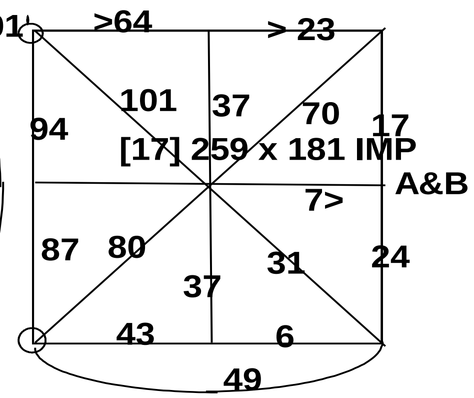


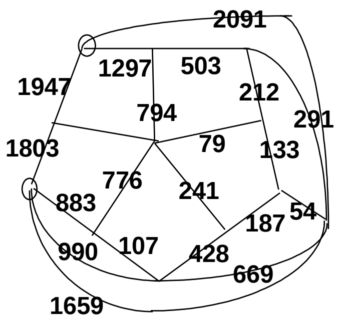


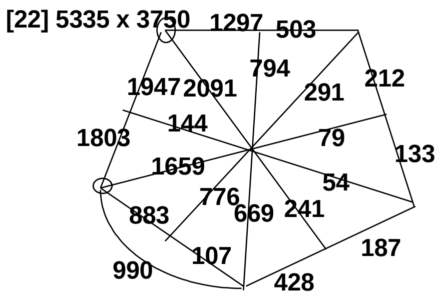












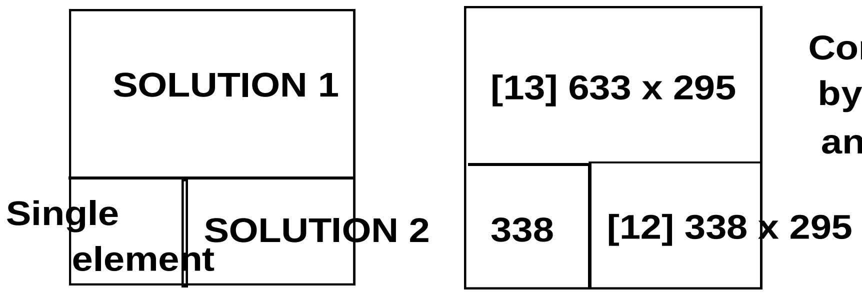
L2.7. ROTOR SQUARES EXAMPLES

The following Perfect squares are possible from the solutions listed.

[19] 1175 x 913 to [39] 1813 x 1813 [19] 2360 x 2001 to [39] 3781 x 3781, [19] 3075 x 2261 to [39] 4639 x 4639.

[22] 4347 x 3760 to [22] 7027 x 7027 [22] 5709 x 3811 to [43] 8299 x 8299 [22] 5800 x 3759 to [43] 8341 x 8341 [22] 6552 x 4033 to [43] 9253 x 9253 [22] 6303 x 5152 to [43] 9937 x 9937 [22] 8509 x 6720 to [43] 13219 x 13219. [22] 116x9805 to [43] 18583x18583. All other pairs give Imperfect Squares.

Yet another way of finding Compound Squared-squares is by adding two solutions and a component Element together as below. Unlike the other means already described, the construction does not depend on known Twin rectangles.



Construction of [26] 633 x 633 by drawing two Squared-rectangles and adding an extra corner element.

Observation shows that if solution 2 has dimensions (a + b) x b. Now a + b is the Semi-perimeter of Solution 1, and as there is a tendency for Semi-perimeters at one Order to be repeated as sides at the Order one higher, it is sense to consider solution 1 with one more Element than Solution 2, firstly. (But there are no restrictions to the Orders which may be employed, of course).

Using all the Valid solutions for Orders 9 to 13, it is found that there are four possible solutions by this method, and all of these employ an Order 13 solution with an Order 12. It is evident that few solutions up to Order 13 are elongated enough to be likely possibilities for this type of construction.

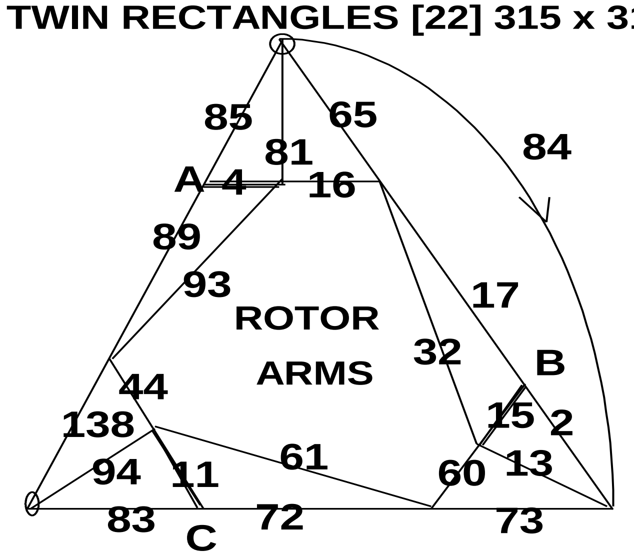
These four solutions are found by listing Order 13 solutions which have higher dimensions which are Semi-perimeters for Order 12, namely

560 x 368 560 x 375 560 x 377 576 x 352 576 x 416

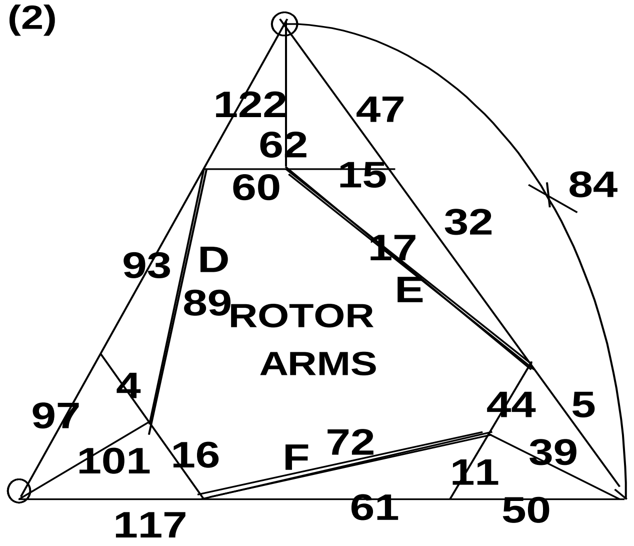
576 x 482 672 x 430 672 x 450 and 672 x 498. The four solutions are above.

The solutions are - 1. [13] 633 x 295 and [12] 338 x 295 2. [13] 608 x 377 and [12] 377 x 231 3. [13] 45 x 29 and [12] 29 x 16 ((both Imperfect) 4. [13] 51 x 29 and [12] 29 x 22 (both Imperfect). The constructions for the last four are -

TWIN RECTANGLES [22] 315 x 312 (1) & (2)



ELEMENTS AT A,B,C, 15 =11 + 4



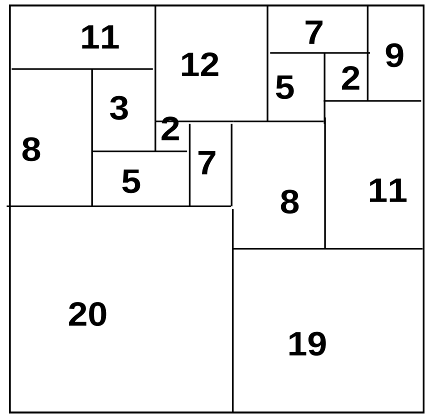
ELEMENTS AT D,E,F, 17 + 72 = 89

M. SIMPLE SQUARED-SQUARES

M1. INTRODUCTION TO SIMPLE SQUARED-SQUARES

M1.1. DEFINITION OF SIMPLE SQUARED-SQUARES

A Squared-square is a Square divided into a number of smaller squares called Elements, without any smaller rectangles present. Duplications in the Elements usually occur to a lesser or greater extent as in this one:-



EXAMPLE OF A VALID SQUARED-SQUARE
ORDER 15. 39x39 (1)
NOTE THE SMALL SIZE ELEMENTS.
THIS ONE IS IMPERFECT.
THIS WOULD MAKE AN EXCELLENT PUZZLE FOR CHILDREN. CUT UP SOME CARDBOARD & SEE IF THEY CAN ASSEMBLE IT CORRECTLY!

This whole subject is a fascinating with many facets, and it is fitting that the most profitable features are those most difficult to find.

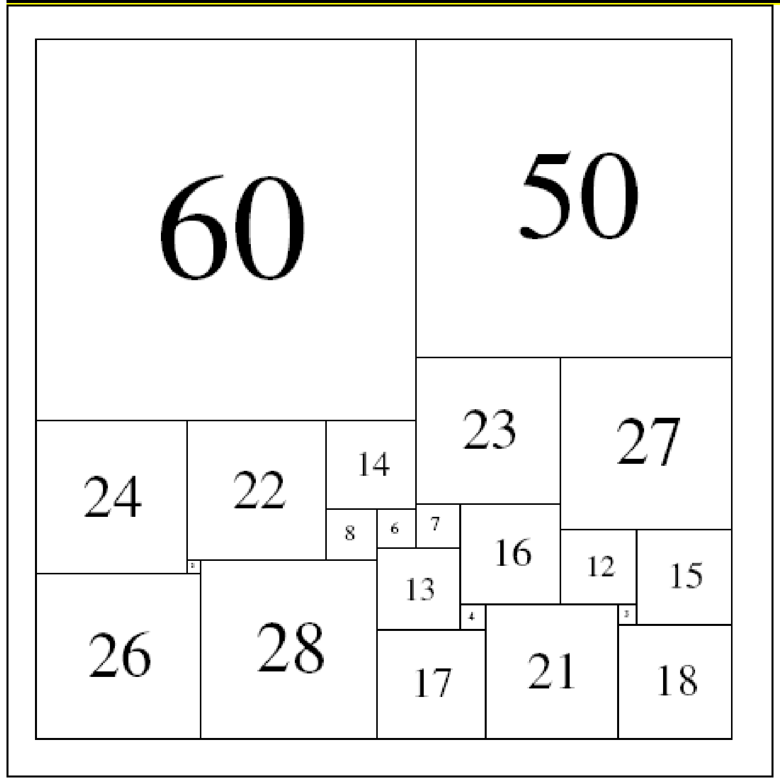
M1.2. HISTORY OF SQUARED-SQUARES

The subject of Squared-squares has seemingly received little attention until the 1930's!
Without computers, calculating them was hard work, and many must have considered Simple Perfect Squared-squares impossible anyway!
Long ago the Russian mathematician Lusin wrongly claimed that to dissect a square into unequal Elements was impossible. , famous for puzzles, also came to the same conclusion.
It was not until 1939 that four Cambridge male Students showed that Squared-Squares which are both Simple and Perfect do exist, and their first discoveries were two of the rather high Order of 69.
Roland Sprague first published such a square in 1939 of Order 55.
R. L. Brooks also published an Order 38 Simple Perfect Square of 4920 x 4920.
In 1959 this was bettered by T. H. Willcocks of Bristol with Order 37 square of 1947 x 1947.
Later Muncey found a much smaller Order 24, 503 x 503 square.
In 1978 A. J. W. Duijvestijn found the spectacular Order 21, 112 x 112 Square.
I independently found the same solution on computer in 1997 unaware of Duijvestijn!
It is the only Order 21 Simple Perfect Squared-square. [22] 110 x 110 is the smallest by Dimension.
Huge progress in the finding of Simple Perfect Squared-Squares has happened in 2013 and 2014. Over 15,000,000 Solutions are known.

M1.3. PROBLEMS IN FINDING PERFECT SQUARES

Constructing the most desirable types of Simple squares is so difficult that most attention in this book is given to known Compound solutions (containing at least one rectangle), and other inferior types which are much easier to find, especially if a collection of Twin or other rectangles as discussed in this book are available for reference.
Many solutions can be found with the sides varying by just one, most of which are Perfect, giving a false impression that a Simple Perfect Square will soon be found just as easily!
At first it seems that all that has to be done is to draw rough square-shaped patterns in the hope that one will turn out to be completely square when calculated, but anyone trying this will soon give up in desperation!
It is now known that we could look for ever in Orders 5 to 20 without finding one Perfect square!
With 2609 Perfect rectangles up to Order 15, increasing by a multiple of 3.3+ for each Order above it, it is easy to think there is a good chance by Order 19, but the reason for such rarity will be revealed later!

M1.4. ELEMENTARY FACTS ABOUT THE SIDE ELEMENTS IN SQUARED-SQUARES



In Above the Side Elements are 24 27 15 28 17 & 21. We could imagine a 'side' of 0 at the top.
Now Adding top & bottom we get T+B = 0+28+17+21 = 66 and left & right we have L+ R = 24+27+15 = 66. In fact in any SS T+ B always equals L+ R! Or we can say (T+B) - (L+R) = zero. It also stands to reason that L+R+T+B will always be an even number. However the total of the four corners may be odd or even.
In 2233 solutions the only two Sides are the same value. In 2234 solutions there are three Sides two equaling the largest.
If we were told a SS has sides of 19 24 29 35 37 what could we deduce from this? They total 144, half of which is 72.
Now 37+35 = 72 and the remainder 19+24+29 = 72.
Possibilities 1. A 2245 solution (Imperfect)

or 2. A 3334 solution (actually the case).

M2.DESIRABILITY OF SQUARED-SQUARE TYPES

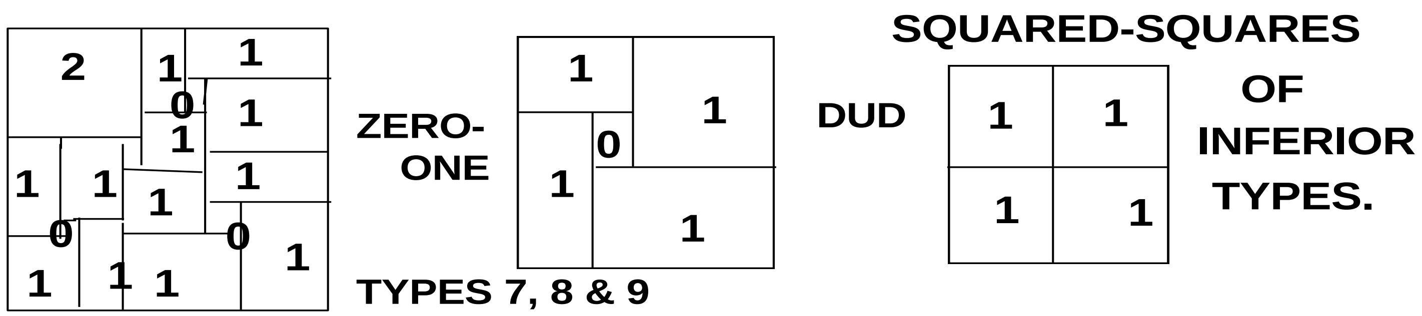
M2.1. VARIETY OF SQUARED-SQUARE TYPES

There is a large range of different types which vary greatly in desirability.
In general the less desirable the type, the greater the number of solutions possible and the far easier they are found!
Many solutions are very Imperfect in the sense that the Elements have heavy duplication, and so not particularly attractive. For the sake of theory, it is appropriate to study all types however.

M2.2. LEAGUE OF DESIRABILITY

Starting with the most desirable these are:-
1. Perfect Section M10
2. Imperfect with sides 3333 or greater. Some sides 3333 are of inferior type.
3. Imperfect with sides 2333 or greater (but one side with 2 elements)
4. Imperfect with sides 2234 or greater (but two sides with 2 elements)
5. Imperfect with sides 2233 which are very common.
6. Non-zero. Unusual but not desirable. Section M3
7. Zero with Elements of 2 or more. Section M3
8. Zero-one type.
9. Duds. Examples of 1 to 5 are shown later, but examples of 7 to 9 are shown below.

They may be useful in the theory of the subject, but uninteresting.



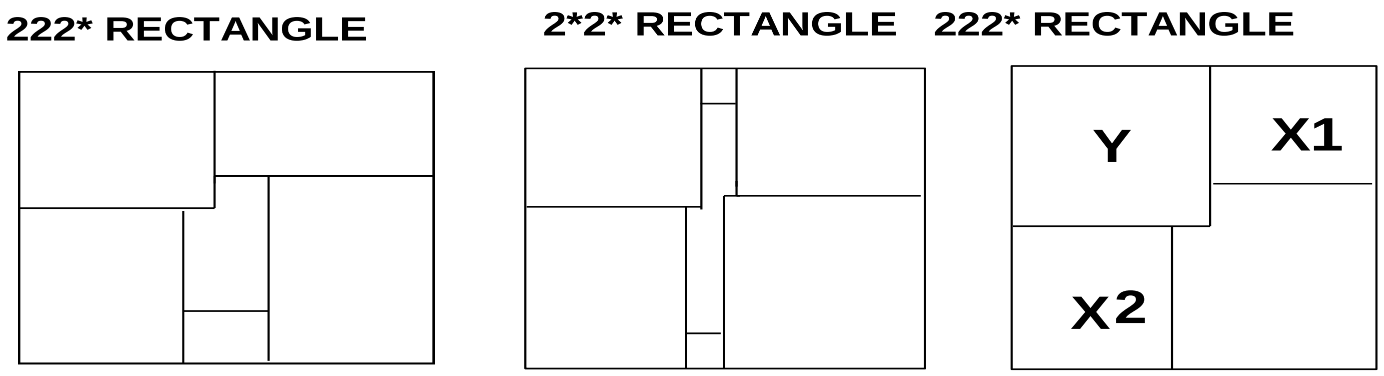
M2.3. DIFFERENT SYMMETRY TYPES

Another division according to desirability is according to the symmetry involved:-
a. Asymmetric or one-fold symmetry (most desirable) Section M4 L5 and L6.
b. Two-fold symmetry. Symmetry 2 and Symmetry 3 Section M7.
c. Four-fold symmetry. Symmetry 4 (least desirable) Section M8.

The last two must always contain a large amount of duplicated Elements, i.e. highly Imperfect.
Discussion on these types and construction is dealt with in Sections M7 & M8.

M2.4. NO SQUARES WITH SIDES 2222* OR 2*2*

1. Firstly it is pointed out that no Squared-square can be S2223 or S2224, S2225 ...S2229 etc. since apart from [5] 2 x 2 (Above 2) All solutions can only be rectangular.
2. Further investigation shows no S2#2# solution can be a square either. (for # read >2)
3. Next, many S22## squares exist but are all Imperfect, as x1 and x2 Below must be equal for the pattern to be square.



NO MATTER HOW SMALL ELEMENTS A,B,C.. ARE, THE RECTANGLE NEVER BECOMES A SQUARE!
IN 3RD DIAGRAM THE ELEMENTS X MUST BE THE SAME PREVENTING THE POSSIBILITY OF PERFECT SQUARES.

M2.5. SQUARES WITH SIDES 2333, 2334, 2335, 2336. . .

See under M16.1. Where it is shown that these do exist - albeit very rarely.

M3. FINDING INFERIOR SQUARE TYPES

M3.1. INTRODUCTION TO FINDING SOLUTIONS

Improbable as it may seem to the Reader, it is generally easier to calculate lots of Squares than Rectangles! But the ones easily found do not have the best status: However calculating Perfect Rectangles is vastly easier than calculating Perfect Squares.
It is sense to start at the worst types of Squared-squares and work up. However class 9 duds can be dismissed,
Next is Class 8 with -

M3.2. ZERO-ONE SQUARES

These are of no great use but are easy to construct and the theory can be looked at.
Based on Section G2 a table can be shown:-

Reduced Size	Order	Sides Index	Reduction

2 x 2	5 = 4 (ones) + 1 (zero)	S2222	4
3 x 3	13 = 9 + 4	S3333	192
4 x 4	25 = 16 + 9	S4444	
5 x 5	41 = 25 + 16	S5555	
6 x 6	61 = 36 + 25	S6666	
7 x 7	85 = 49 + 36	S7777	
8 x 8	113 = 64 + 49	S8888	
9 x 9	145 = 81 + 64	S9999	

M3.3. SYMMETRY 4 SOLUTIONS

These are dealt with fully in Section L6, and so far it will simply be stated that there are several easy ways to produce these using Simple algebra.

M4. REDUCTION INDEX AND SQUARES

M4.1. RELATIONSHIP BETWEEN SQUARED-SQUARES AND REDUCTION INDEX

The following reasoning is longwinded but draws out a very important principle step by step, and the considerations arising from them.

1. Whenever Squared-squares are found from computer programs for Squared-Rectangles, Reduction Indexes are always very high compared with those for Rectangles - even Inferior Rectangles.

2. The sides are also remarkably small [11] 7 x 7 [12] 12 x 12 [13] 3 x 3 and 4 x 4 and 12 x 12 and 16 x 16 and 17 x 17 and 23 x 23 are the whole list to Order 13. Only the last one is not Invalid. Incredibly, the Dimension of the Squared-square is never found to exceed the Reduction Index!

In the case of [13] 23 x 23 the Reduction is actually 23 (same as the Dimension) - is this just coincidence?

This is in stark contrast to Rectangles where all solutions with Reductions of 1 or 2 or 3 are always Perfect.

Also why are all but one of the squares to Order 13 Invalid?

3. In view of the Above reasoning, there must be, and is, a very definite reason for this!

4. Gradually clues begin to unfold. Consider Rectangles with Reduction one in say, Order 11.

177 x 176 185 x 151 185 x 168 185 x 183 187 x 166 191 x 162 191 x 177 194 x 159 194 x 183 195 x 191 199 x 169 199 x 178
205 x 181 209 x 127 209 x 144 209 x 159 209 x 168 and 209 x 177.

5. Firstly we observe that all have at least one odd number dimension and sometimes two.

6. Each pair of dimensions has no common factor either - they have a highest common factor of 1 and are prime to each other.

7. I assure the reader this is always found to be true! This means that Squared-squares cannot have Reduction 1. 8. What about Reduction Index of 2? In Order 12 there are 160 x 159 162 x 142 (1) and (2) 162 x 157 163 x 159 165 x 157

169 x 167 177 x 145 193 x 126 193 x 129 and 193 x 143. Most have a highest common factor of 1 but 160 x 142 an h.c.f. of 2... Referring to other examples the h.c.f. of either 1 or 2 and never higher. So no Squared-squares exist for Reduction 2 either
9. The last item means that where the h.c.f. is 2, the Reduction Index must be a multiple of 2.
10. So what happens if we select solutions which have h.c.f.'s of greater than 2? [11] 7 x 7 R28, [12] 12 x 12 R24,
[13] 3 x 3 R192, 4 x 4 R128, 12 x 12 R48, 16 x 16 R32, 17 x 17 R34, 64 x 56 R8, 23 x 23 R23, 72 x 64 (1)(2)(3) R8. are the lot to Order 13. It is now clear that <u>the highest common factor of the two reduced sides is always a factor of the Reduction Index.</u>
11. This important statement means that the <u>maximum</u> dimension for any class of Squared-square will be the Reduction Index itself! Thus in [25] 503 x 503 the Reduction Index will be a multiple of 503 - in fact is 503 x 1 = 503.
12. Expressing the above another way, the square root of the full size will indicate the maximum size a Squared-square could have.
13. With such Reduction necessary it means that the majority of Squared-squares will be Imperfect and frequently Invalid too. Therefore, Perfect solutions are very hard to find!
14. Take Order 15 where full sizes are rarely over 1940. the square root is 44+
This dictates a maximum Element size of say 25. The chances of 15 Elements under 25 happening to all be different?... Yes zero!
15. The reasoning under 14. virtually removes any likelihood of any Perfect Squared-squares up to and including Order 18.
Orders 19 and 20 look doubtful too.
16. As there are well over 200000 Perfect rectangles to Order 18, 500000 ++ to Order 19 and 1500000++ to Order 20 and probably not one square amongst them, the problem of finding Perfect squares is clear!
17. The everlasting Add-on series mentioned later emphasises that Imperfect solutions must be much more common than Perfect ones.
18. It will be seen that most Perfect Squared-squares have four everlasting Add-ons branching from them.
This is interesting since if any one of these is found by computer, the Perfect solution can be worked back from it.
Each such solution found by the computer is to be looked at eagerly for a possible, though unlikely, Perfect solution.

<u>M4.2. LIST OF MAXIMUM SQUARE DIMENSIONS IN RESEARCH STAGE</u>

In an attempt to discover the theory of Squared-squares I ran a computer program to discover the ranges of Dimensions maximum Squared-squares can have. In such cases the side will be the square root of the full dimension.
For example in Order 13 there exist complexities for 21, 23 and 24 squared. Only a square is in fact exists using 23.
This list is far from complete.
Order 11 - 13 14 Order 12 - 16 17 18 19 UPDATE THIS!?
Order 13 - 21/24 Order 14 - 27/28 30/32 34
Order 15 - 33 36/42 Order 16 - 42 45/52 54 58
Order 17 - 61/72 74 Order 18 - 72 76/80 82/88 90/92 94/95 98 100
Order 19 - 94 96 98/102 104/108 110/124 126 128/129
Order 20 - 112 120 122 128 130/162 164 166 168 170

Order 21 - 172 / 174 176/181 183/210 214/216 218 220 226
Order 22 - 188 196 208 210 218 220/221 224 226/238 240/244 246-252 255/266 268/276 278 281/283 288 290
Order 23 - 270 277 280 284 288 290/292 294/298 300 302-371 373-374 376 378 381/382 389/390
Order 24 - 320 335 340/341 344 360 364 366/368 370 372/373 376/381 383/384 387/390 392/404 406 408/412 414/458 460/466 468/472 475-477 479/480 482/483 485 487 490 504 507 510
Order 25 - 437 449 452 458 460 473/474 478/483 486 488/489 491 493/494 496/498 502/504 506/508 510/521 524/526 528 530/546 548/586 588/594 596/608 610/630 632/635 637/638 640 643 645/647 651 657/658 666/667 683
Order 26 - 549 553 566/567 581 588 592 595 597/598 601/602 609/610 615/616 619/620 622 626/630 632 635/641 643/645 647/648 650 652 654/656 659/660 662/664 666/669 672/684 687/690 692/726 728/804 806/808 810 812/816 819/822 824/825 829/830 837 841 845 847/849 851 864 876 878 906
Order 27 - 787 795 799 809 825 828 832 836/839 842 844 846 848/849 852/855 860/861 863 867/870 875 882 887/889 892 895/897 899 902/903 906 908 910/914 917 921 924/927 929/933 935/936 941 943/944 946/948 950/951 953/954 956/961 963/964 966 968 971/972 974 976/980 982 984/988 992/995 997 999/1000 1002/1006 1008 1012/1016 1018 1021 1023/1024 1027 1038 1040/1042 1044/1045 1047 1049/1050 1052/1053 1055/1059 1063 1065 1071 1074 1082/1084 1087 1092/1093 1101/1103 1131 1136/1137 1153.

M4.3. OBSERVATIONS ON THE SQUARE-ROOT SIDES

1. In lower Orders the numbers follow on, later there is overlap (e.g. 270 to 290) in Orders 22 and 23.
2. Parts of the series are tidy consecutive numbers particularly Order 15 with 33 36 37 38 39 40 41 and 42!
3. Squared-square solutions seem to be absent from the upper parts of the ranges e.g. in Order 17 there are squares with 64 65 67 68 69 70. in Order 23 most Squared-squares seem to range from 296 to 353.
It is interesting to note that there are 24 Order 12 rectangles over 90% elongation for upper Complexities 288 to 338 and none for Complexities 339 to 386. Other Orders follow this general pattern.
4. In higher Orders Squared-squares appear to be absent also from the lower ranges. e.g.. Below about 296 in Order 23.
5. The averages are about 13 17 22 30 39 50 67 86 113.. If the 2nd and 3rd below a point are added the series is reasonably compatible at - - 30 39 52 69 89 117…
6. If the idea in 5. is applied to the lowest and to the highest values there is a reasonable but not total degree of compatibility.

M4.4. MORE RESEARCH IDEAS

It is possible to look at a collection of computer solutions where -
1. Full side is 1 to 10 times a large squared number.
2. And where a Complexity is found with at least total horizontal and vertical Elements are 12 or more e.g. sides 3333, 2334. etc. giving a wide choice of poles available.
3. Suppose we found a Complexity 253009 = 503 x 503 being a prime squared and S3333 and asymmetric.

If we were to calculate the network with different poles then nine solutions would arise, all of them R1 unless.. R503 giving a Squared-square. Nine chances! I need to know what kind of percentage of squared number sides actually give squares compared to rectangles.
4. On the basis that a SS cannot exist unless there are at least 2 different complexities for the same number, if we check on them and find only 1 per number, no squares can exist. “Different” complexities may show up by (a) a different number of vertical sides Elements or (b) composition of vertical lines code 33345 etc.
5. It seems to me that a comprehensive (but longwinded check) might help to unearth the Squared-squares.
Where squares exist there must be a quantity of other rectangles having either of the 2 complexities. Where 22## type there is a possibility of something better at the Root. Very ‘high’ Complexities are unlikely to give Squared-squares.

M5. FINDING SIMPLE PERFECT SQUARED-SQUARES

Since writing this Section there has been an enormous amount of recent work done on this subject - and progress on the Computer Internet.
All Order 21 to 35 Squares (inclusive) have now been calculated.
Some 12,000 solutions are found on a single Internet Program - the first one listed under Google “Squared-Rectangles”.
So much for my sustained efforts to find nine over many years!

M5.1. FINDING PERFECT SOLUTIONS BY COMPUTER PROGRAMS

All solutions found to date by me have been lucky finds from computer programs in BASIC programming.
As far as establishing the theory of how to find them is concerned, this offers no help!

M5.2. FINDING PERFECT SQUARES BY OTHER METHODS

See later where
1. Suitable Triad Rectangles, or
2. Suitable Octad Rectangles can be converted into Squares - which occasionally turn out to be perfect.
3. Two twin Diad Squares can sometimes be amalgamated to obtain a high Order Perfect Square.

M5.3. THE MAXIMUM SQUARE SIZE RULE

There is a Simple rule concerning the maximum size a Squared- square can be for a particular Order.
It has been already seen that the side of a Squared-square cannot exceed the square root of the relevant Complexity.
If the very largest known Complexity for an Order is taken, and the maximum integer square-root then taken, is this near the maximum size a Squared-square can be?
The answer is No, since with rectangles one Element can be much more than the other.
A little thought corrects this idea. For a square the Semi-perimeter has to be double the dimension.
So, if the maximum possible Semi-perimeter for a given Order is ÷ by two and the square root then taken, the result is the maximum size a square can be in theory for that Order.

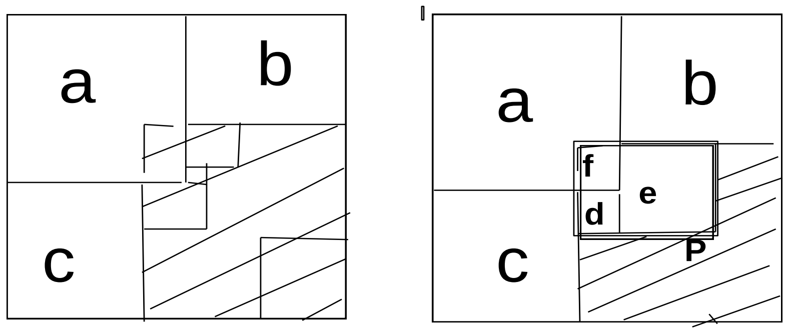
An example makes this much clearer. The maximum SP for Order 13 is 1176.
Half this is 588 and the square-root about 24.
But out of 298 solutions for Order 13 there happen to be 104 with a full side of over 588 and Complexities running up to 672.
Now imagine a square of say 589 x 589 exists. A Semi-perimeter of 1178 would have to exist, but the maximum happens to be 1176.
This is a very important rule as it can be said without a doubt that there will not be any squares of Order 13 with Complexities from 589 to 672.
Returning to the theoretical maximum value of 24 there is a solution [13] 23 x 23!

M5.3.1. SQUARED-SQUARES AND MINIMUM UNKNOWNS NEEDED

The Writer has observed that although some xy Squared-Squares are uncommon compared to the many xy Squared-Rectangles.
We have seen that
1. The Sides Index indicates a minimum low of Unknowns. E.g. If Sides start with “3xxx” then there will be at least 3 Unknowns (x, y, z). For side 2233 there will be at least 2 (x & y), and
2. The Order indicates a maximum high of Unknowns.
In view of 1 we might expect many 2233 cases to be calculated with only x and y. However there is a reason why most Solutions require 3 or more Unknowns.

As we add “Branches” of 2 extra Elements existing Squares, it is not surprising that the Unknowns does not decrease.

For any Squared-Square any “Branches” which exist can be stripped off until a basic construction with Sides of at least 3333 are found. (Some of these are inferior having One or Two Repeater Elements making it Compound and with Adjacent Elements but that does not matter). This being so, we would expect all Squares to be at least xyz -3 Unknowns, and to remain with at least 3 Unknowns when Branches are added. So why is it so few can be calculated using 2 Unknowns?



Stripping of the Branch b and c and reducing the size of a gives a Valid Square in the first diagram but an Invalid Repeater (& Adjacent) in the second.

Look at the shaded area P in above 2 where it is possible to construct a pattern with just x and y. When d e and f are added still only x and y are needed. By adding b and c and replacing f with a the Square is then Valid and still only needs x and y! This is not so in above 1 where the basic square 3333 or greater must require at least 3 Unknowns whilst adding a b & c will require the same Unknowns.

If there a Rectangle in the smaller Square then 1. How many Unknowns are required to produce P? then 2. The Rectangle will sometimes need a further Unknown, or as above no further Unknown. 3. The Solution with b and c added will require the same number of Unknowns.
Where there is no Rectangle in the smaller Square at least 3 Unknowns are needed with the same number for the larger Square.

M5.4. SEMI-PERIMETERS AND MAXIMUM SQUARE SIZES

Below is a table of maximum SP's known to me and although higher ones do exist the table is useful as if the 'maximum' is suitably increased (10% is probably enough) to allow for this, then the range of complexities which can give squares is still considerably reduced.

Order	Max Semi-P	÷ by 2	Square Root		Order	Max Semi-Perimeter	÷ by 2	Square root
9	130	65	8		23	292,088	146,044	382
10	224	112	10		24	497,640	248,820	498
11	392	196	14		25	885,312	442,656	665
12	672	336	18		26	1,466,752	733,376	856
13	1,176	588	24		27	2,453,760	1,226,880	1,107
14	2,037	1,018	31		28			
15	3,792	1,896	43		29			
16	6,800	3,400	58		30			
17	13,665	6,832	82		31			
18	19,200	9,600	97		32			
19								
20								
21								
22								

M5.5. SEMI-PERIMETERS and MINIMUM VALUES

The Above idea also applies the other way round with the minimum. Finish this!

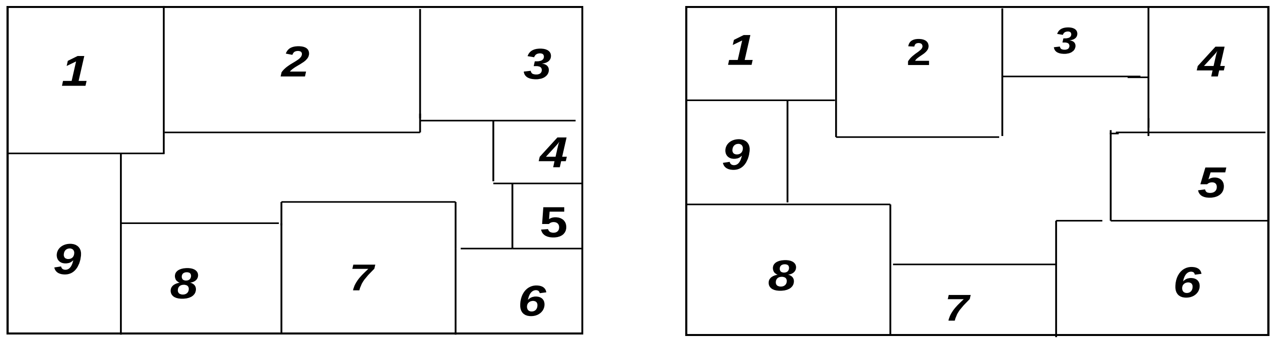
M5.6. SOME IDEAS ON TRACKING DOWN SQUARES

Take details of rectangles from computer programs where the sides chosen are a large squared number times say 1,2,3,4 or 5. We look for two (or more) Complexities with the following properties -

1. Duplicated Complexity values required but -
2. Different Complexities are essential
3. The vertical codes must differ

4. The vertical code of one must equal the horizontal code of the other and vice versa.
5. Both with same amount of external Elements, and at least 8, preferably 9 or more, i.e. sides codes of S2334+ or S3333+
6. The Sides codes of the ultimate choices must of course be identical.
7. The quantity of outer Elements must be the same.

RECTANGLE ONE — COMPLEXITY 400000A | RECTANGLE TWO — COMPLEXITY 400000B

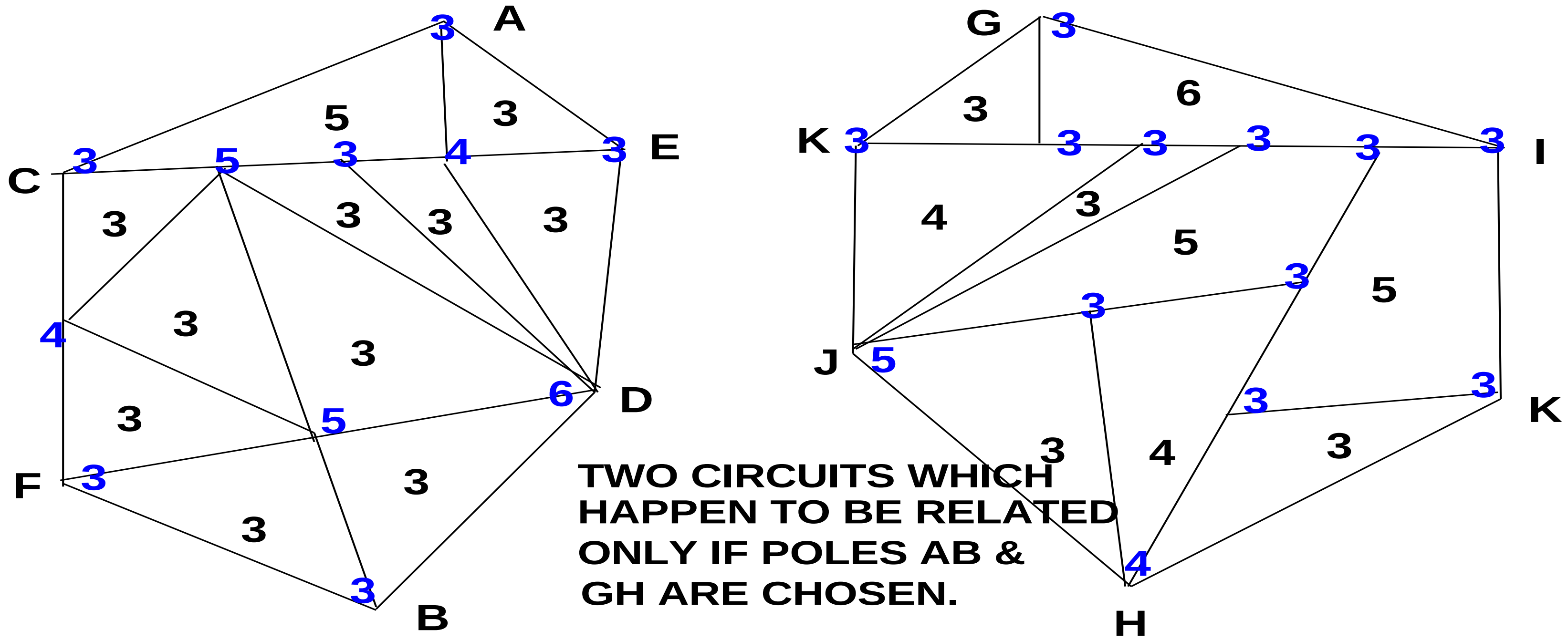


V43 H403 | V403 H61

SAME COMPLEXITY NUMBER (400000) BUT DIFFERENT (A,B).
& SAME NUMBER OF EXTERNAL ELEMENTS (9).

M5.7. TESTING TWO SUITABLE COMPLEXITIES FOR A POSSIBLE SQUARED-SQUARE

Refer to B9.5. Below are the two Networks chosen by me knowing a Squared-Square is obtainable from them.
It will now be shown why they are related, once the correct choice of poles is established.
Let suppose that by computer two rectangles have been found with the same suitable Complexity number.
If a square is possible then there must be at least one pair of suitable Poles.
Is it possible to track these down? It appears so.



As in B9.5 compare the blue terminal numbers of one with the black circuit numbers of the other. Circuits in one - 33333333335 terminals in two - 33333333335 plus 34. Circuits in two - 333344556 terminals in one - 333344556 plus 33.
OK so far, but look at the extras of 34 and 33 which have to relate to the poles.
We must find poles of 3 and 3 in Above 1 and 3 and 4 in Above 2.
We also need to find a rectangle of sides S3334 (note that the 3 of "34" is donated by the second not third "3" in 3334, the "33" part by the 1st and 3rd numbers in S3334).
In Above 2 the "4" in "34" can only be pole H. However poles IH GH and KH all give the required "34" but since the sides have to be S3334 there must be three wires between H and the other Pole so only GH is possible.
OK so far. (M9.7}
Now, in above 1 we are looking for two "3" poles three and four apart so CE and BE are out, however AF and AB and BC and EF may be in - but which one? The next clue is to look at the circuits surrounding G and H which are 3-6 and 3-4-3 respectively.
These need to be the wires between the Poles in above 1. Between E and F we have 3-6 but 3-3-4. Only AB has the required 3-6 and 3-4-3.
So the resultant rectangle must have complexities of the same number, which means it will be a Squared-square.
In fact it is [21] 112 x 112.
There are many instances where a square cannot be found by the above methods:
1. If the two Complexities are actually the same one, it will fail.
2. If the wires adjoining the Poles of one Complexity vary from the number of external wires in the other, it will fail.
No solution can have two different side indexes!
3. Also if the Complexity is not a large squared number times 1,2,3,4 etc... or times a relatively small number.
The rules for choosing a suitable Complexity have been given earlier.

M6. SIMPLE PERFECT SQUARED-SQUARES SOLUTIONS

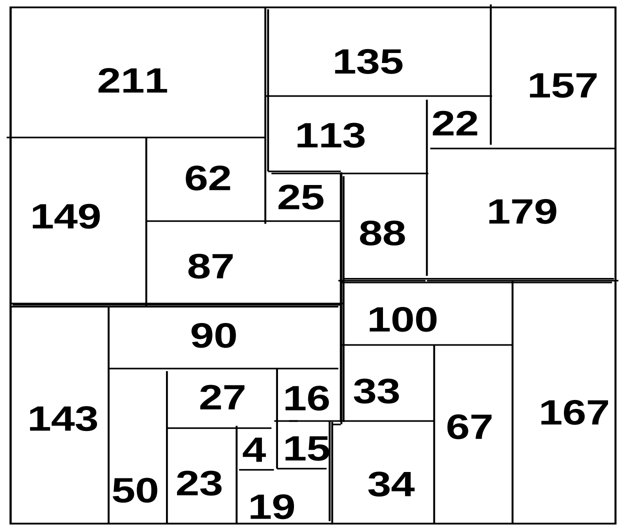
M6.1. SIMPLE PERFECT SQUARED-SQUARES

1. A Simple Perfect square of [25] 503 x 503, found by J. C. Wilson is shown later.
2. After calculating many thousands of solutions on computer, on 13.1.96 I ultimately found a Perfect square of Order 22 with dimensions 172 x 172 with a Reduction Index of 344 being 172 times 2.
3. A further Perfect and Simple Squared-square was discovered by me in July 1996 [23] 332 x 332 RI 332.
Imperfect solutions of Order 25 are easily obtained from this of 541 x 541 544 x 544 552 x 552 573 x 573, plus many Branches.
4. In February 1998 another solution was found for Order 23 of 215 x 215 R430 (= 215 x 2).
5. I discovered in 1997 [21] 112 x 112 the ultimate Perfect solution, but later discovered from Bromley Library that it was first found by Duijestijn in 1978.
6. Another Square of [22] 154 x 154 was discovered by me on 14.1.2002 by Computer.

It could be used as a good dissection puzzle. As usual, four sets of Branches can be found from this.
7. A solution of [22] 139 x 139 was found from the Internet in 2000. This means there are three or more Perfect Squares for Order 22. 8. A solution of [23] 257 x 257 was found by me on 22.1.2002.
9. A solution of [28] 953 x 953 was found by me on 22.1.2002.
All 9 spectacular solutions are listed here.
As will be seen later, there are several hundred Solutions in just the range Order 21 to 27 inclusive.

M6.2. SIMPLE AND PERFECT SQUARED-SQUARES DISPLAYED

ORDER 25 SIMPLE PERFECT SQUARE - 503 x 503.



DISCOVERED BY J.C.WILSON IN 196

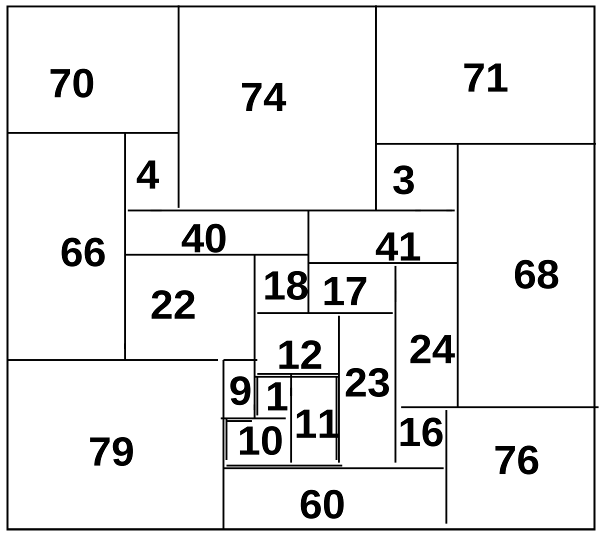
REDUCED 503 TIMES

THE SOLUTION CAN BE DIVIDED INTO
FOUR SECTORS AS INDICATED.

M6.3. OBSERVATIONS FROM [25] 503 x 503

1. The square can be divided into four sectors as above with 4, 5, 5 and 9 Elements.
2. Lots of trios exist - 62 + 149 = 211 25 + 69 = 87 25 + 88 = 113 22 + 113 = 135 22 + 135 = 157 22 + 157 = 179 67 + 100 = 167 33 + 67 = 100 33 + 34 = 67 4 + 15 = 19 4 + 19 = 23 4 + 23 = 27 23 + 27 = 50 50 + 90 = 140 only 16 is not involved!

ORDER 23 SIMPLE PERFECT SQUARE - 215 x 215. REDUCTION 645.



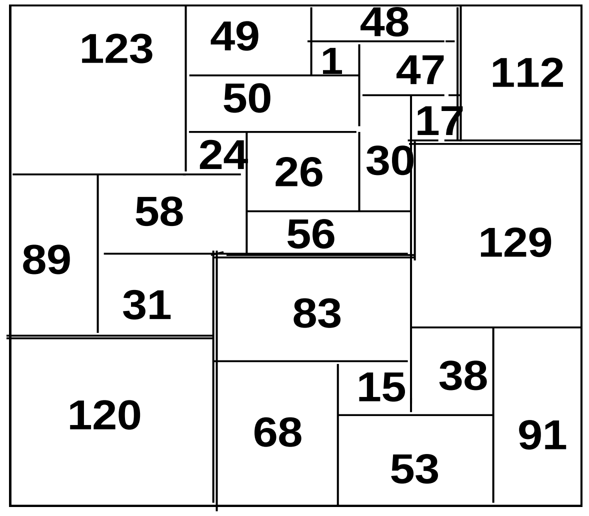
DISCOVERED BY BERNARD MOSS
ON COMPUTER, FEBRUARY 1998

(see also [23] 208 x 208 later which
is related)

M6.4. ANALYSIS OF [23] 215 x 215

Apart from 1 10 11 the Elements have a symmetric placement 70 and 71 4 and 3 40 and 41 66 and 68 17 and 18 22 and 24 12 and 23 9 and 16 76 and 79 with 74 and 60 centralised. possibly 12 and 23 should be excluded. Differences of 1, 1, 1, 2, 2, 3.

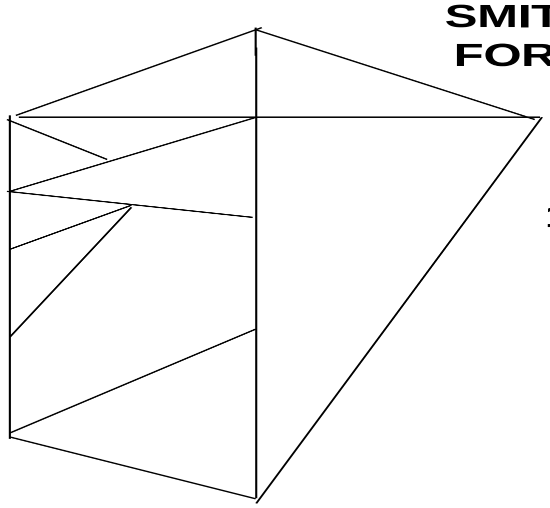
ORDER 23 SIMPLE PERFECT SQUARE - 332 x 332



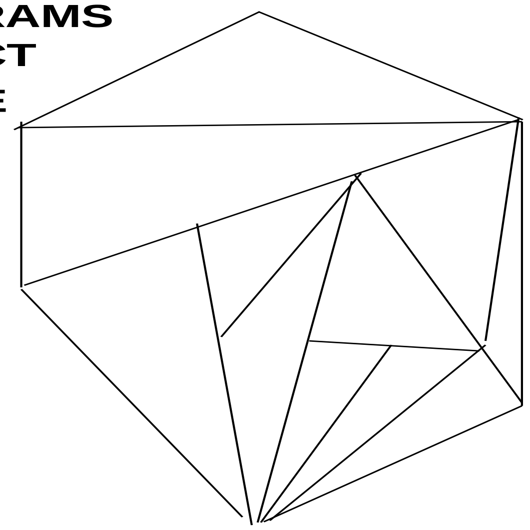
DISCOVERED BY COMPUTER IN
JULY 1996 BY BERNARD L. MOSS
REDUCTION 332.
LARGEST ELEMENT ON SIDE.

M6.5. OBSERVATIONS FROM [23] 332 x 332

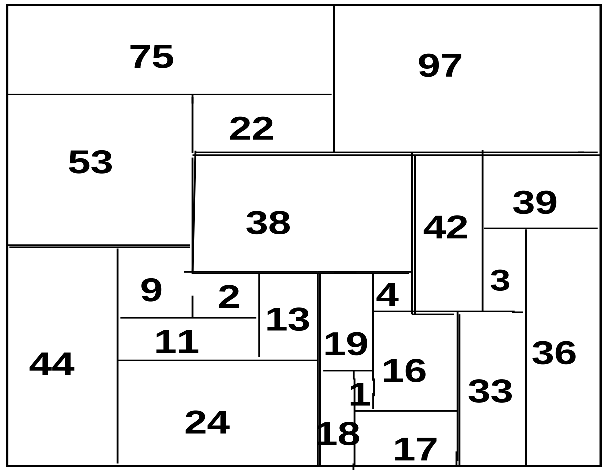
1. The first 14 numbers are only represented by the Element 1! Most unusual!
2. 1 15 17 24 26 30 31 38 47 48 49 50 53 56 58 68 83 89 91 112 120 123 129
3. Would make a good dissection puzzle for children
4. The simplest division is in four parts as above.



SMITH DIAGRAMS
FOR PERFECT
SQUARE
[22]
172 x 172



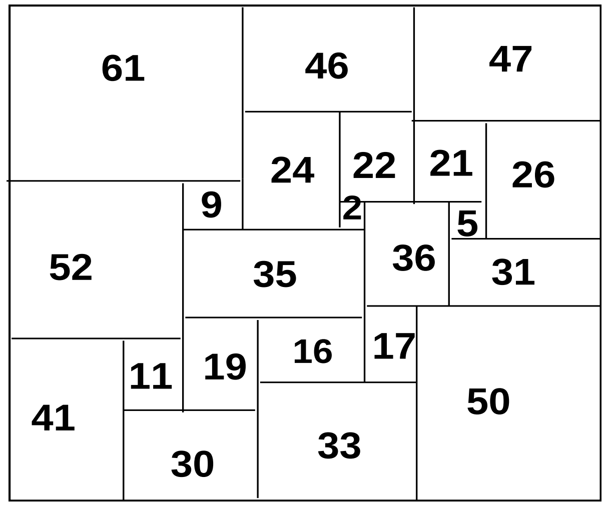
ORDER 22 SIMPLE PERFECT SQUARE. 172x172.



DISCOVERED BY BERNARD MOSS
ON 13.1.1996
REDUCTION 344
FULL SIZE 59168 x 59168.

M6.6. OBSERVATIONS ON [22] 172 x 172

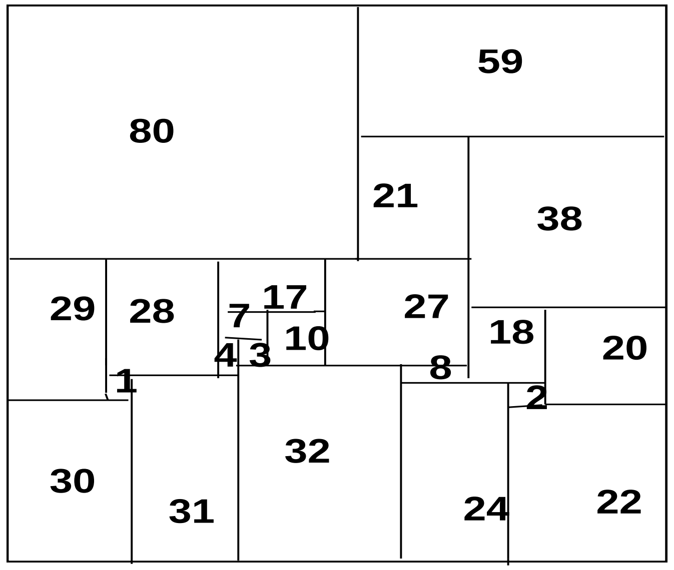
1. Easiest dissection is into five as shown above, or three or four blocks.
2. Note 1 2 3 4 then 9 11 13 16 17 18 19 22 24 33 36 38 39 42 44 53 75 97
3. Every Element forms part of a trio e.g. 36 = 3 + 33 42 = 4 + 38 etc.



[22] 154 x 154 PERFECT SQUARE
DISCOVERED BY BERNARD MOSS
ON 14.1.2002
REDUCTION 308

M6.7. OBSERVATIONS ON [22] 154 x 154

1. Arithmetic Progressions 21 26 31 36 16 33 50 19 30 41 52
2. Does not dissect well into blocks.
3. Would make a good Dissection Puzzle - possibly with Elements 2 and 5 joined to Element 36 for convenience - making twenty pieces. The answer can be displayed in eight different ways due to rotations and reflections.



[22] 139 x 139 PERFECT SQUA

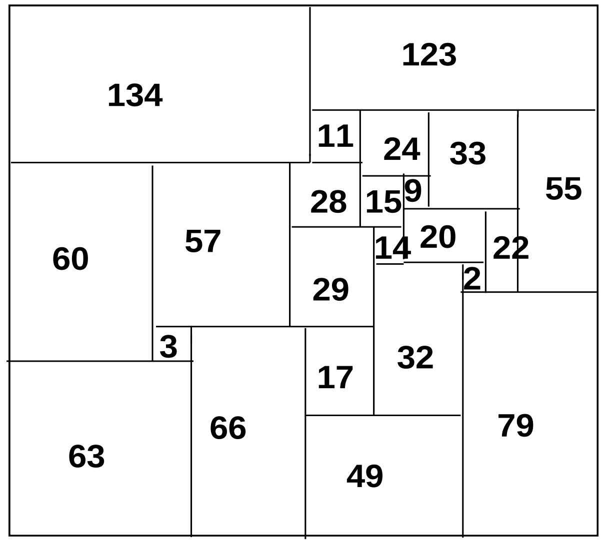
From the Internet. Sides 2 3 5 4

M6.8. ANALYSING SOLUTION [22] 139 x 129

1. Contains 1 2 3 4 and is not a useful Dissection Puzzle.

M6.9. PERFECT SQUARE [23] 257 x 257

ORDER 23 PERFECT SQUARE - 257 x 257



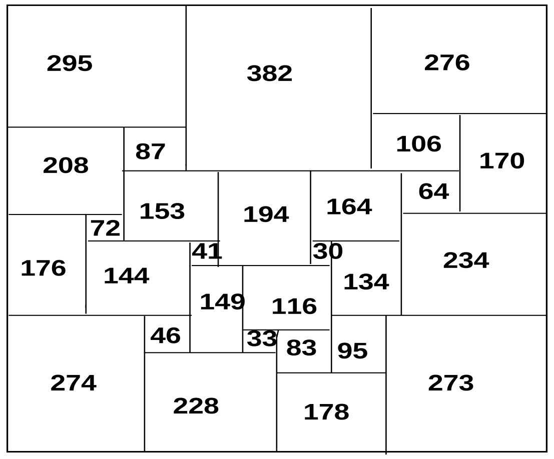
SIDES 2343

Elements

2 3 9 11 14 15 17 20 22 24 28 29

32 33 49 55 57 60 63 66 79 123 134

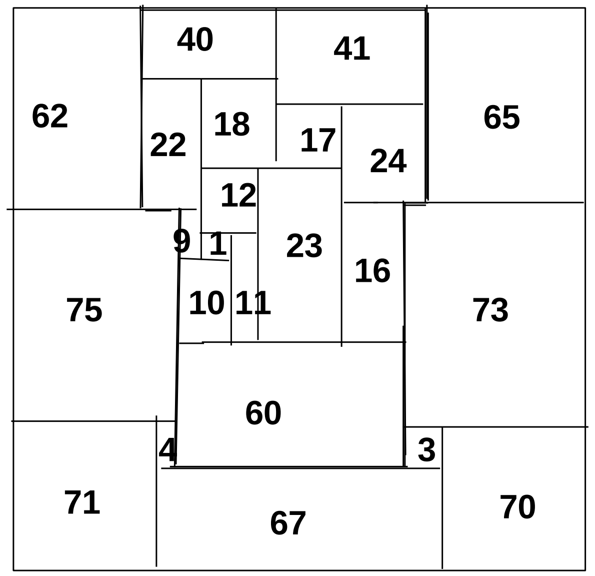
M6.10. PERFECT SQUARE [28] 953 x 953



PERFECT SQUARE
[28] 953 x 953 sides 3444
Discovered by B.L.Moss 22.1.2002
by Computer.

M6.11. PERFECT SQUARE [23] 208 x 208

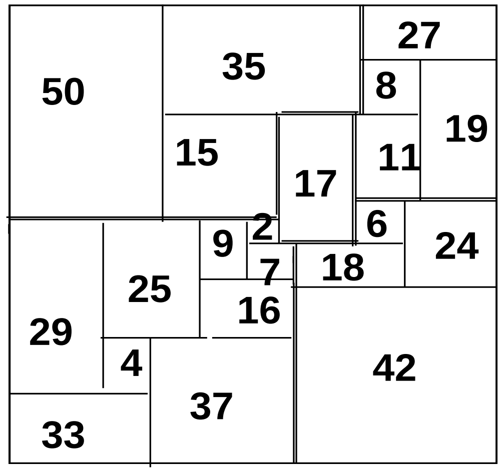
I realised that the Up-down Link could be applied to [23] 215 x 215 Perfect Square! [23] 208 x 208 was easily obtained and also proved to be Perfect!



PERFECT SQUARE
[23] 208 x 208 sides 3334
Discovered by B.L.Moss 22.4.2002
by applying Updown Link.
Refer to [23] 215 x 215 also Perfect
with the Elements within the darker
lines.

This would make an ideal Dissection Puzzle

THE UNIQUE ORDER 21 SIMPLE PERFECT SQUARE - 112 x 112.



CONTAINS THE ELEMENTS
2 4 6 7 8 9 11 15 16 17 18 19 24 25 27 29
33 35 37 42 50 42% OF ALL NUMBERS TO 50
NOT 1 3 5 10 12 13 14 20 21 22 23 26 28 30 31 32 34
36 38 39 40 41 43 44 45 46 47 48 49
REDUCED 336 TIMES. (i.e. 3 x 112)
IT IS NOT QUITE THE SMALLEST
SIZE PERFECT SQUARED-SQUARE, AT 21 IT IS
THE SMALLEST POSSIBLE ORDER.
FOUND BY B.L.MOSS IN 1997.

M6.12. ANALYSING SOLUTION [21] 112 x 112

1. If 17 is isolated, four segments can be drawn around this Element.
2. There is one segment 4 25 29 33 37 16 and five lots of three, 2 + 7 = 9, 18 + 24 = 42, 6 + 11 = 17, 8 + 19 = 27 and 15 + 25 = 50. Elements total 237.
3. The series can almost be divided into 7 arithmetical progression sets 2 4 6 - 7 8 9 - 11 15 19 - 16 29 42 - 19 27 35 - 24 37 50 - 17 25 33 with 18 missing and 19 in twice! This may be insignificant.

1. The Largest Element is an incredibly small 50.
2.

M7. QUANTITIES OF SQUARED-SQUARES FOR EACH ORDER

M7.1. QUANTITIES OF SIMPLE SQUARED-SQUARES FOR EACH ORDER

In recent years an enormous expansion of SS's has appeared on the Internet

1. For Imperfect Squared-Squares:
   Apart from Order 1 (Singlend) there are no Valid Squares from Orders 2 to 12 inclusive or for Order 14.
2. There is 1 Square for Order 13 (23 x 23)
3. For Order 15 three, Order 16 five, Order 17 fifteen, Order 18 nineteen and Order 19 fifty-eight.
4. Not one of these 101 Solutions mentioned are Perfect!
   Nor are there any Perfect Squares for Order 20.
5. There is one Perfect Square for Order 21 112 x 112.
   This is slightly larger than the smallest Dimension possible of 110 which occurs in Orders 22 and 23.
6. For Simple Perfect Squares. Confirmed as all Possible Solutions-

Order 20 and below		0 up to Order 20
Order 21	1 Solution	1 up to Order 21
Order 22	8	9 up to Order 22
Order 23	12	21 up to Order 23
Order 24	26	47 up to Order 24
Order 25	160	207 up to Order 25
Order 26	441	648 up to Order 26
Order 27	1152	1800 up to Order 27
Order 28	3001	4801 up to Order 28
Order 29	7901	12702 up to Order 29
Order 30	20566	33268 up to Order 30
Order 31	54541	87809 up to Order 31

Order 32 144161 231970 up to Order 32

Many more Solutions Exist for the following Orders -

Order 33 37819 Order 34 21329 Order 35 25978 Order 36 48308 Order 37 100202 Order 38 205862 Order 39 414504 Order 40 803483 Order 41 1506848 Order 42 2663241

Order 43 4242554 Order 44 5100736!

8. Much smaller quantities are known from Orders 38 and higher (up to Order 120+).

I have personally found three!

M8. SQUARES WITH SIDES 2333 DO EXIST

M8.1. SOLUTIONS WITH SIDES S2333, S2334, S2335 etc.

In drawing out all known Perfect Simple Squared-squares up to and including Order 27 - many hundreds - I observed that none contained Sides 2333, nor any for Sides 2334, 2335, 2336, 2337 ... etc.

Upon juggling with imaginary Elements, I found it theoretically possible to place random Elements in a Square of 2 3 3 3 sides; however the internal space had to be thin and elongated.

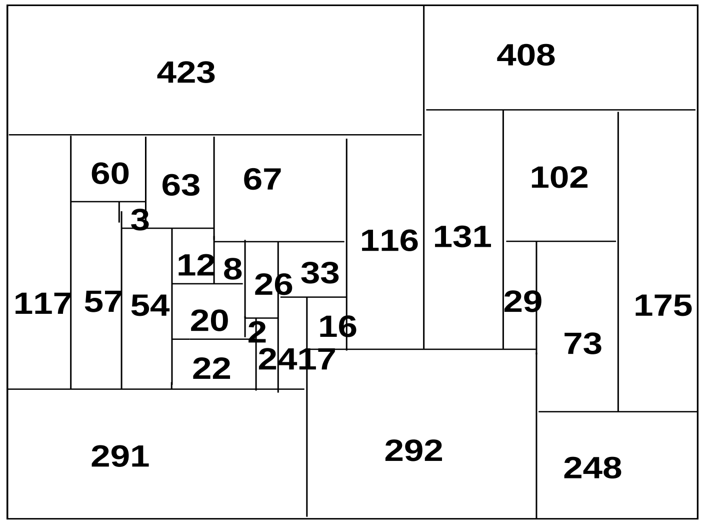
This meant if 2333 Solutions exist they would be unusual and hard to find. Also they would need to have quite high Orders.

With reference to my Computer database of over 640,000 Solutions, it was observed there were hardly any Solutions of Elongation between 96 to 100% rather suggesting that 100% Elongation was impossible for Sides 2 3 3 3+.

Furthermore there are loads of Solutions with Sides 2 3 4 3 (which have 8 Outer Elements) and loads of Sides 3 3 3 3 (Also with 8 outer Elements).

Suddenly a Perfect Simple Solution of Order 28 came to light - with Sides 2 3 3 3 - and only SEVEN OUTER ELEMENTS! However having recorded all known Order 29 and 30 Perfect Squares no more have been seen (but there may be some Imperfect solutions). No doubt Sides 2334 2335 2336 2337 . . . . are definitely possible also, 2333 being the most restrictive shape of this range i.e. observe how elongated the pattern between Elements 117 and 116 has to be to fit in with the other outer Elements.

See Below.



THE ELUSIVE
SQUARED SQUARE
WITH SIDES
2 3 3 3.

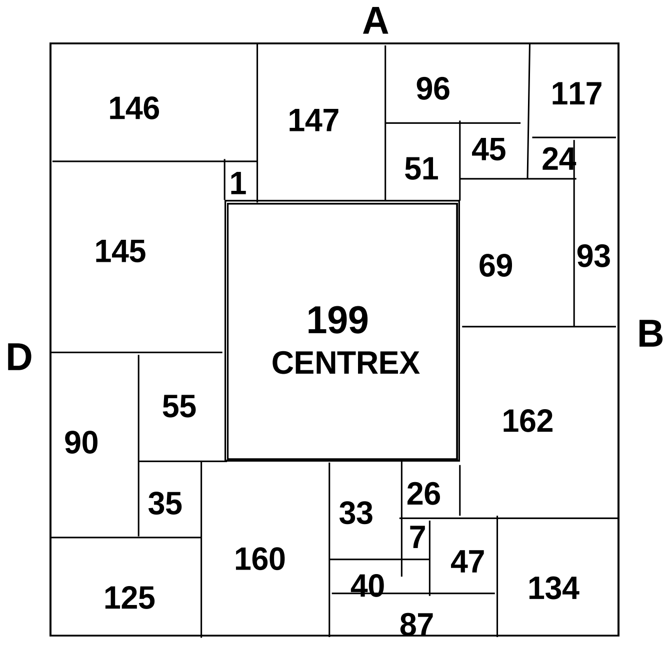
It has only SEVEN outer
Elements which is the
least possible for any
Squared-Square

M9.CENTREX SQUARED-SQUARES

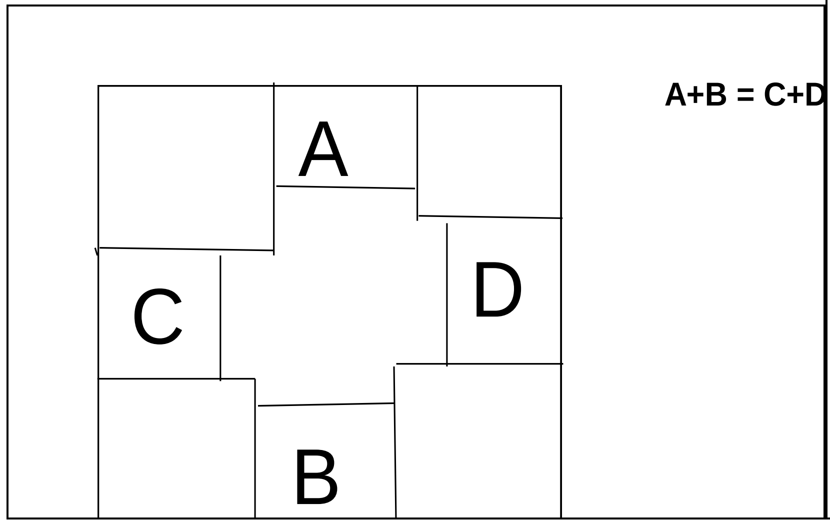
M9.1. CENTREX SQUARED-SQUARES

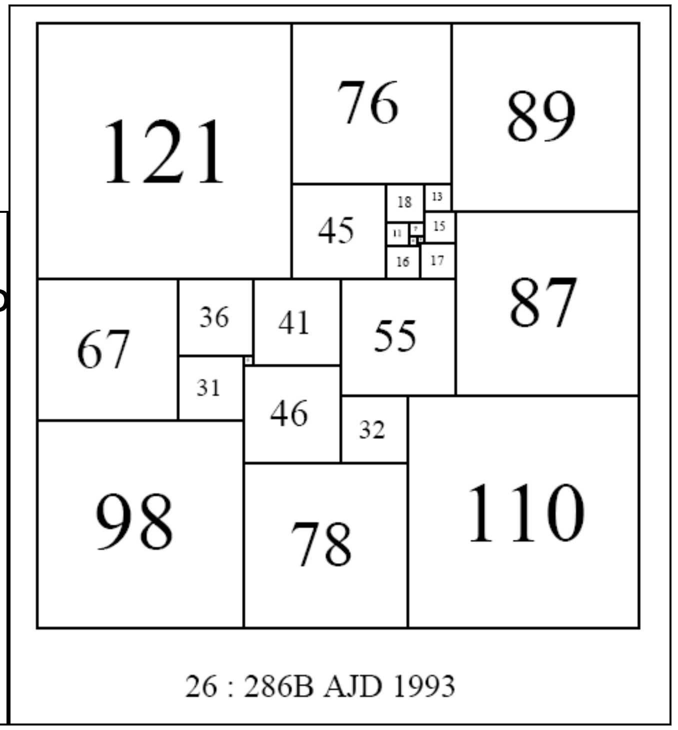
Simple Perfect Squared-squares may be Cornex, Middex of Centrex. As usual, the Centrex Solutions are rather Uncommon, but a really good example is shown below. It is [25] 506 x 506 and the central Element 199 is easily the largest.

In this example the Square has four segments apart from the central Element at A B C and D, but this fact may not be significant.



M10.1. PROPERTIES OF SIDE 3333 SQUARED-SQUARES





In comparing the Unit Area of A & B with C & D the difference is often a lot smaller than we might think.
e.g. 76 squared + 78 squared = 11860 and 67 squared + 87 squared = 12058 a difference of 198.
The differences in the Linear sizes are 2 (78 - 76) and 20 (87 - 67).
Now 198 = 18 x 11 is found to be (20 - 2) x (20 + 2)/2. This relationship can be proved by algebra.
So, <u>ABS(Difference in Unit Areas of A&B and C&D) = ABS (A - B - C + D) x (ABS(A - B) - ABS (C + D))/2</u>
Although this is true for all Squared-Squares (not just S3333 ones) it does not make much sense in practice!

M11. SQUARED-SQUARES & PRESENCE OF SLIDES

This Section looks at the prevalence of Slides in Squared-Squares. What causes Slides to occur? Are there hard & fast rules determining this? Are there any tendencies?
It seems reasonable to expect Slides where the Elements are rather small, and less likely where they are huge.
This indicates that very reduced Solutions are more likely to have a Slide, i.e. where the Reduction Indexes are huge.
Reduction Indexes are huge in the case of Squares, which is why the subject is discussed here.
Those Solutions which small dimensions tend to be those of the lowest Orders. But low Orders are not ideal for Slides as several Elements have to appear in a line in Slides.
In Squared-Squares of Orders 13 and 15 no Slides occur in the 4 Solutions.
In Squared-Rectangles up to Order 12 there appears only one Solution with a Slide!

Now a surprise! In the five Order 16 Squares all five contain Slides!

For Order 18, seventeen have Slides & two have Crossovers. None are without them!

But Order 17 with seventeen Solutions has only three with Slides, none with crossovers, and fourteen without!
Very small Squares can contain several Slides, and even Vertical as well as Horizontal.

For some unexplained reason, Horizontal Slides are more frequent than Vertical Slides

M11.1. SLIDES & EVEN ORDERS

There is a marked tendency in favour of Even Order Squares having Slides, and Odd Order Squares not having Slides!

	PLAIN	SLIDES	CROSSES	ORDER	PLAIN	SLIDES	CROSSES
13	1	0	0	14	NO	SOLUTIONS	EXIST
15	3	0	0	16	0	5	0
17	14	3	0	18	0	17	2
19	46	9	0	20	7	55	8
21	224	47	3	22	C96	C348	53
23			48	24			48

The results though startling, are not consistent. Sides Indexes seem to have little or no bearing on Slides.
Sometimes most Solutions with the same Reduced Dimension have Slides, with other sizes not having Slides. But there is considerable inconsistency!

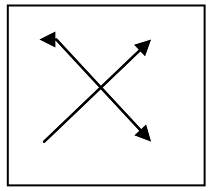
Could it be that the Full Dimensions have a greater bearing on Slides/ Non-Slides? (Up-rating the Elements does not alter the existence or otherwise of Slides of course).
Could the Reduction Index have some bearing? Not convincing so far.

N. SQUARED-SQUARE LINKS

N1. SYMMETRY 2 SQUARES

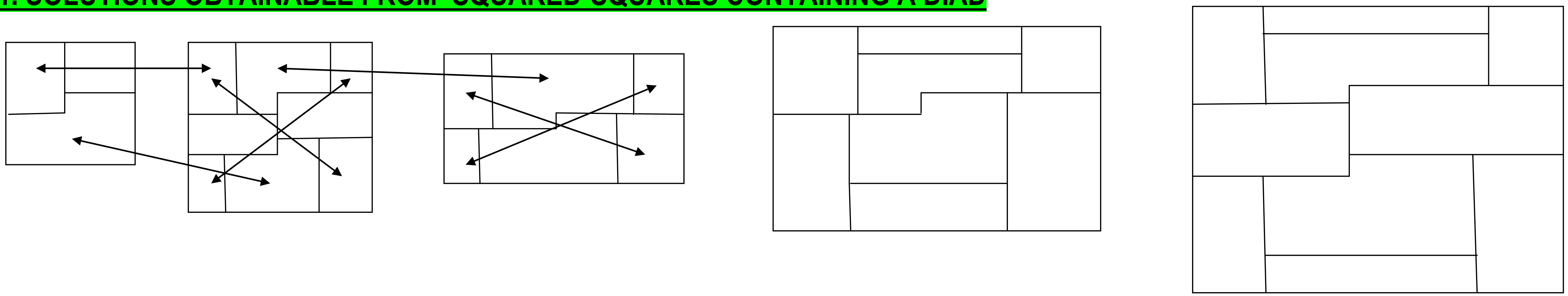
N1. SYMMETRY 2 SQUARES AS A GROUP

Symmetry 2 solutions may be squares as well as rectangles. (Symmetry 2 means that the lhs is repeated up-side-down on the right hand side, or that the left to right top side is repeated at the bottom, right to left.)



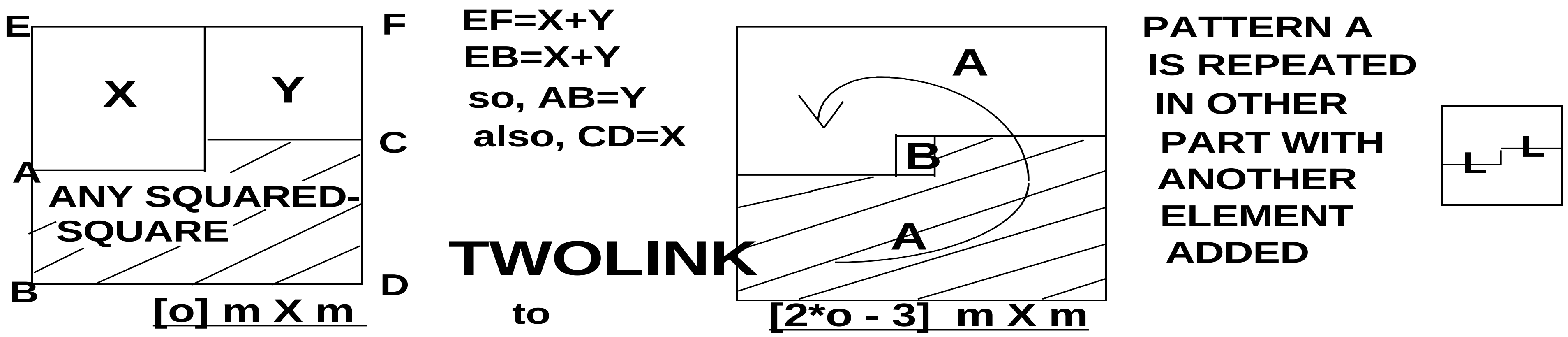
It is also easy to think as I did at first that Symmetry 2 squares are rare as there are very few solutions in the lowest Orders. But there are multitudes of them with higher Orders.

N1.1. SOLUTIONS OBTAINABLE FROM SQUARED-SQUARES CONTAINING A DIAD



We start with a Square with a Diad (NOT a Triad) as in the 1st diagram where the bottom section is the rest of the Elements. The arrows simply show which Elements or blocks of Elements are the same in the 2nd and 3rd diagrams.
2nd diagram forms a Symmetry 2 Square. In the 3rd the middle Elements are removed to form a 2 by 1 Symmetry 2 Rectangle, the Elongation being exactly 50%! In 4th Diagram two Elements have been added and in 5th two more (of equal size) giving yet more Symmetry 2 Solutions.
For the original Solution of Order O, the Orders of the other diagrams are 2*O + 2, 2*O, 2*O + 2 and 2*O + 4 respectively. 00
(Examples of the above types are not shown, not being particularly interesting).

Yet another format is now considered termed “TWO-LINK”.
Consider any Squared-square with a Diad side of 2 Elements shown as x & y below. (There must not be an Element x – y). The Elements within ABCD can be repeated within ACFE. But ACFE happens to be larger (the reader can verify this) and the pattern only fits when Element B is



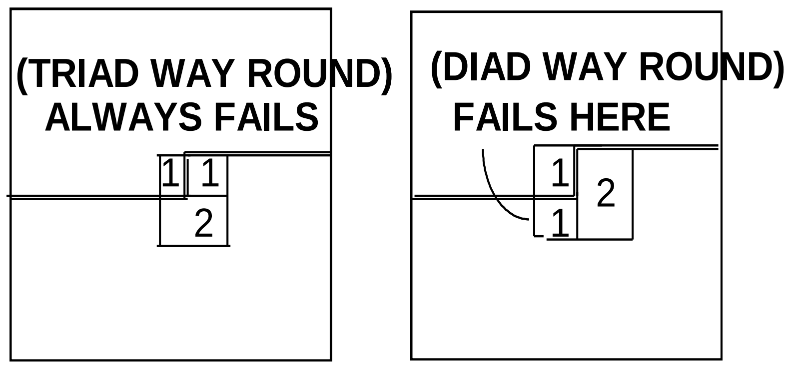
added.

N1.2. ORDER SIZES FOR SYMMETRY 2 SQUARES

Beyond doubt, these squares exist for all odd Orders of 27 plus by the construction above.
But if squares of above 3 type exist, these would have even Orders!

N1.3. SOME SOLUTIONS FAIL

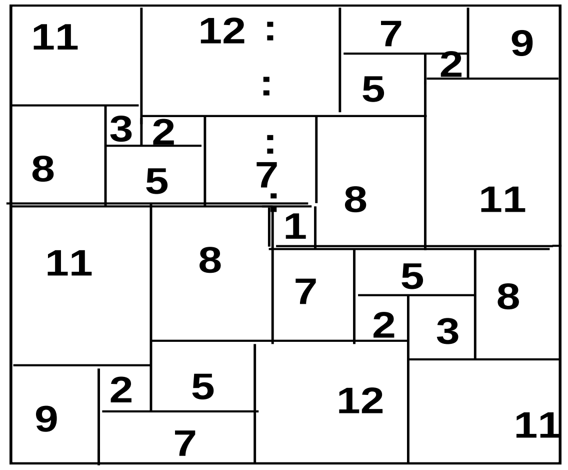
The 2234 solution [13] 23 x 23 fails as the extra Element of 1 lies adjacent with the existing 1.
Only the parts which fail are shown here.



N1.4. A SYMMETRY 2 SQUARE EXAMPLE

As no good Order 14 Squared-squares exist, [15] 39 x 39 has been utilised below.
As Elements 19 and 20 drop out and 1 is added, the resulting square is Order [2 x 15 - 3] or in algebra Order [2n - 3].

Order 27 appears to be the lowest possible for any symmetry 2 square. By its very construction, the largest Element is always small compared to the dimension - 12 against 39 here - and may be in the corner, side or internally.



[27] 39 x 39
FROM
[15] 39 x 39
(which has 19 & 20
left & right of the
dotted line)

TWOLINK

N1.5. REDUCTION INDEX IN TWO-LINK 2 SQUARES

How incredibly small the above Square is! 39 is the 2nd largest dimension for Order 15 (after 41).
Yet the dimension remains the same in the resultant square of Order 27!
For Squares of this type for Order O, the resultant Order is 2 x O – 3 being double the Order minus three.
Now [15] 39 x 39 has a Reduction Index of 39 (& Square Index of 1).
Could the RI for [27] 39 x 39 be a multiple of 39? It might be. But is it 39 x 39 = 1521? No!
The numerical increase in the Order is significant. It is up 27 – 15 = 12. The higher the Order the higher of course is this increase.
The Fibonacci series is clearly a guide to the sort of increase we might expect in the Square Index. 1 2 3 5 8 13 21 34 55 89 144 233 377
Counting 12 numbers after 1 we reach 377 (i.e. 13th in the list). Could the RI then be 39 x 377 or 14703?
Though this needs much more investigation 14703 should represent the sort of R.I. we could expect.

N1.6. FINDING PERFECT SQUARED-SQUARES USING TWO-LINK

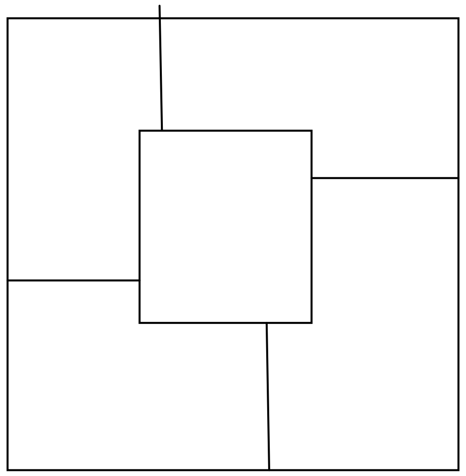
We have seen how Symmetric Squares are produced, but repeating the same Squared-Square and adding an extra central Element (1 in N1.5 above).
But if two different SS's are found with the same two Elements across the top and the Two-Link applied, a solution is found which is no longer Symmetric. However the result is likely to be Imperfect, particularly if the Element sizes are small.
The Internet can be searched for pairs of Diad (NOT Triad) Perfect Squared-Squares with identical two top Elements.

1. We first need to examine the Central Element, and be sure that it is NOT repeated in EITHER of the found Solutions. (e.g. The Central "1" in N1.5. above is not found in either the top or bottom sections).
2. We then examine the Two Solutions to see if any duplication of Elements occur, and very often there is!
3. Where the Two Solutions are of the same Order, there is quite often some relation between them with a bank of Elements repeated.

There is a better chance where Orders vary.

4. However the Two Solutions are not bound to be the same size, provided that the largest Element divided by the Side is the SAME PROPORTION. Say we found SS Solutions of 580 largest Element 296 and 290 largest Element 148. By multiplying the Solutions by 1 and 2 respectively (and checking the Central Element is NOT repeated), we then test for Duplicated Elements.
5. But suppose the Solutions have to be multiplied by say, 34 and 37 respectively to match the size. (These two numbers are always prime to each other, after any cancelling down has been done). The chance of a Perfect Squared-Square is hugely increased though rarely absolutely guaranteed.
6. The Writer has produced over 40 Simple Perfect Squared-Squares by this method – and probably most are new to science!

N1.7. FINDING SQUARED-SQUARES WITH KNOWN DIAD SOLUTIONS WITH THE SAME PROPORTIONS



1. With just ONE known Diad Squared-Square it is possible to display them four times as shown above. It is necessary to reverse the Elements in two of these sectors in order to fit. Note the large Central Element (the size of which is never duplicated).
   The Result is of course 4-Fold Symmetry and highly Imperfect.
2. With TWO known different Diad Squared-Squares a 2-Fold Symmetry Square can be produced, or a Non-Symmetric Square according to how the Sectors are placed. Of course, either is highly imperfect.
3. With FOUR known different Diad Squared-Squares a Non-Symmetric Square is produced. The chances of a Perfect Square resulting is tiny however, but increased if the four solutions have to be highly up-rated, making a large Dimension.

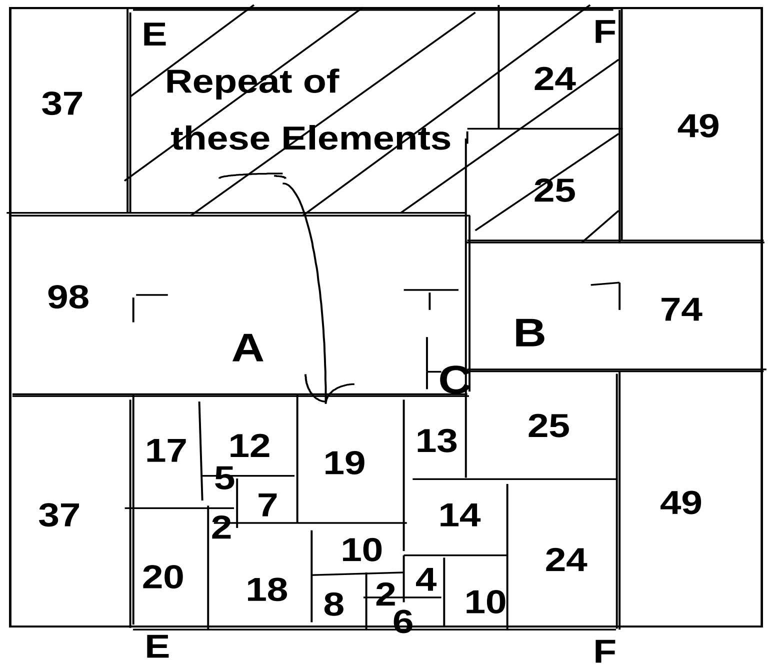
The Writer has not tried to find a Perfect Square of this type. The theory is fine however, and any Perfect Square found would be at least Order 95 and probably over 100 – very small for such an Order and really spectacular!

4. Where the result is Perfect, THREE different solutions can be formed! If the Sectors are ABC&D, then A could adjoin B&C, B&D or C&D.

N2. SYMMETRY 3 SQUARES

N2.1.1. RAILINK

For squares which contain a TRIAD (but not a DIAD) a Link can be formed to give a Symmetry 3 Square of exactly twice the Order! Firstly the three Triad Elements are removed, then ends added (37 and 49 Below),



[42] 172 x 172 SYMMETRY 3 SQUARE
Constructed from {21} 86 x 86 which is indicated by ABEF (A=49 B=37 C=12) -
Then by adding 37 & 49 at the ends, 98 & 74 as shown & repeating the rest.
<u>Formula [o] m X m to [2*o] 2m X 2m</u>

RAILINK
& OPPLINK

Then 98 and 74 are added in the manner shown and all other Elements then repeated up-side-down.
If two different Squares with similar Triads are employed then an Imperfect Solution is obtained which is not Symmetry 3. Perfect Squares are unobtainable from this Link.

N2.2. OPPLINK

A further Link is now possible by adding Elements of 86 at EF and EF Above. You may think it cannot stay a square, but as six Elements all have to increase by 86, it does!

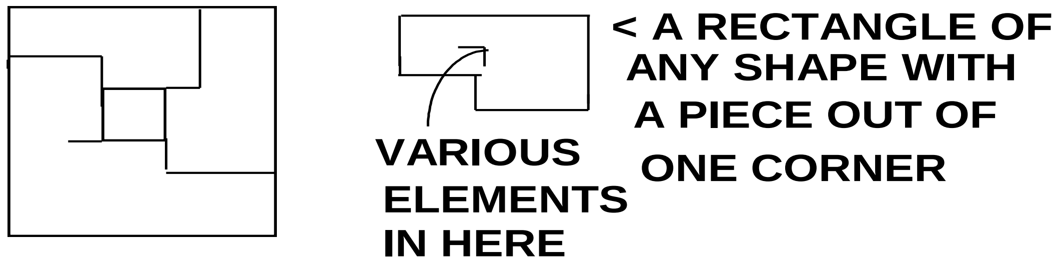
The Order goes up 2 to 44 but the Dimensions again double - now [44] 344 x 344.
Opplink can only be applied to this type of Symmetry 3 Square (with two Elements only straddling the middle).
<u>The OPPLINK Formula is from [o] m X m to [2 + o * 2] 4m X 4m</u>

N3. SYMMETRY 4 SQUARES

N3.1. SYMMETRY 4 SQUARES AS A GROUP

By definition, symmetry 4 means that a given pattern of Elements is repeated four times in such a manner that a four-sided figure is produced. It is easily seen below that such figures cannot be Rectangles - only squares.

A gap arises in the centre and so an extra central Element is necessary to complete it.
So with x Elements are in the pattern the Order will be [4x + 1] and always be Odd.



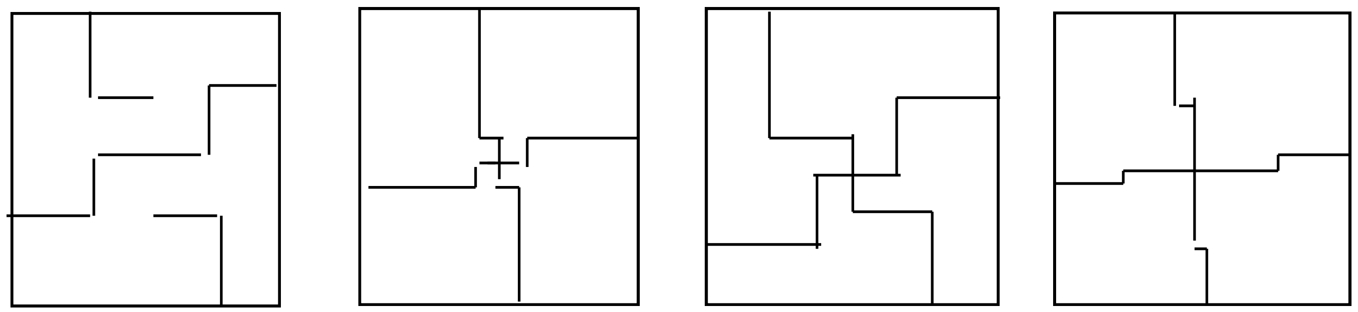
N3.2. ORDERS POSSIBLE FOR SYMMETRY 1 2 and 4 SQUARES

1. In symmetry 1 squares Orders can be any number, odd or even.
2. In symmetry 2 squares Orders can be any odd number, but not even?and,
3. In symmetry 4 squares otherwise called “4-fold squares” Orders can be each alternate odd number only.

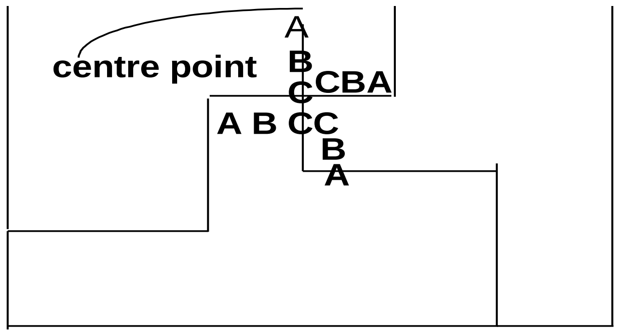
Therefore it is not surprising why Squared-squares for odd Orders appear to be more numerous.

N3.3. SYMMETRY 4 SQUARES ALWAYS REQUIRE A CENTRE ELEMENT

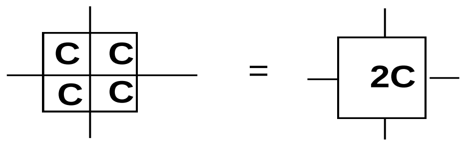
Because the Order for these squares is in the form [4x + 1], I was surprised to discover that a square can be divided into four segments without a centre Element! If the centre point inside a square is taken and a cross drawn from it, by adding two further lines to each, the square can be divided into four equal segments. Here are some of many possible shapes:-



These would suggest that 4-fold squares of even Orders divisible by four exist, but this is an illusion! As seen Below, they always default to the standard type with a central Element, and the Order remains [4x + 1].

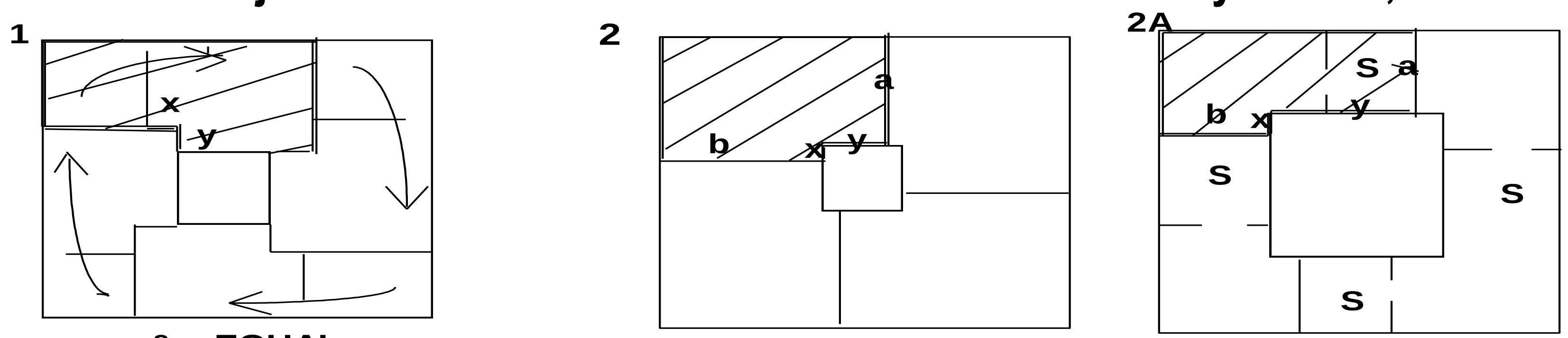


A NUMBER OF ELEMENTS MUST BORDER THE ARMS RADIATING OUT FROM THE CENTRE POINT.
ALTHOUGH THREE ELEMENTS ARE ASSUMED HERE IT IS CLEAR THAT HOWEVER MANY ARE CHOSEN THOSE ADJOINING THE CENTRE POINT WILL BE THE SAME FOUR!



N3.4. GENERAL FORMAT OF SYMMETRY 4

There are just two basic formats 1 and 2 below which always rotate, but in 1 below x and y are equal.



x & y EQUAL

a & b EQUAL BUT x & y UNEQUAL

.

NOTE: the shape and proportions of the shaded segment often vary greatly from those show

In 2 and 2a above a and b are equal instead (x and y usually unequal).

Although extra Elements S can be added to above 2 making above 2a the basic pattern (times four) is still there but including this Element, or to put it differently, imagine the broken lines are not there.

So there are always 4 rotating equal patterns plus an inner Element, and if this statement is born in mind as always applying, no matter what Add-ons are shown later it will help in understanding 4-fold squares, which though not difficult, can be confusing.

Of course patterns drawn anticlockwise are equally Valid, but since the solution does not change, direction is irrelevant.

Although in theory eight rotations and reflections exist for any given pattern, only one is actually distinct.

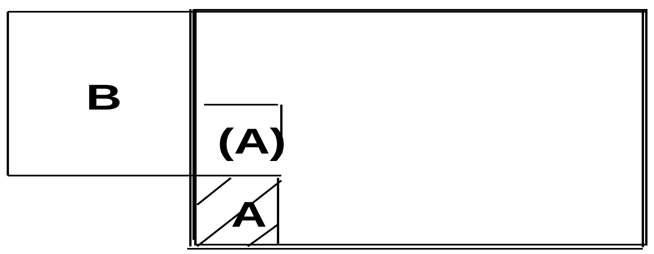
It is easy to observe that the four corner Elements are always identical in a symmetry 4 square.

It is important to notice that the segment in 1.

Above is like a Squared-Rectangle with a corner Element removed (A Below) and another Element added (B Below).

Two or more Elements must adjoin B however.

But a solution with a repeated end (see A and (A) below) will suffice and there are many more of these.

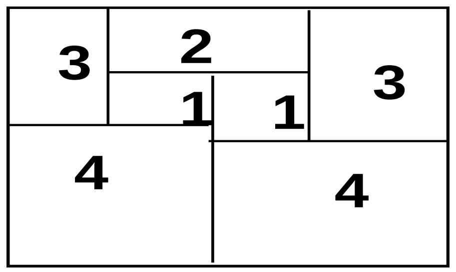


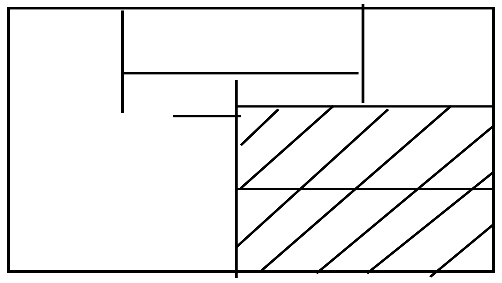
N3.5. FORMAT WITH x AND y EQUAL - CONSTRUCTING SOLUTIONS BY ALGEBRA

The procedure is as follows -

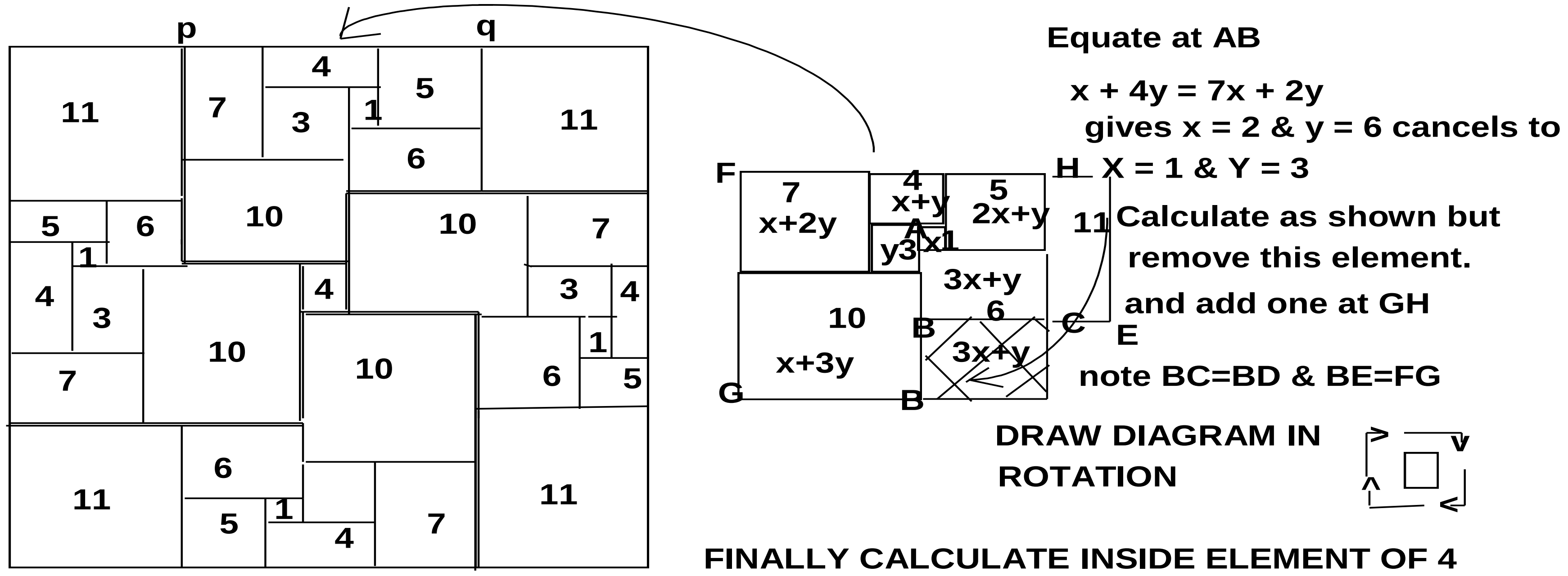
1. Choose any known rectangle, say, [7] 8 x 7 below. *
2. Select any corner - say 4 at bottom right. Redraw pattern with this corner shown twice vertically (shaded).
   Or it can be done horizontally []] with an equally acceptable different result.
3. Recalculate the rectangle using Simple algebra as shown below.
   The reason for using a Repeater solution with a double end is to ensure BC = BD.
4. Shed the unwanted double Element 6 at BCD.

5. Add four extra corner Elements of the same size and draw the pattern four times rotationally.
6. A central Element appears. This is found to be 4 in this example.





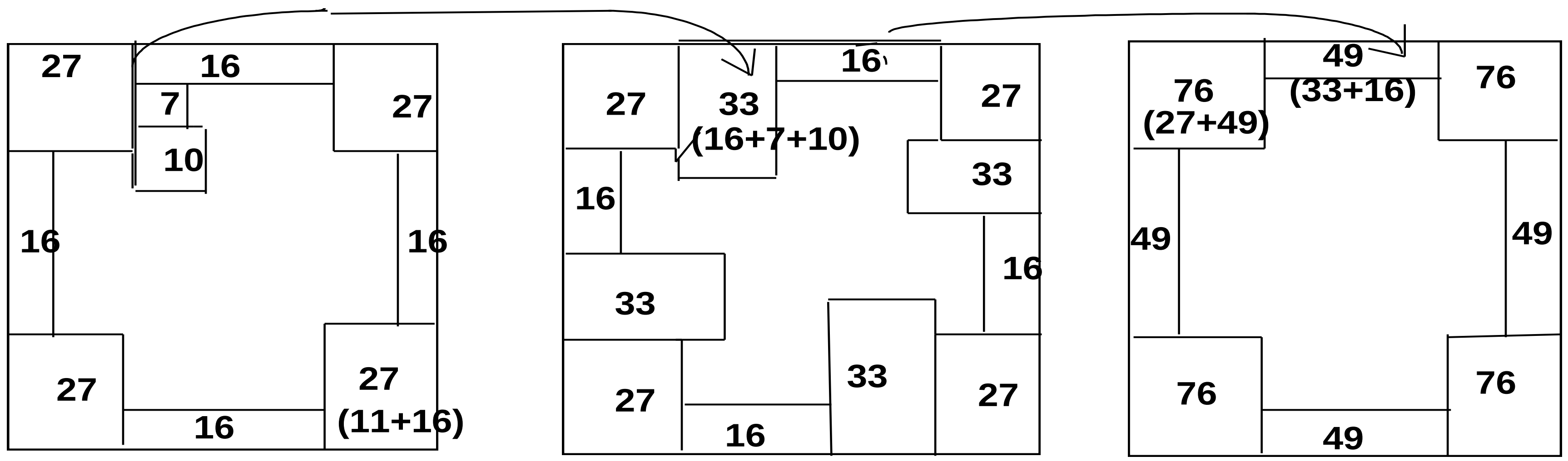
Alternatively choose any pattern with a double end, calculate and drop the unwanted corner Element.



The solution above is [33] 38 x 38 and can be usefully referred to arising from the Source rectangle, in this case [7] 8 x 7, although the actual Elements in [7] 8 x 7 are irrelevant to the destination 4-fold solution.

N3.6. FIRST FORMAT WITH x AND y EQUAL - MANY MORE SOLUTIONS EXIST

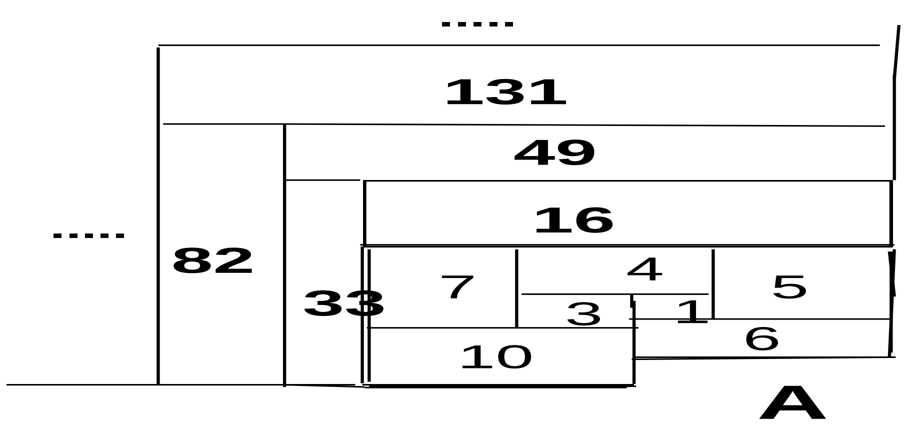
(See L8.12 onwards for the second format)
Using [33] 38 x 38 above, further solutions may be constructed as follows.
The results could be reached direct from a segment calculated from appropriate rectangles.

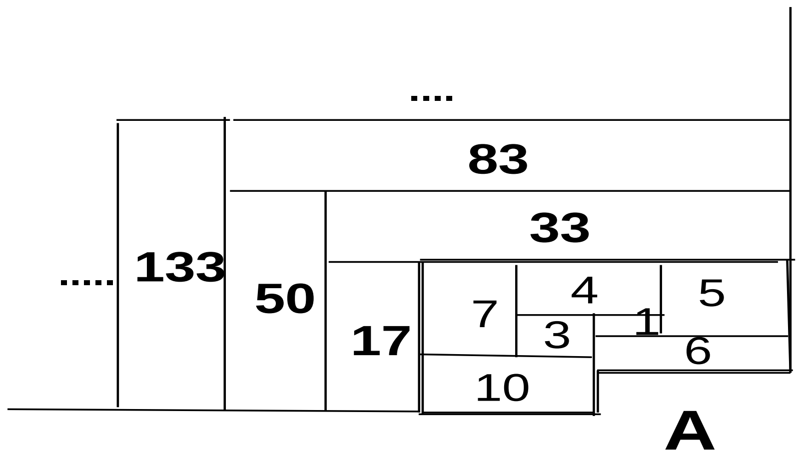


[37] 70 x 70 [41] 103 x 103 [45] 181 x 181

CROSSLINK **SIDESLINK**

Elements not shown are identical to those before. This is less complicated than it seems!
An Element is simply inserted at the solid line, repeated four times and certain Elements suitable increased in size.
The series continues indefinitely by adding sets of Elements internally then externally alternatively.

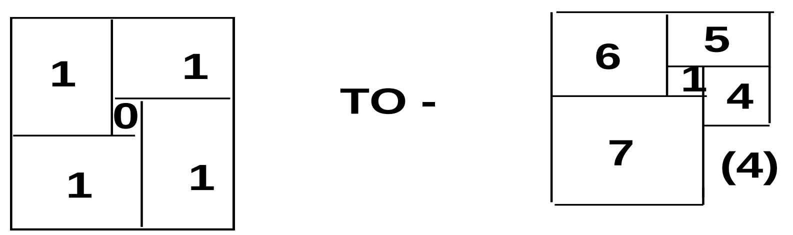




Above 2 shows another series possible. Now provided that the kink at a remains internal the series (1 3 4 5 6 7 10) can be turned round at various angles to provide more solutions, provided instances where adjacent Elements occur, are avoided.

Now as all Root solutions can be extended by adding Steps to create Branch solutions, this means an amazing quantity of solutions can be found from any one Repeater solution.

N3.7. A LIST OF SOLUTIONS FROM JUST ONE REPEATER



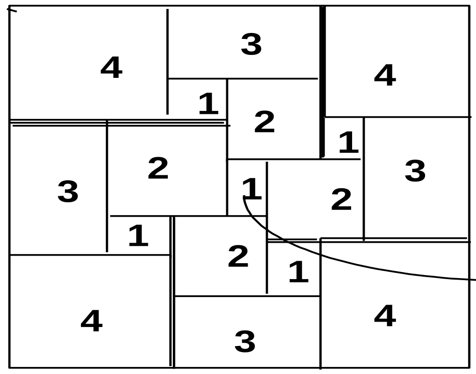
SOURCE RECTANGLE DESTINATION RECTANGLE

Just from the above feeble pattern I have constructed 4 fold Root squares plus Branches of the following Orders and sizes, and only those with dimensions under 1000 are shown:-
[25] 29 Branches [27] 49 [29] 89 [31] 149 [33] 249 [35] 409 [37] 669 (1st lot)
[29] 42 Branches [31] 75 [33] 141 [35] 240 [37] 405 [39] 669
[33] 77 Branches [35] 134 [37] 419 [39] 704
[37] 127 Branches [39] 221 [41] 409 [43] 691
[41] 212 Branches [43] 367 [45] 677
[45] 347 Branch [47] 600
[29] 51 Branches [31] 82 [33] 144 [35] 240 [37] 395 [39] 646 (2nd lot)
[33] 86 Branches [35] 141 [37] 251 [39] 416 [41] 691
[37] 145 Branches [39] 235 [41] 415 [43] 685
[41] 239 Branches [43] 388 [45] 686
[45] 392 Branch [47] 637
[29] 53 Branches [31] 86 [33] 152 [35] 257 [37] 416 [39] 680 (3rd lot)
[33] 90 Branches [35] 147 [37] 261 [39] 432 [41] 717
[37] 151 Branches [39] 245 [41] 433 [43] 715
[41] 249 Branches [43] 404 [45] 714
[33] 75 Branches [35] 130 [37] 240 [39] 405
[41] 680 (4th lot)
[37] 121 Branches [39] 211 [41] 391 [43] 661
[41] 204 Branches [43] 353 [43] 353 [45] 651
[45] 333 Branch [47] 576

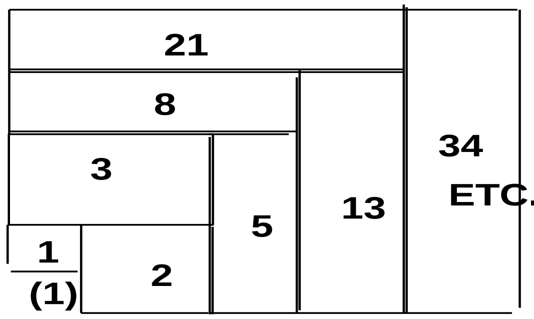
But some solutions give 8 patterns and thus 8 times this amount, but the above has only one as shown above (the other seven being duplicates)! The 4 lots indicated are from presenting the pattern on its sides etc.

N3.8. FIBONACCI SERIES AND 4-FOLD SQUARES

Since rectangles can be drawn from the Fibonacci Series 1 1 2 3 5 8 13 21 and 4-fold squares constructed from most rectangles, it is clear that the f series can be used to construct 4-fold squares. The simplest of these is [17] 11 x 11
which uses a rectangle with 1 1 2 3 with 1 removed and 4 added. a centre Element of 1 is necessary.
This solution is remarkable in only containing Elements of 1 2 3 and 4!



[17] 11 x 11
5'S ADDED
AT THICK
LINES GIVES
[21] 19 x 19
INCREASING
THE CENTRE
ELEMENT



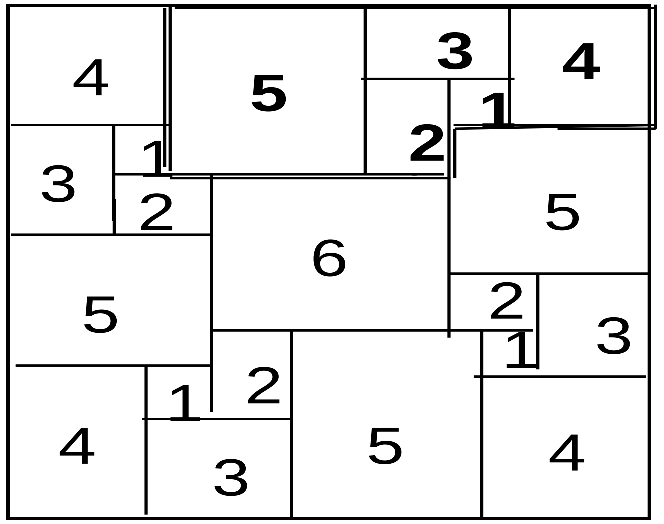
ETC.

WITH 5 ADDED [21] 19 x 19
8 ADDED [25] 32 x 32
13 ADDED [29] 53 x 53
21 ADDED [33] 87 x 87
34 ADDED [37] 142 x 14
& SO ON

Amazingly squares of 11 19 32 53 87 142 231 375 608 985 can be calculated for Orders 17 21 25 29 33 37 41 43 45 49 53 respectively and each has a string of branch solutions possible as 1-fold solutions. Although the differentials in the series are 8, 13, 21 … the F. Series, the key number is found to be 2. e.g. 375 + 608 + key 2 = 985.

N3.9. 2ND FIBONACCI SERIES

Very similar to the 1st series, this one gives Invalid results using 1 2 and 3 only so starts with 1 2 3 5.

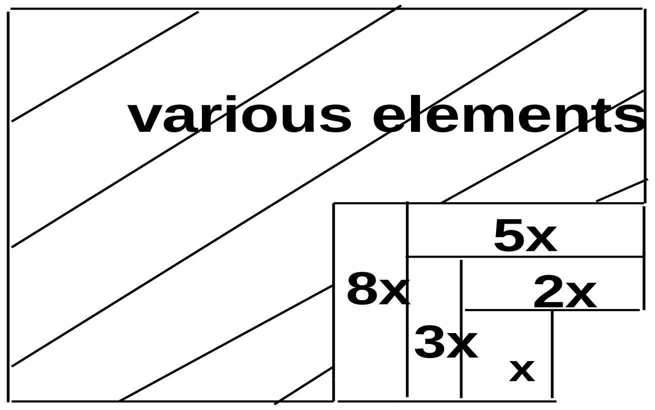


2ND F-SERIES
WITH 1 2 3 5
CORNERS 4
& CENTRAL 6

Squares of 16 27 45 74 121 197 320 519 and 841 are found for Orders 21 25 29 33 37 41 45 49 and 53 respectively, and the Key number is again 2. As always Step solutions are extra.

N3.10. EXTENDING THE FIBONACCI IDEA

Apart from those solutions already dealt with involving additions to top and left alternately, look at the following.
In the kink area two Elements x and 2x can be put, or x 2x 3x, or x 2x 3x and 5x or as shown x 2x 3x 5x 8x and so on.
The area “various elements” can follow the pattern of a normal rectangle, but recalculated to give the required proportions in the kink.



Completion
of rectangle
gives a
Non-simple
solution

N3.11. LIST OF THESE SQUARES TO ORDER 33 (INCOMPLETE)

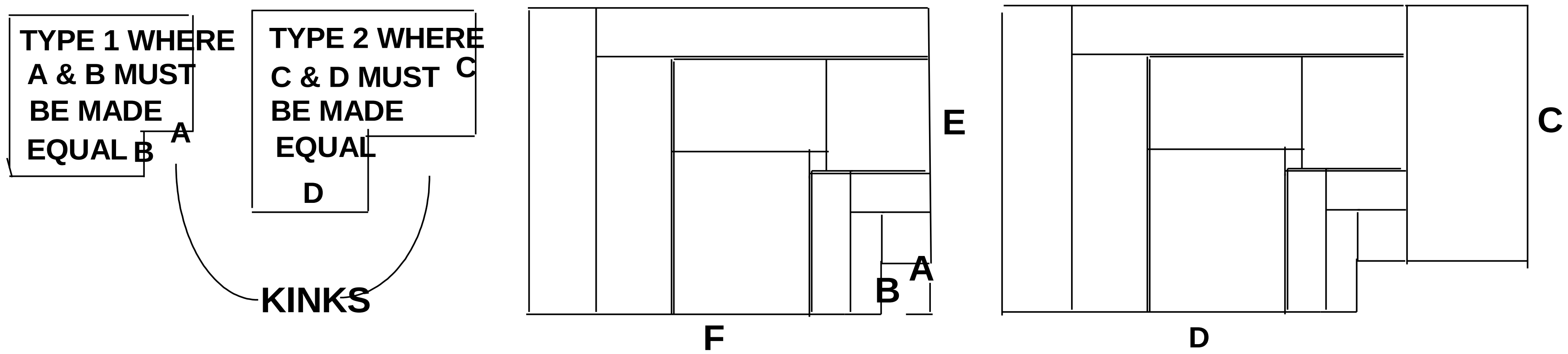
[17] 3 1 4 2 1 solution
[21] 5 2 7 3 1 [21] 1 2 3 4 5 2 solutions
[25] 5 2 7 3 1 8 3 5 8 2 1 12 1 2 3 5 4 5 4 1 5 6 7 9 4 solutions
[29] 6 1 7 5 4 13 20 1 9 10 11 8 4 15 8 1 9 17 10 11 4 6 1 7 5 8 4 1 2 3 5 1 8 7 8 4 1 5 9 13 6 7
- 3 2 5 1 4 1 4 7 solutions

[33] 1 6 1 7 5 4 13 20 24 6 1 5 7 4 13 24 33 2 3 5 8 13 20 21 1 11 4 15 7 19 16 9 26 4 1 3 2 1 7 4 12
- 1 3 4 5 6 7 10 11 9 10 19 1 8 36 11 4 11 3 14 8 5 12 19 7 25 7 18 11 4 15 19 7 6 1 7 5 11 4 24 31
- 5 2 3 8 1 13 12 13 4 1 3 2 5 1 4 8 8 1 7 9 6 4 2 15 1 7 8 9 6 4 13 2 4 3 7 1 2 1 3 11
- 1 9 10 11 19 8 4 15 10 3 7 17 24 13 28 12 4 9 13 17 21 22 16 15 18 solutions

N3.12. CATEGORIES OF RECTANGLES WHICH MAY BE USED TO CONSTRUCT 4-FOLD SQUARES

It first seemed to me that in Order to construct 4-fold squares that it was necessary to use only normal Imperfect or Perfect solutions or patterns which look similar to these, and certainly these usually work.
But the range of “rectangles” which may be used is so wide that it includes many forms which are easily overlooked.
Firstly if source rectangles are being recalculated with Repeaters, it is the status of the Destination rectangles which matter. For instance, as long as the destination rectangle is OK, the source one may be Invalid with or without zeros.

Concentrating on the destination rectangles now, it is true that a few constructions will fail particularly if the result gives adjacent Elements and / or zeros. But note the following which usually result in acceptable 4-fold squares -
1. Repeater solutions - actually essential for one of the two groups.
2. Singlend solutions and ranges of such.
3. Containing smaller rectangles which need not be Valid solutions in themselves.
However it is necessary for the Compound part to form part of the kink - see Below.



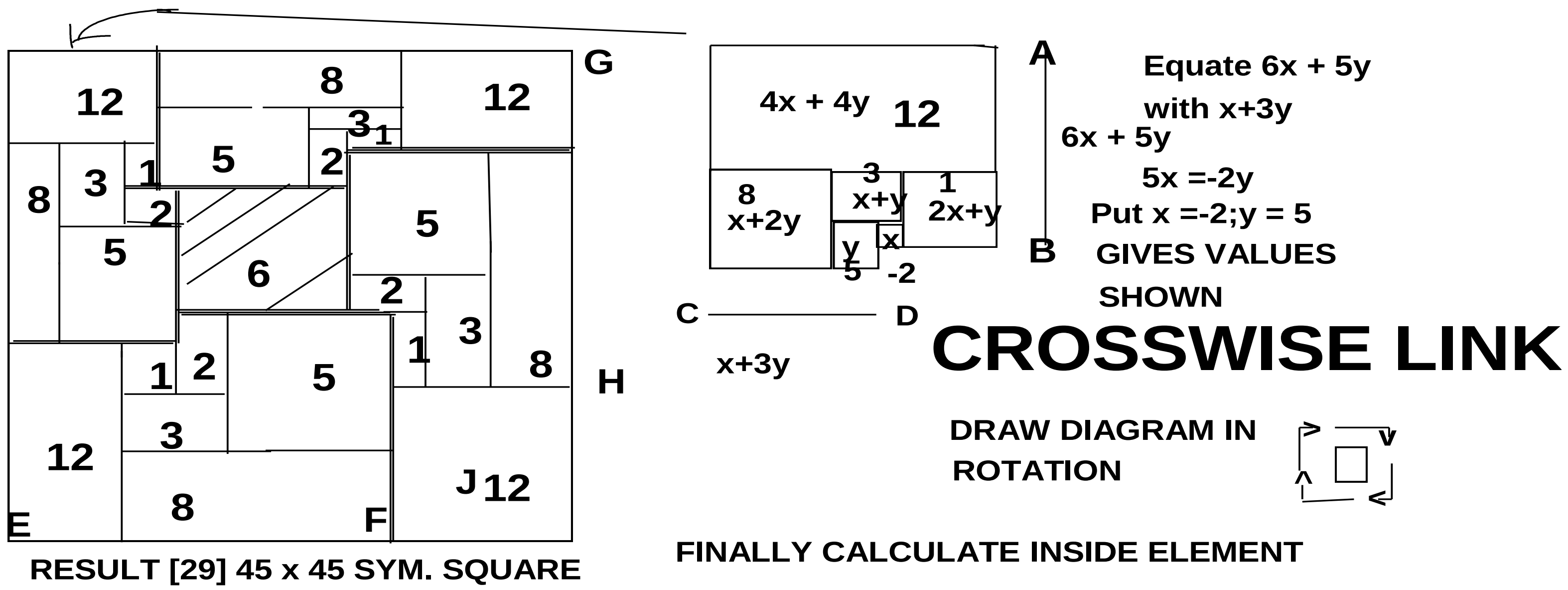
Rectangles from the above patterns would be unacceptable, but despite the repeated single ends and rectangles within rectangles, good 4-fold squares may be found. A is shown as a single Element but need not be.
E and F both have several Elements.

The only restrictions are:-

1. In type 1 A must equate with B.
2. In type 2 C must equate with D
3. There must not be single Elements at both C and D.

N3.13. SECOND FORMAT - x AND y UNEQUAL - SOLUTIONS USING ALGEBRA

This second type is shown below. Firstly draw a pattern of squares in an L shape.
Then calculate the algebraic values using x and y (and z, a … if required). After equating AB with CD and calculating numerical values, the pattern can be repeated four times such that AB and CD are drawn adjacent to each other.

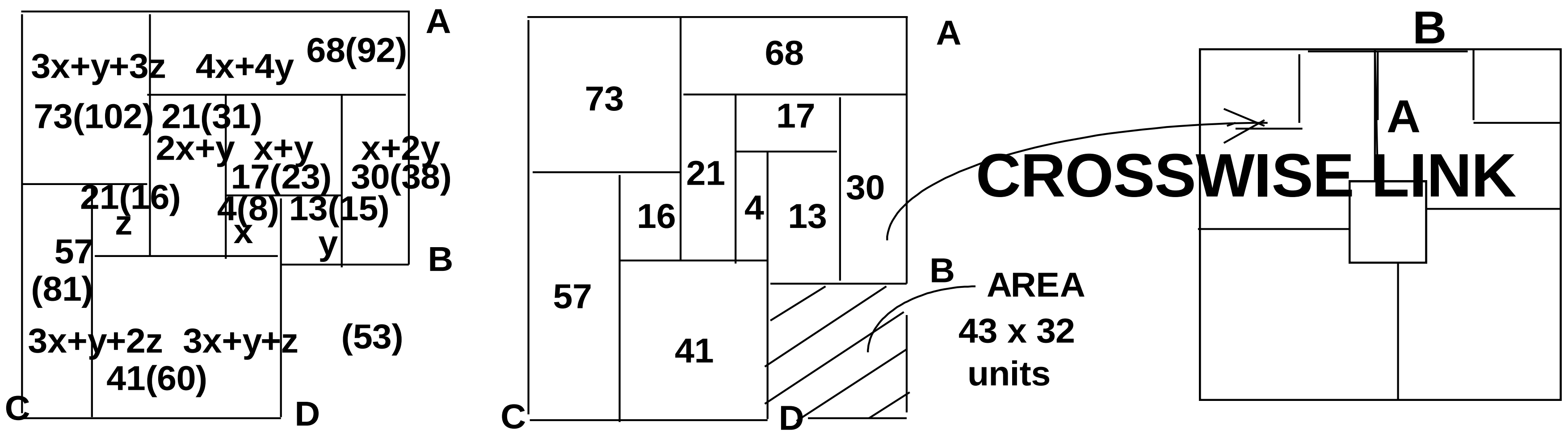


Note at least two Elements are required at both AB and CD. Also, slight modification to the pattern is needed when negatives occur (see above). The 4-fold square [25] 28 x 28 results.
An internal Element is formed and easily calculated - 6 in above case. If required, an asymmetric square can be drawn by adding Elements at EF and GH (each 20 in above) and expanding Element J to 32.

N3.14. USING KNOWN RECTANGLES

Instead of the chosen pattern Above, if rectangular solutions are already known, one with sides S2233 or greater can be taken, a suitable corner removed and the resulting pattern recalculated by algebra and ensuring that AB and CD are equal.

The pattern for S2223 and xy solution [11] 194 x 183 has been utilised below. Often a further Unknown, here z, is needed for the algebra.



The original Elements (194x183) are shown in brackets, and the new Element values can be calculated from simultaneous equations from the algebra shown. A number of features emerge which remain true when other patterns are investigated:-

1. The shaded area becomes a rectangle, no longer a square.
2. The Elements have shrink in size for each position in the pattern.
3. The size 194 x 183 reduces to 141 x 130 and the differences in each pair are the same- 11.
4. Note 43 - 32 in the shaded area is also 11.
5. The corner Element removed is 53 and if this is deducted from 194 x 183, 141 x 130 is obtained!

This last find is important as it means that the dimensions of the new pattern and the 4-fold square arising from it are easily found without resorting to algebra. But annoyingly, I has found so far no means of supplying the actual Elements without using algebra or trial and error values, as there seems no quick route to find the numerical value of AB (= CD), or the inner Element value - pity!

N3.15. FOUR FOURFOLD SQUARES (AND STEPS) ONLY

There are four root solutions arising from each solution -

1. The original in the form of above 3.
2. With Elements added at “A” Above.
3. With Elements added at “B” Above.
4. With Elements added at “A” above then at “B” also.

Now if [9] 33 x 32 is used and represented algebraically as [o] m x n with a corner Element of 9 = c, a = 17 and b = 21 by using algebra. The four resulting Fourfold squares can be calculated with general formulae as follows -

1. [33] 47 x 47 [o x 4 - 3] m + n - 2c x m + n - 2c.

2. [37] 64 x 64 [o x 4 + 1] m + n - 2c + a x m + n - 2c + a.
3. [37] 89 x 89 [o x 4 + 1] m + n - 2c + 2b x m + n - 2c + 2b.
4. [41] 140 x 140 [o x 4 + 5] m + n - 2c + 2b + 3a x m + n - 2c + 2b + 2a.

It appears only the dimensions for No. 1 can be found without using algebra. The usual everlasting step series can be applied to each for more solutions (asymmetric ones).

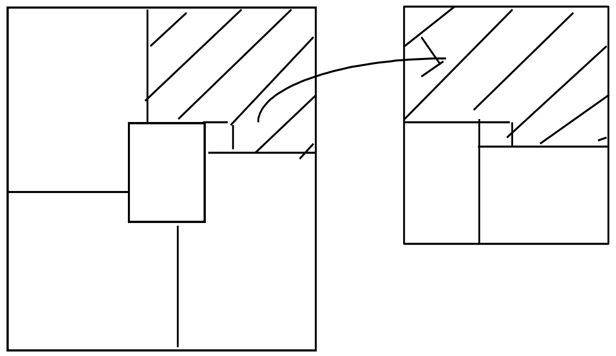
N3.16. REDUCTION INDEX AND 4-FOLD SQUARES

It is clear that even maximum size dimensions of Fourfold squares are tiny when compared to one-fold squares. For example, 16 for Order 21 whereas about 170 is typical for Order 21 one-fold.
It is also obvious that Reduction indexes are always very large for fourfold squares.

So far I have very little information on Reductions for 4-fold solution.
[17] 11 x 11 has an RI of 374 or 11 x 34 but gives little clue as to other solutions. Despite this and some other solutions involving Fibonacci type numbers such as 5, 21 and 34 other solutions do not, so this does not help.
I have drawn blanks on this subject so far. **********much more work needed**********

N3.17. SQUARES WITH A DIAD OR TRIAD END CAN BE CONVERTED INTO 4-FOLD SQUARES

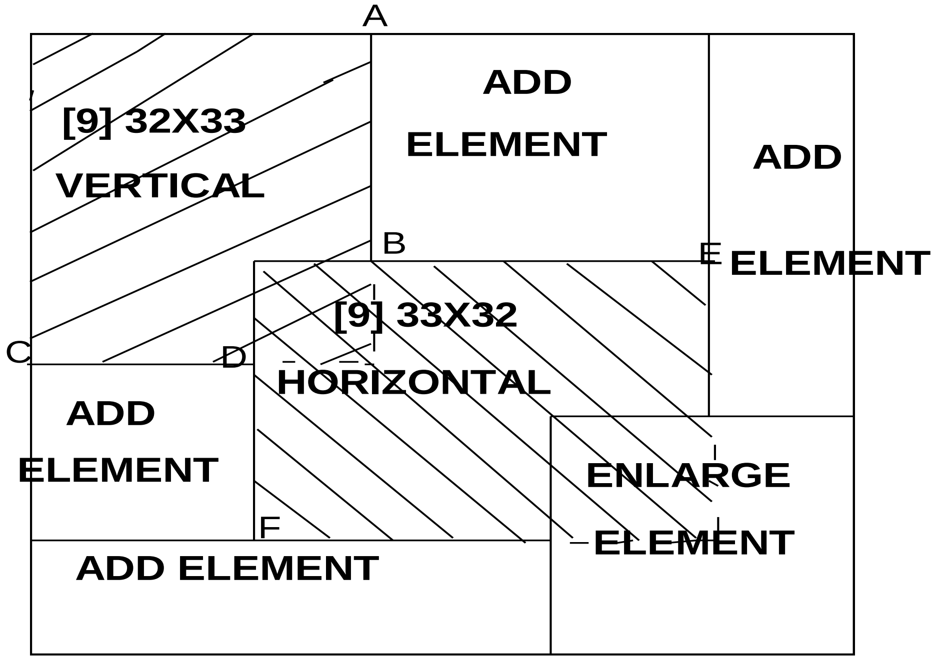
It was shown how Squares with a Diad (but not Triad) may be converted into 2-Fold Squares. However all of these - and also those with Triad Ends - can be converted into 4-Fold Squares by removing the Diad, and circulating the rest four times as below 1. A last Element then appears in the Centre. Conversely, we can take any 4-Fold solution and add a Triad to create a smaller Squared-square, although some will have repeated ends.
CHECK THIS - SEEMS WRONG!!!



N4. CONSTRUCTING SQUARED-SQUARES FROM RECTANGLES

N4.1. USING KNOWN RECTANGLES TO MAKE SQUARED-SQUARES ***THIS WAS MENTIONED EARLIER!!

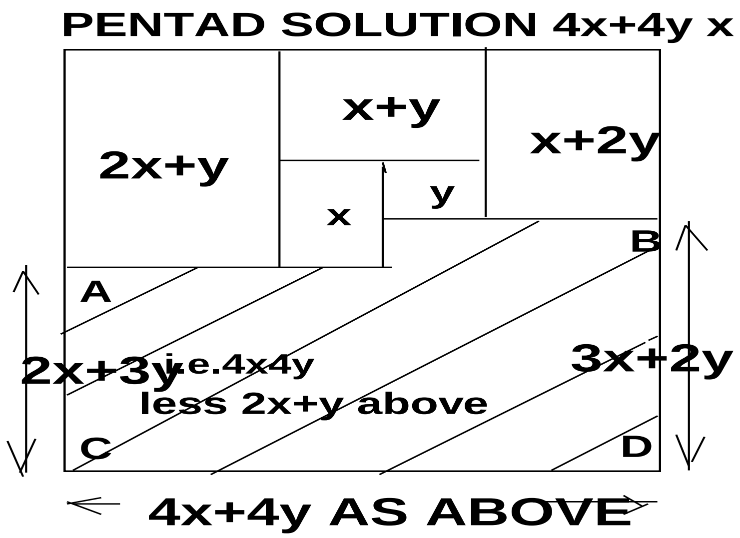
There is another way of constructing Squared-squares easily using a pair of rectangles of the same size.
A pair of twins may be used, but so may any solution used twice, as in below with solution [9] 33 x 32.

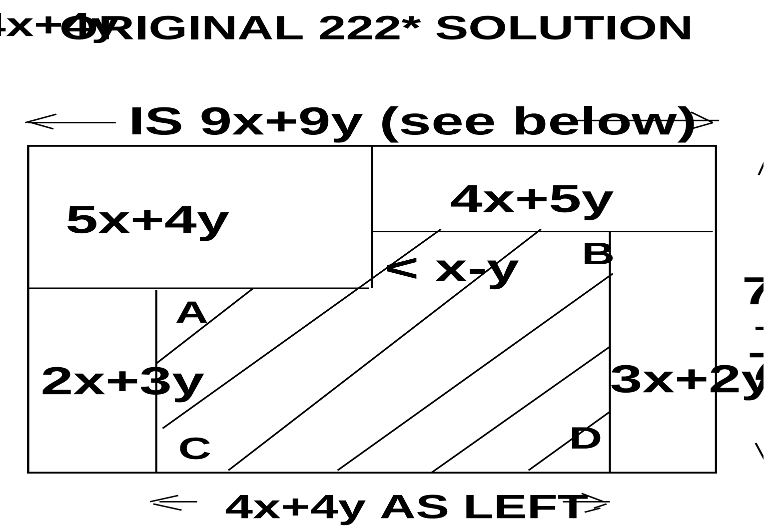


ANY SOLUTION OF 2233 OR GREATER CAN BE USED PROVIDED AB CD BE & EF EACH HAVE 2 OR MORE ELEMENTS BORDERING.
ELEMENTS AT AB AND CD ARE ADDED TO GIVE A COMPLEX SQUARE, BUT BY ENLARGING THE BOTTOM RIGHT ELEMENT AND ADDING TWO MORE AN IMPERFECT 22-- SOLUTION RESULTS!

N4.2. USING PENTAD TYPE ADD-ON

Previously it was shown how any S222# solution could be made into a Pentad solution.
What sort of S222# solution is necessary to ensure that the corresponding Pentad solution is a Squared-square?
Below shows that the S222# solution has to have the ratio of 9:7 from which a Square of Pentad type is guaranteed.
Where a suitable original S222# solution is Perfect, the hope is that the Pentad square will be also - but usually is not! Imperfect originals give rise to Imperfect squares.



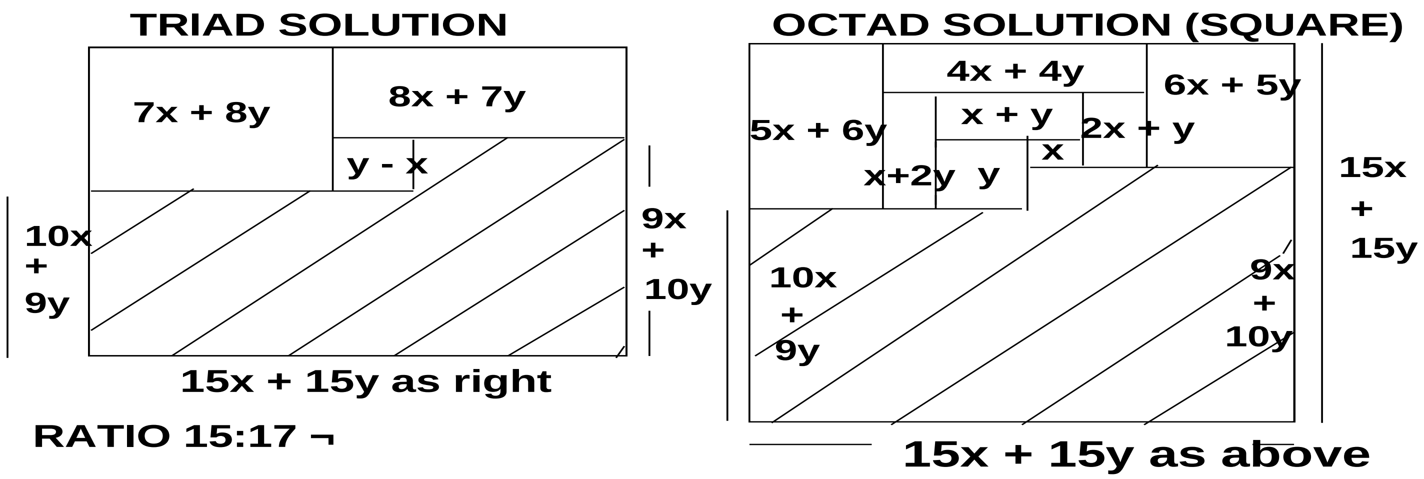


Above 1 is explained as follows. Calculate the algebra for the Pentad as shown.
To be a square the depth must also be 4x + 4y (2x + y + x + y + x + 2y) so AC has to be 2x + 3y and BD 3x + 2y.
In Above 2 we transfer the 4x + 4y, 2x + 3y and 3x + 2y as shown. This means the length of rectangle has to be 9x + 9y.
The two Elements of 5x + 4y and 4x + 5y can be deduced making the depth of the rectangle 7x + 7y.
This is interesting as it means any S222# solution of ratio 7:9 or 77.777…% elongation can be converted into a square!

So a 222* of Ratio 7:9 gives a Squared-Square Ratio 9:9 (or 1:1). Of course if the 222* Solution is Imperfect - so is the Square. If the 222* Solution is Perfect the Square will only be Perfect if none of the five Elements in the Pentad clash.
In my database of over 600,000 Rectangles just 18 222* Solutions of Elongation 7 / 9ths were found and just one gave a Perfect Square! From the Square to the Rectangle the Ratio goes from 9:9 to 9:7.

N4.3. USING OCTAD ADD-ON SQUARES

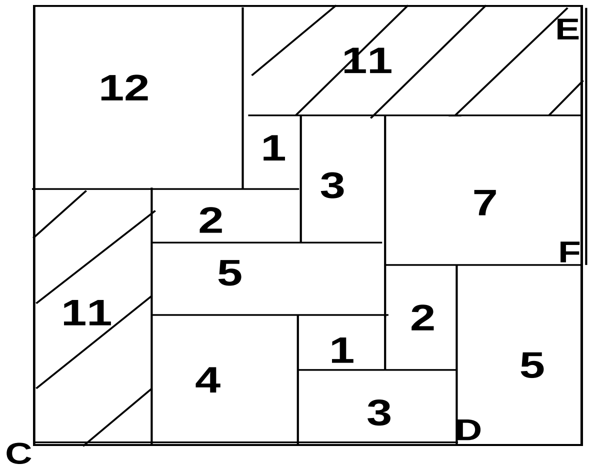
Another Add-on similar to the last is the Triad to Octad one. Again the conditions for a Triad solution can be found so that the resulting Octad solution will be a square. The reader is left to decipher the algebra, but briefly the Triad solution must have an elongation of 88.23…% and Ratio of 15:17.



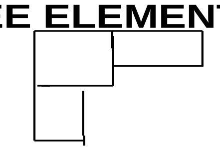
A TRIAD RECTANGLE can be converted to an OCTAD SQUARE if it has the Ratio 17:15
It can be shown also that -
An OCTAD Rectangle can be converted to a TRIAD SQUARE if it has the Ratio 13:15

N5. BRANCHES AND ROOTS

N5.1. SMALLEST ORDER IMPERFECT SQUARE



EXAMPLE OF A "TWO-TWO" ROOT
SQUARED- SQUARE
[13] 23 x 23 R23
SIDES 2234 - IMPERFECT AS SHADED
ELEMENTS HAVE TO BE THE SAME.
A ROOT SOLUTION IS ONE WHERE IT IS NO
POSSIBLE TO STRIP OFF THREE ELEMENTS
OR WHERE TO DO SO WOULD
CREATE AN DOUBLE CORNER
AS IN THIS EXAMPLE.



BRANCHLINK

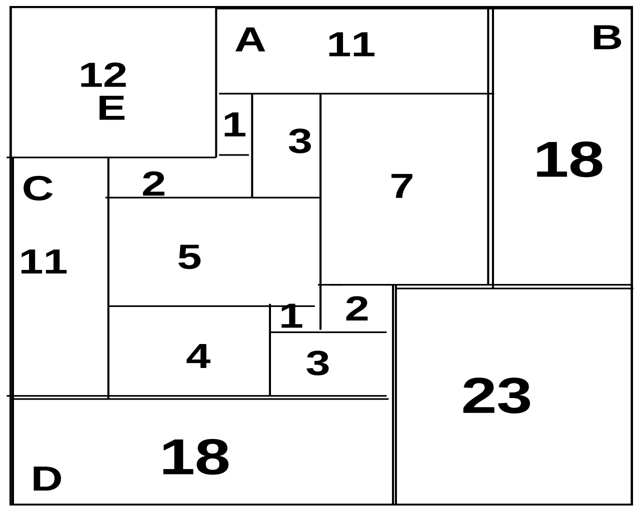
Above is the smallest Order Imperfect Squared-square possible with no zero Elements - [13] 23 x 23 with a Reduction Index of 23.

N5.2. BRANCHES

Below a Branch solution is shown. From any Squared-Rectangle it is possible to apply the Add-on Below in an infinite series. All such solutions are termed Branches.

It is interesting to observe that if Elements are drawn at CD and EF and the shaded Element suitably increased, another Squared-square is produced [15] 41 x 41 and is found to have a Reduction Index of 41. Similarly a further square is produced by drawing Elements at G-H and E-F and enlarging the Element “12”.

In effect, using the top left and bottom right corners alternatively, an infinite series is possible- See Below.



BRANCH SOLUTION
[15] 41 x 41 IMPERFECT
A FURTHER BRANCH OF ORDE
IS OBTAINED BY DRAWING ELEME
AT AB & CD AND INCREASING THI
ELEMENT AT E. IN FACT BRANC
OF ORDERS 19,21,23,25.....
COULD BE CONTINUED FOR EVEF
CONTINUALLY CHANGING ENDS!

Although the Squared-squares are always Imperfect, they are extremely reduced in size, and here have the Same Reduction Index as the side! The Complexity is equal to side squared or the Reduction Index squared!

It is further found that each side number is the sum of the previous two plus 6, e.g. 41 + 70 + 6 = 117, 70 + 117 + 6 = 193, and the series can be continued backwards! e.g. [11] 12 x 12 and [9] 5 x 5 although these are Compound solutions.

N5.3. ROOT SOLUTIONS

All Squared-squares are either Branches or Roots. If any Branch solution is taken and series of 3 Elements are stripped off as many times as can be done the resultant square is a Root. Sometimes, as above, there are three Elements left, but to strip these off will create a repeater solution.

Or in some cases stripping off the last three Elements will give rise to a Compound solution. Often, however, the Root solution will be of better status than this one. Whenever a computer calculates a Branch solution S2233 it is worth remembering the Root might be, but rarely is, a Perfect Simple Square.

In passing it should be mentioned that if a repeater solution is known, it can be easily converted into a Valid Root solution (plus Branches if required).

N5.4. ROOT SOLUTIONS PREFERABLE TO BRANCHES

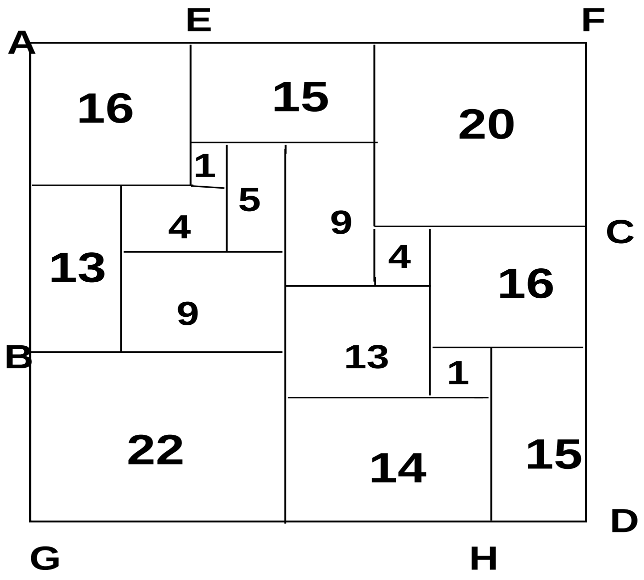
Branches are obviously numerous whereas there is only one Root derived from any given Branch. In fact a S3333 Asymmetric solution gives rise to four series of Branches. Although the Root solutions vary considerably in status, some might wish to catalogue only Root solutions as these are the better squares.

N5.5. FINDING KEY NUMBERS

The number 6 in the last paragraph is termed a KEY NUMBER. If the series is commenced from a square of sides S3333 or greater, there are in fact four distinct everlasting series possible. But there are only two if there is 2- fold symmetry applies, and only one for 4-Fold Symmetry. In [16] 51 x 51 which has sides S3333, the four series are as follows -

Order	Followed	By the	Dimensions
16-51	16-51 as before	16-51 as before	16-51 as before
18-80 key no. 9	18-82 key no. 9	18-86 key no 20	18-87 key no 20
20-140	20-142	20-157	20-158
22-229	22-233	22-263	22-265
24-378	24-384	24-446	24-443
26-616 and so on...	26-626	26-729	26-728

In this case the Key numbers are 9 9 20 and 20. The first series is extended thus
378 + 616 + 9 = 1003, 616 + 1003 + 9 = 1628 and so on...



KEY NUMBERS IN SQUARED- SQUARES

1. ELEMENT 20 AB - 20 = 9 } SAME
2. ELEMENT 22 CD - 22 = 9
3. ELEMENT 16 GH - 16 = 20 } SAME
4. ELEMENT 15 EF - 15 = 20

(or if preferred 16 + 15 - 22 = 9, 14 + 15 - 20 = 9
20+16-16=20 & 22+13-15=20)

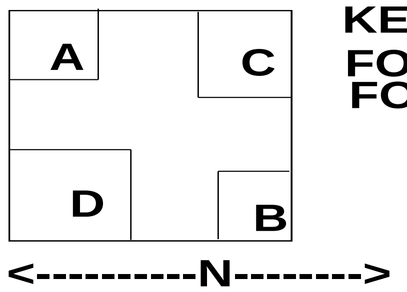
NOTE THE SAME KEY NUMBERS FOR
DIAGONALLY OPPOSITE CORNERS.

Note AB CD EF GH represent al l elements along the sic excluding the corner.

[16] 51X51 IMPERFECT.

N5.6. EVALUATING THE KEY NUMBER

From a given square of S3-3-3-3+ two key numbers emerge. One from one opposing pair of corners and one from the other pair. But why 9's and 20's in the Above Example? Above 1 explains. An alternate and easier explanation is below-



KEY NUMBERS
FOR A & B, N-A-E
FOR C & D, N-C-I

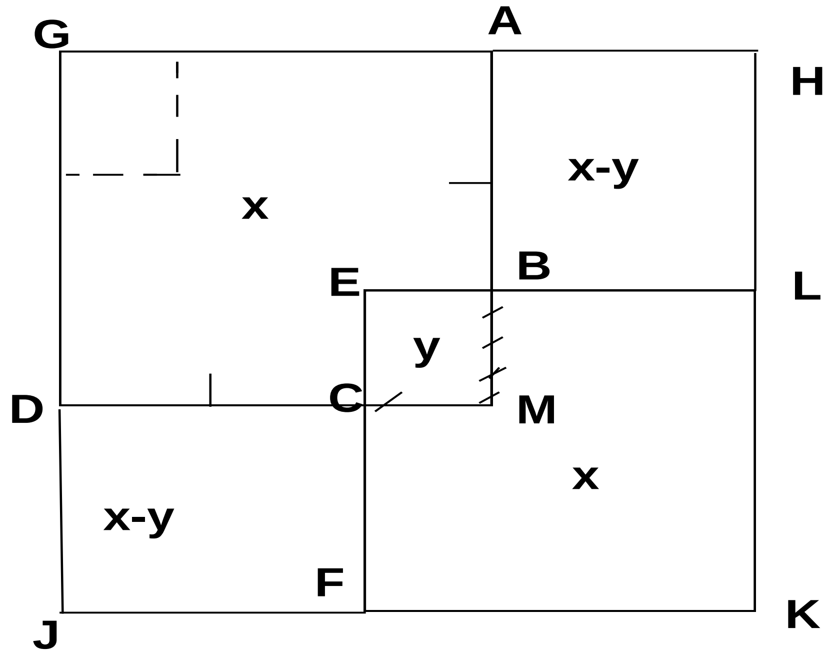
Interestingly, the series can be continued by applying the Key number backwards but chaos ensues!

In theory values can be given for smaller non-existent squares, some of them negative in size!

N6. SQUARED-SQUARES OBTAINED FROM SERIES

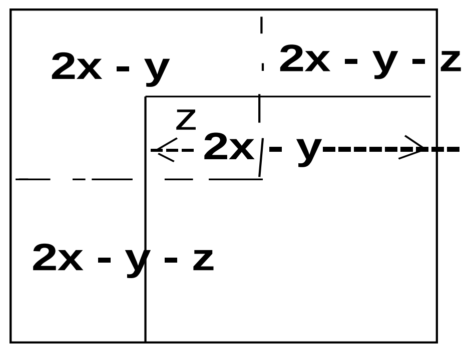
N6.1. AN ADD-ON SERIES IS POSSIBLE FROM ANY SQUARED-SQUARE!

Ignoring useless Invalid examples such as [5] 2 x 2 it is easy to see that at least two sides of any Squared-square must have at least three Elements even if the other two only have 2 (in which case the solutions are automatically Imperfect).



THE SQUARES ADD-ON REQUIREMENTS ARE A SQUARED SQUARE HAVE 3+ ELEMENTS ALONG AM & DM.
1. INCREASE ELEMENT y TO SIZE OF x AND
2. ADD TWO ELEMENTS EACH x-y IN SIZE.
THE ADD-ON CAN BE REPEATED AT ELEMENT z THEN BACK AT END K, AND FOR EVER!

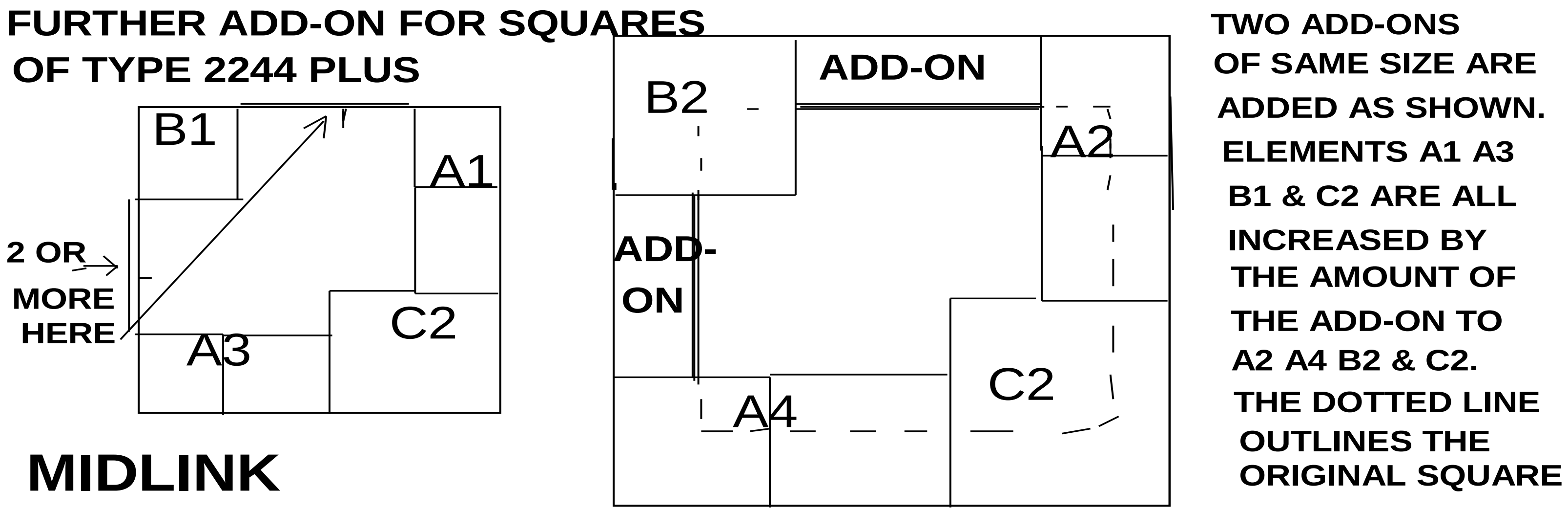
Let the dimension of the square be x and the corner Element y as in above. Let the Order be n, and the original solution will be [n] x by x. AB is clearly x - y as in C - D, and E - F = x, so by adding two Elements of x - y and replacing Element y with Element x we have a new Squared-Rectangle of [n + 2] 2x - y by 2x - y.
Note that in the new solution the sides with more than two Elements are G - H and G - J which means we must use Element z (top left corner) in Order to calculate the next in the series. See Below. The next solution is [n + 4] (4x - 2y - z) by (4x - 2y - z).



USING THE ADD-ON FOR 2ND TIME.
EFFECT ON RESULT -
4x-2y-z X 4x-2y-z

N6.2. ANOTHER ADD-ON FOR SQUARES AND STEM SOLUTIONS

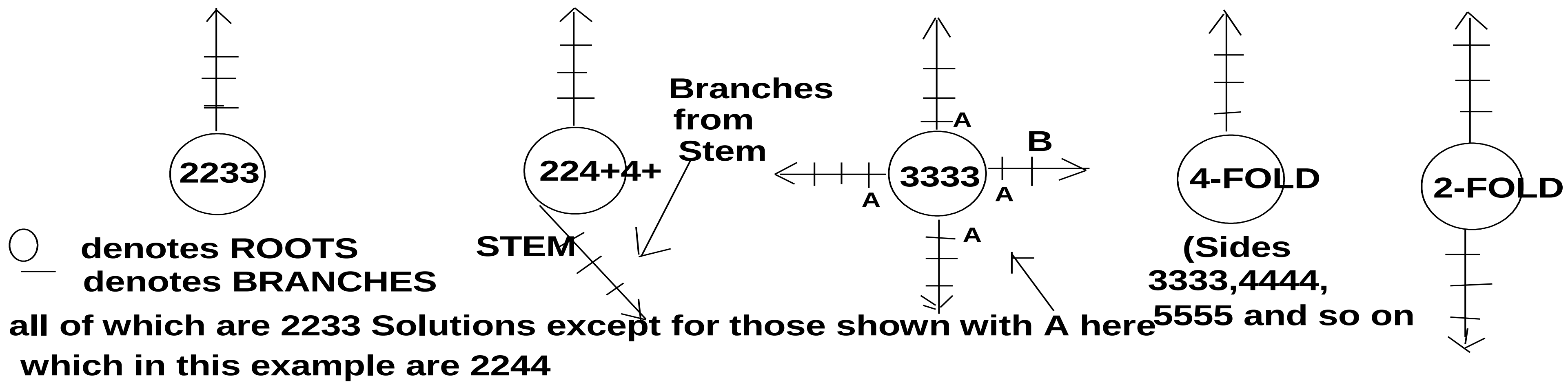
A similar Add-on involving Sides 2 2 4+ 4+ solutions is shown Below. This addition is less useful than the last - It occurs once only but Branch solutions exist ad lib from them. These will be termed STEM Solutions.



It may be argued that these STEM solutions are BRANCH Solutions as they come from a Solution two orders lower. But from another viewpoint they can be considered ROOT solutions since no more pairs of Corner Elements can be removed! I feel they are better regarded as a type of BRANCH Solutions for the following reason - he has catalogued over 3000 Squared-squares from Computer workings and found out from all known ROOT solutions all BRANCH solutions available with Dimensions below 1000. But to ensure no STEM Solutions, and their resulting BRANCH Solutions are missed, each ROOT solution of sides 2 2 4+ 4+ must be separately considered.
It is easier and more logical to group sets with the ROOT Solution with the lower Order.

It is true that some Branch Solutions have sides 2244, 2245, 2255 etc. but the Solutions available by the above method are not STEM Solutions! This is because they are already obtained (duplicated if you like) from the main ROOT.

As it is confusing, the chart below has been included.
It is necessary to define the ROOT Solution as that Solution obtained when no more pairs of Elements can be removed without either a Non-Simple or a Repeater Solution resulting.

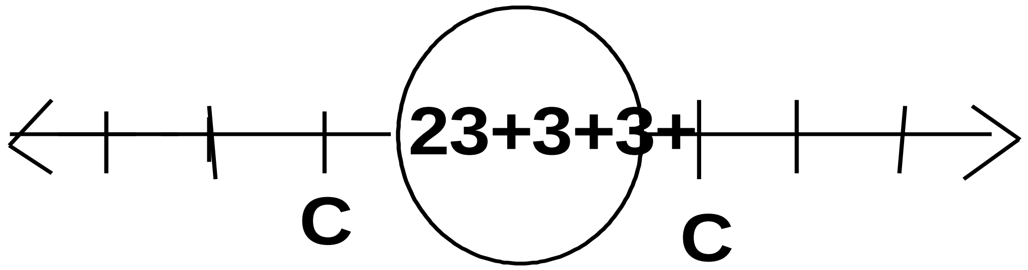


In Above 1, only one series of Branches is possible. In Above 4 the four series of Branches are all duplicated and there is really only one series. In Above 2 there would have been only one series of Branches but the Stem addition gives a second series.

In Above 3 Four series of Branches are possible the first in each series having sides 2244 (In a 4444 case these would read 2255, and 2266 for 5555 and so on).

It is intriguing to have a Solution placed at say B Above which can be worked back to a ROOT of say 3333 and four sets of Branch Solutions also found.

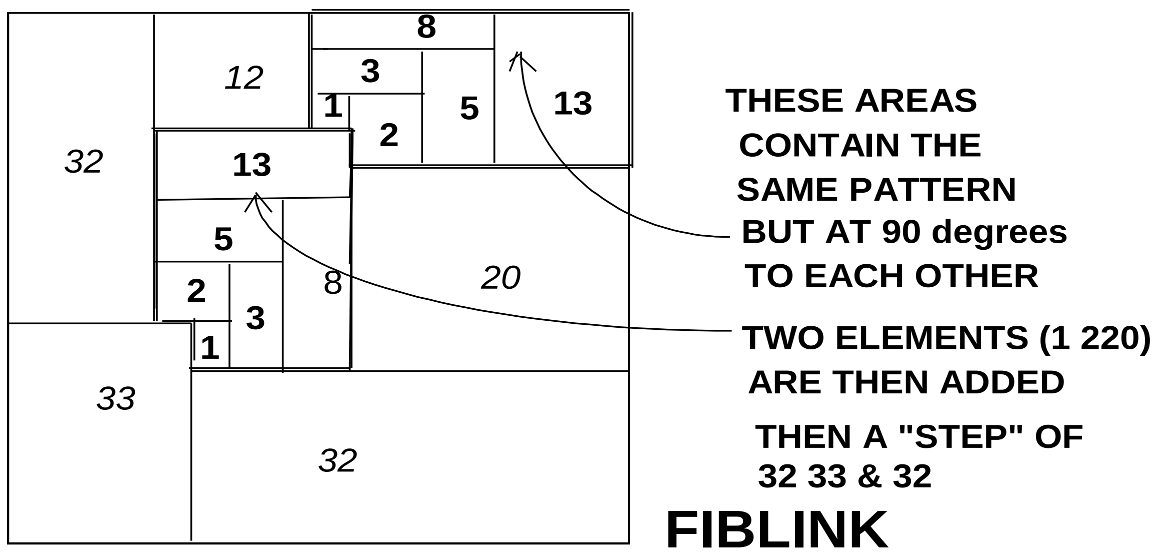
Even better if a Perfect Solution is found this way - I found three using this method!



With a Root of say 2333, 2334, 2344, 2343 of general format 2 3+ 3+ 3+ , there are Two series of Branches possible of which the first pair marked C are not sides 2233 though the rest are..

N6.3. SOLUTIONS FOUND FROM FIBONACCI SERIES MENTIONED EARLIER!!

The Fibonacci Series runs 1 1 2 3 5 8 13 21 34 55 89 …… each number the sum of the previous two.
Look at the series 1 2 3 5 8 13 repeated below-



FIBLINK

It is interesting that five Elements namely 12 20 32 33 and 33 can be added to obtain the solution [17] 65 x 65.
If the two 13's are omitted and the five Elements suitably reduced [15] 39 x 39(1) is discovered!
If two 21's are added then [19] 107 x 107 can be constructed.
Using 1 2 3 only leads to Compound solutions and the minimum able to be used is always 1 2 3 and 5.
In fact the following series can be found provided that one lot of Elements is always drawn at 90° to the other.

The series actually runs-

[13] 23 x 23 [15] 39 x 39 [17] 65 x 65 [19] 107 x 107 [21] 175 x 175 [23] 285 x 285 [25] 463 x 463 [27] 751 x 751.

It is observed that any dimension is the sum of the previous two plus 3, e.g. 285 + 463 + 3 = 751.

Now each of these can be extended by adding a series of STEPS as mentioned elsewhere, and these are listed below as they are of interest.

13-23 15-41 17-70 19-117 21-193 23-316 25-515 27-837

The first numbers show the Order and the

-------- 15-39 17-70 19-120 21-201 23-332 25-544 27-887 second the dimensions of the square

----------------- 17-65 19-117 21-201 23-337 25-557 27-913 found e.g. [27] 913x913 which appears twice

------------------------- 19-107 21-193 23-332 25-557 27-921 In fact two solutions for most sizes

----------------------------------- 21-175 23-316 25-544 27-913 appear, the solutions being twins. Notice the

--- 23-285 25-515 27-887 rise and fall, viz. 837 887 913 921 913 887 837 and

--- 25-463 27-837 751 for Order 27. The last is at the

--- 27-751 low end of the range, 463 being particularly small for Order 25 and 751 for Order 27. In fact these two and those for higher Orders may be the lowest possible, and this puzzles me knowing solutions of [19] 106 x 106 and [17] 62 x 62 and 64 x 64 exist which are lower, whereas 23 and 39 for Orders 13 and 15 are the lowest.

Amazingly all the above squares can be so readily constructed by simply drawing a set of Elements 1,2,3,5... then a similar set at 90° and adding five extra Elements which always fit perfectly to form a square. Clearly the results are always Imperfect. The Reduction Index is always the Dimension of the square in size e.g. 837 for [27] 837 x 837.

These Squares only require 2 unknowns x and y to calculate them.

ARE THEY THE ONLY XY SOLUTIONS POSSIBLE FOR ANY SQUARED-SQUARES? check ****

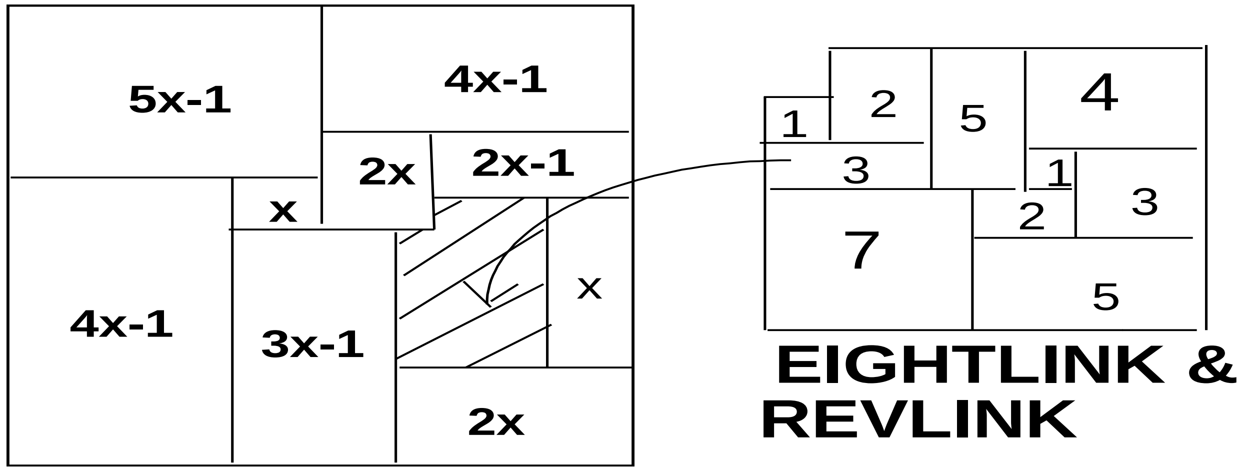
N6.4. REVERSIBLES

The pattern which forms a square but for a 1 by 1 kink in one corner will be referred to as a REVERSIBLE since it is always possible to completely reverse the pattern, one horizontal and one vertical. Although this means that two solutions exist in theory, sometimes they will be the same. This always happens where the rest of the pattern is symmetric.

Otherwise the solutions are Twins with identical Elements.

N6.5. USING OTHER PATTERNS WITH THE FIBONACCI SERIES

I observed from a selection of calculated Squared-squares discovered several other patterns which can be substituted for the Three Elements. For example -



EIGHTLINK & REVLINK

SOLUTIONS FOUND
[19] 106x106
[21] 178x178
[23] 295x295
[25] 484x484
[27] 790x790 .....
(those below 999 in si:

The algebra above proves that the results will always be a perfectly matching square i.e. (9x - 1) x (9x - 1). In the Above x= 4 + 3 + 5 or 12 making the solution [19] 106 x 106. When the series 1 2 3 5 is increased to 1 2 3 5 8 [21] 178 x 178 is obtained. With 1 2 3 5 8 13 [23] 295 x 295 is obtained, and so on.

Since the reversible (above 2) can be reversed (i.e. showing 1 3 7 horizontally instead of 2 5 4) Twins with Identical Elements also exist.

Furthermore each solution has Branch solutions by adding Steps.

Thus in compiling solutions, many complete groups of solutions can be found quite quickly and easily.

The Above pattern may be shown by the formula 5x - 1 4x - 1, 2x 2x - 1, 4x - 1 x, x, 3x - 1, 2x.

So far I have found the following patterns from which the solutions shown (plus their branches to 999) may be found:-

0. [2x - 1] x [2x - 1] x x-1, x-1 See L6.5 one of each

[13] 23 x 23 [15] 39 x 39 [17] 65 x 65 [19] 107 x 107 [21] 175 x 175 [23] 285 x 285 [25] 463 x 463 [27] 751 x 751

1. [9x - 2] x [9x - 2] 5 x-1 4x - 1, 2x 2x - 1, 4x - 1 x, x, 3x - 1, 2x Two of each

[19] 106 x 106 [21] 178 x 178 [23] 295 x 295 [25] 484 x 484 [27] 790 x 790

2. [7x - 2] x [7x - 2] 2x 2x - 1 3x - 1, x, 2x - 1, x 4x - 1, 3x - 1 one of each all with Crossovers

[18] 82 x 82 [20] 138 x 138 [22] 229 x 229 [24] 376 x 376 [26] 614 x 614 [28] 999 x 999

3. [12x - 5] x [12x - 5] 7x - 3 5x - 2, 2x - 1 3x - 1, 5x - 2 2x - 1 x - 1 x, 4x - 2, 3x - 1 Two of each

[20] 139 x 139 [22] 235 x 235 [24] 391 x 391 [26] 643 x 643

4. [12x - 7] x [12x - 7] 4x - 2 3x - 2 5x - 3, x - 1 2x - 1, 3x - 2 x - 1, 2x - 1 7x - 4, 5x - 3 One of each

[20] 137 x 137 [22] 233 x 233 [24] 389 x 389 [26] 641 x 641

5. [15x - 2] x [15x - 2] 8x - 1 7x - 1, 2x 5x - 1, 7x - 1 x, 3x, x 3x, 3x - 1, 2x two of each

[21] 178 x 178 [23] 298 x 298 [25] 493 x 493 [27] 808 x 808

6. [20x - 4] x [20x - 4] 11x - 2 9x - 2, 2x 2x - 1 5x - 1, x, 9x - 2 4x - 1, 2x, x 6x - 1, 5x - 1 Two of each

[22] 236 x 236 [24] 396 x 396 [26] 656 x 656

7. [25x - 2] x [25x - 2] 23x - 1 12x - 1, x 3x 8x - 1, 12x - 1 2x, 5x, 2x 5x, 5x - 1, 2, 3x two of each

[23] 298 x 298 [25] 498 x 498 [27] 823 x 823

8. [32x - 7] x [32x - 7] 18x-4 14x-3, 5x-1 9x-2, 14x-3 5x-1 2x x 2x-1, 2x, 3x-1, x 10x-2, 9x-2 Two of each

[24] 389x389 [26] 653x653
9. [43x - 7] x [43x - 7] 24x - 5 19x - 4, 5x - 1 14x - 3, 19x - 4 10x - 2, 2x 2x - 1 10x - 2, 9x - 2 x, x, 3x - 1, 2x, 5x - 1 Two of each
[25] 507 x 507 [27] 851 x 851

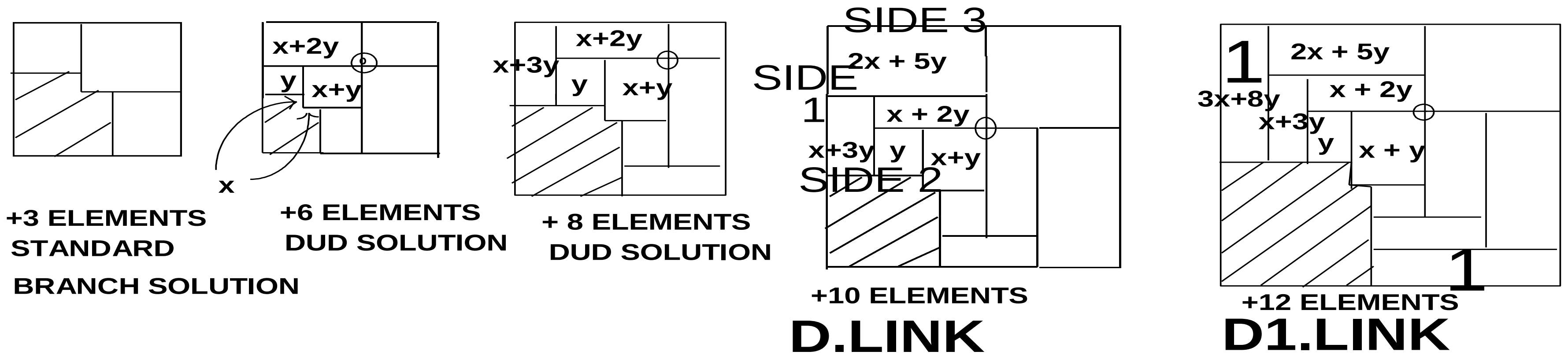
N6.6. VARIOUS BRANCH TYPES

It has been seen how Branch solutions can be obtained from absolutely any given Squared-square, but it is interesting to see that there are an unlimited number of other "Add - On" patterns which may be applied.

In applying these, the Order is raised - sometimes considerably, and often - but not always - it is necessary to up-rate and / or down-rate the Elements on the Original Square.

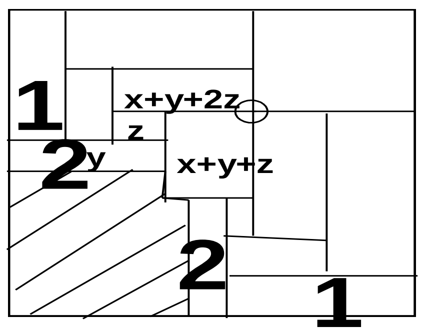
Below are four patterns with 6,8,10 and 12 added Elements. Note that each has 1 crossover point (circled), and that each can be split diagonally into two identical halves. Patterns 2 and 3 are ignored as they form Duds - Adjacent Elements of value x + y appear three times - but by adding further Elements as in Patterns 4 and 5 valid patterns are formed.

Subject to employing algebra and applying up-rating as required, the Patterns will always fit perfectly.
Starting with any given solution, an equation of the *type* e.g. "7x + 9y = 34" is formed, and since x is already known y can be readily found and turned into an integer. Thus the patterns always fit exactly.

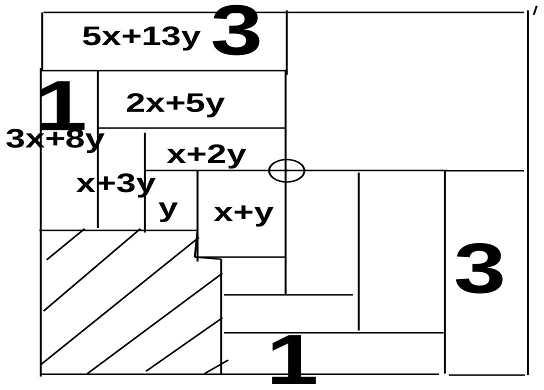


With Above 3 regarded as the Base Addition, it is found (that Elements can be added at Sides 1 or 2 or 3 (whenever there is more than one Element bordering the side so B1 and B2 is possible but not B3), with valid Solutions possible for each.

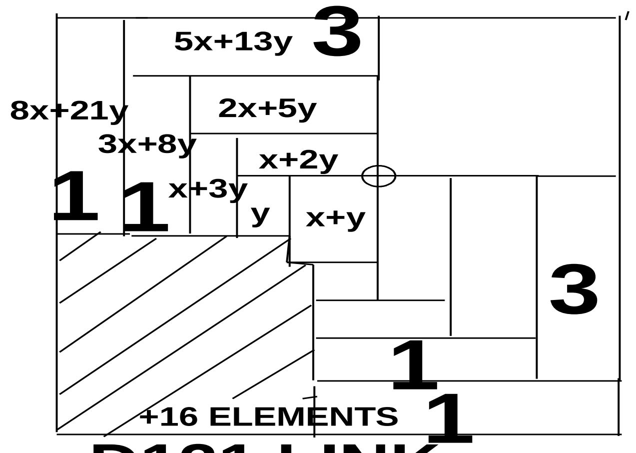
There are patterns for D D1 D2 D12 D13 D21 D131 D123 and so on, in fact an infinite number of them including D131313 ...ad infinitum D121212… ad infinitum. Furthermore ordinary Branch Solutions may be calculated at any stage as well! For each number following B the Order naturally rises by 2.



+14 ELEMENTS
D12.LINK



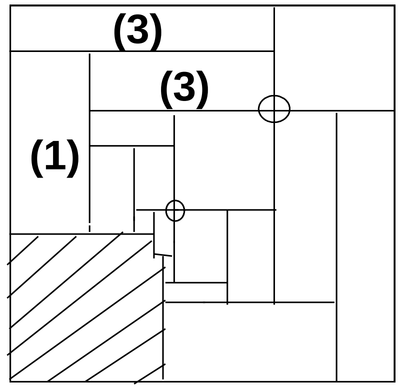
+14 ELEMENTS
D13.LINK



+16 ELEMENTS
D131.LINK

N6.7. CONTINUED BRANCH SERIES

Returning to the Base Addition B, a further layer can be added - giving two Crossovers. In fact the pattern can be extended to 3, 4 and as many Crossovers as we wish (indefinitely).

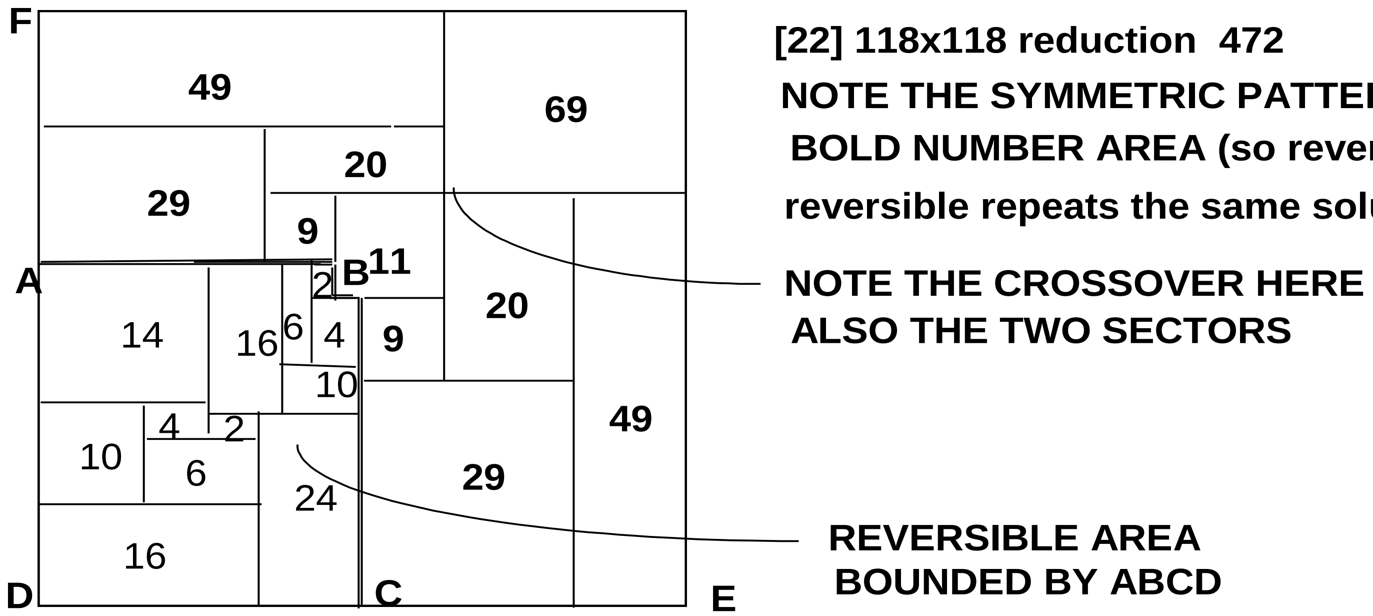


+17 ELEMENTS

D313.LINK

Multiple of a "D" Addition and further down a "D1 addition (see previous section).

There is another group of Squared-squares employing the growth Fibonacci Series 1 1 2 3 5 8 13 21 34 55 89 except that sometimes this series needs up-rating by 2 3 or some other number is sometimes necessary (e.g. 3 6 9 15 24 39 63…)
This is better explained below:-



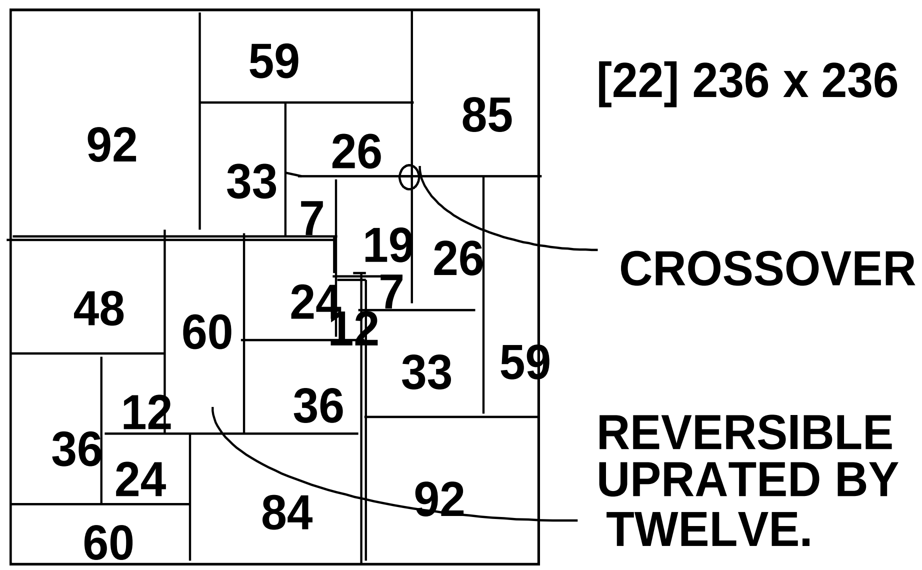
[22] 118x118 reduction 472
NOTE THE SYMMETRIC PATTERN IN THE BOLD NUMBER AREA (so reversing the reversible repeats the same solution)

Unlike previously, the Reversible is situated in one corner. Providing a suitable pattern is added (which must effectively form a square), a Squared-square is always possible.

But in above note the up-rating by two found to be necessary in the Reversible to avoid fractions arising elsewhere. The series found from this pattern is [20] 70 x 70 [22] 118 x 118 [24] 392 x 392 [26] 161 x 161 [28] 526 x 526 which involve varying up-ratings. All other Orders give dimensions over 999. The solutions all have sides 2245.

Now, suppose Elements are added at FA and CE in above. Then [22] 236 x 236 [24] 132 x 132 [26] 164 x 164 [32] 476 x 476 are obtained which again involve various up-ratings.

These all have a more desirable sides Index of S3334 and contain Crossovers. All higher Orders give dimensions over 999. Here is the example [22] 236 x 236 -

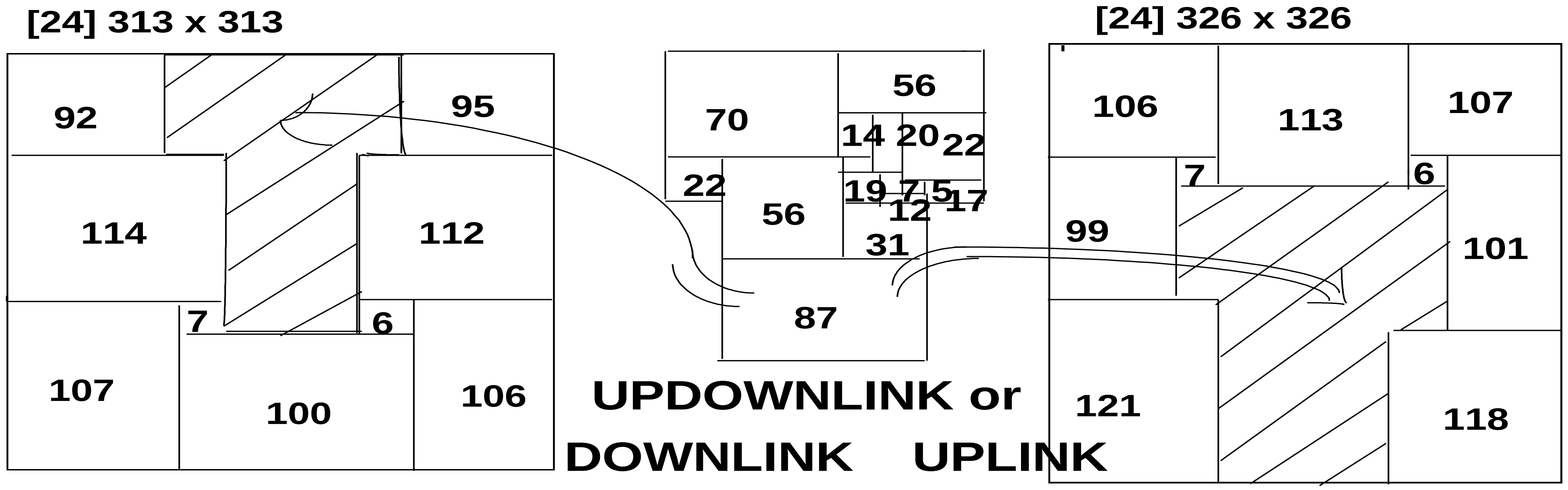


[22] 236 x 236

In all cases Simple algebra has to be used to create these solutions.

N6.8. AN INTERESTING LINK

Whilst assembling a catalogue of Squared-squares I came across two Squares with a big chunk of identical Elements as shown in below 2 -

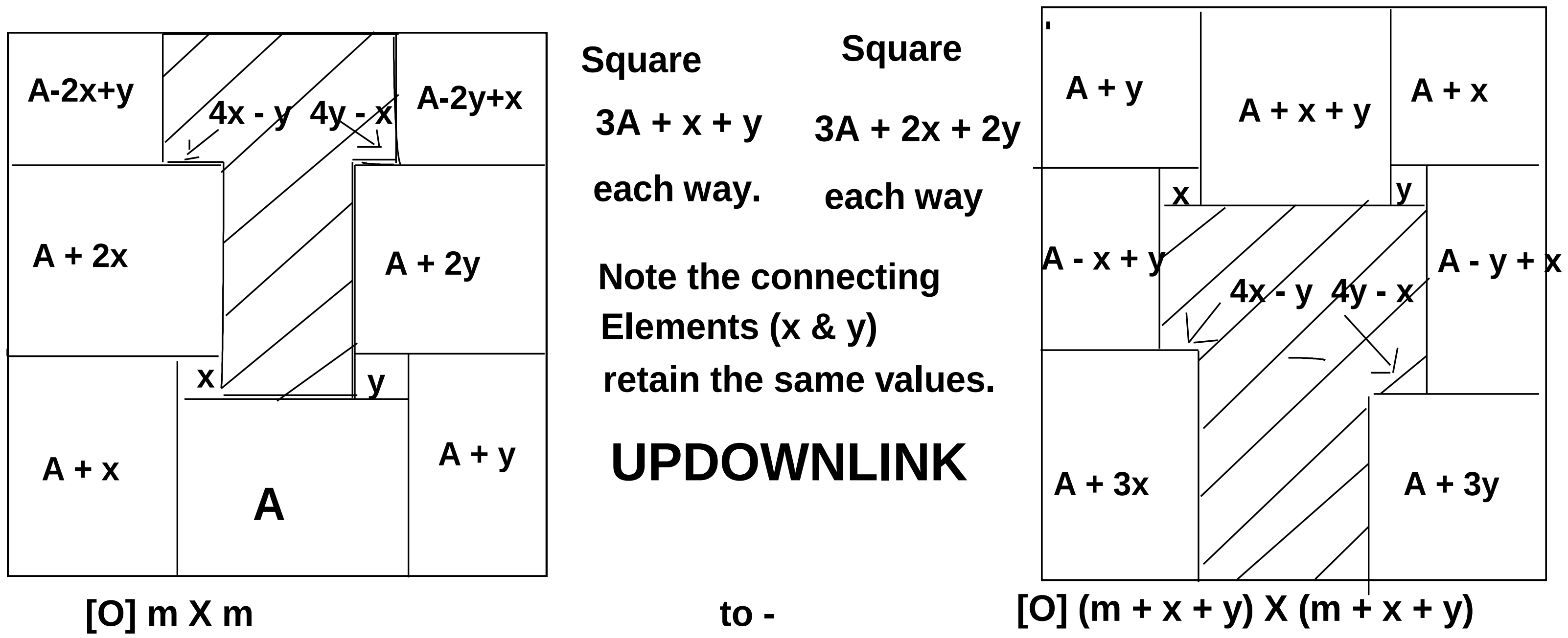


Above 3 is created from Above 1 by removing the bottom Elements 107 100 106 - replacing 6 and 112 with 118 - replacing 7 and 114 with 121 - 99 from 121 - 22 - 101 from 118 - 17 and adding the other Elements as shown. The result is another Square but a slightly larger one.

Note the positions of the two connecting Elements 7 and 6 in above 1 and above 3.

N6.8.1. PROOF THAT THIS LINK WORKS

Simple algebra can be employed in one pattern and transferred to the other, proving the relationship is always true -



The second Square is of different size, in fact x + y larger in dimension.

The Elements of sizes A + y and A + x remain though in different places.

The Formula Above may be improved by replacing x + y with M - 3A (A being the middle Element). Thus in Down-link to Uplink <u>[O] m X m becomes [O] (2M - 3A) X (2M - 3A)</u>

For this property to work it is necessary for x to link with A + 2x as well as A + x, and y to link with A + 2y as well as A + y. Sometimes the Resultant solution is a Repeater (Solution with a double ended Element) and is not useful.

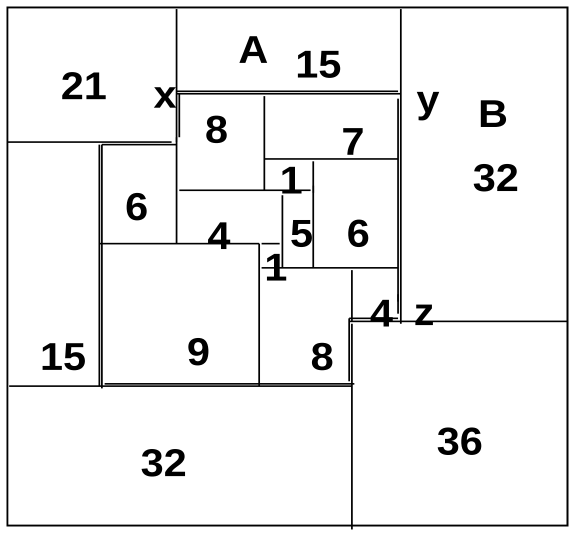
Some Examples of this not too frequent Link are:-

[23] 57 x 57 and [23] 60 x 60 [24] 313 x 313 and [24] 326 x 326 [24] 322 x 322 and [24] 335 x 335.

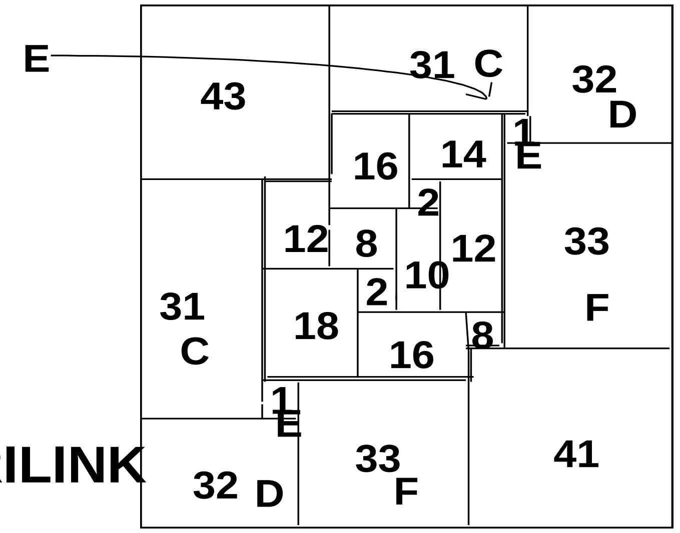
The Sides Index is 3333 or 3334 or 3335 or 3336 ...etc.

N6.9. A NEW LINK - TRILINK

An amazing link discovered by the Writer. Explanation of how these are found is given later -



TRILINK



[17] 68 x 68 (1) SIDES 2233 [21] 106 x 106 SIDES 3333

A Squared solution with Sides 2233 has been chosen in above 1.

Now look at the Elements xy and yz. 8 + 7 is 15 but 7 + 6 + 4 is different at 17.

In a few solutions xy and yz are found to be equal in which the Resultant Square is of Zero type and unacceptable - <u>otherwise absolutely any Squared solution of Sides 2233 can be employed!</u>

Next we replace Elements A B with Elements C D E F which is not difficult with a little practice.

With xy and yz deduct the smaller from the larger and see if it is divisible by 4.

17 - 15 is 2. 2 is not divisible by 4 until doubled. So in above 2 it was found necessary to double all central Elements.

Now 34 - 30 is 4 and 4 = 4 x 1 and E is assigned the value 1. C D and F are easily calculated as 31 32 and 33 and found to fit, but in half of cases the value E will need to appear where indicated to fit correctly.

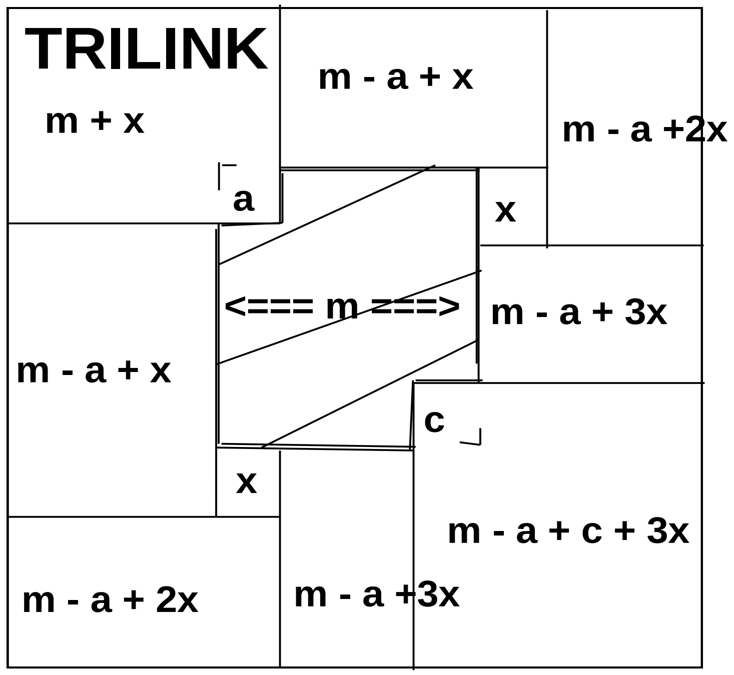
Though the Resultant Square is effectively a ROOT Solution, the Original may be a ROOT, STEM OR BRANCH Solution - so we can not only convert a Branch Solution (the lion's share of which are Sides 2233 anyway) to a Root Solution!

Also, if required, <u>four</u> Series of Branch Solutions can be calculated from Above 2 as well!

Note that the Order increases by 4, also that two of the Corners are the same (32) and two different. As these outer Elements are Symmetric when viewed on their own, these Solutions be considered a more inferior type of Sides 3333 type.

N6.10. FORMULA FROM SOURCE TO TRILINK

The algebra for this important and very extensive Link is calculated below, and is less daunting than it looks! It is possible to remove all the x's, but the expressions are better left as shown -



Given: Original Square m X m with Corners a and c3 (removed), & an Element termed x, the Added Elements assume the values shown, BUT if x is a fraction, uprate everything by 2 or by 4 as necessary.

<u>The value of x is found to be (a - c) /4</u>

<u>[o] m X m becomes</u>

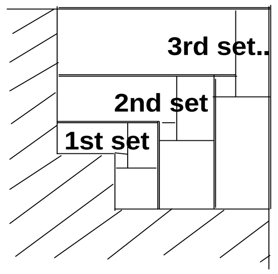
<u>[o+8] (3m - a - c) X same</u>
<u>Uprated by 1, 2 or 4 as necessary.</u>

In arriving at the Formula [o + 8] (3m - a - c) times itself, note that the 4 branch Elements (previously shown) are irrelevant. Also that this Formula dispenses with the value x.

To calculate the individual Elements it may be simpler to use the value <u>x which is (a - c) ÷ by 4, 2 or 1</u>. x may be positive or negative. The construction fails when a = c, which makes x zero, otherwise any Source Square will do - even if both a and c are Repeaters, or one forms part of a Non-simple rectangle.

N6.11. EVERLASTING TRILINK SERIES

Without showing full workings, the Reader can verify that adding four Elements in this way can be done again and again, and fit!



EVERLASTING TRILINK

Above three sets of Additions have been shown. Each Addition adds 8 (not 4) to the Order - remember that identical Elements will show also in the south-west corner as well, and that the north-west and south-east Corner Elements need to be suitably increased to fit. (Also no Elements are actually replaced this time).

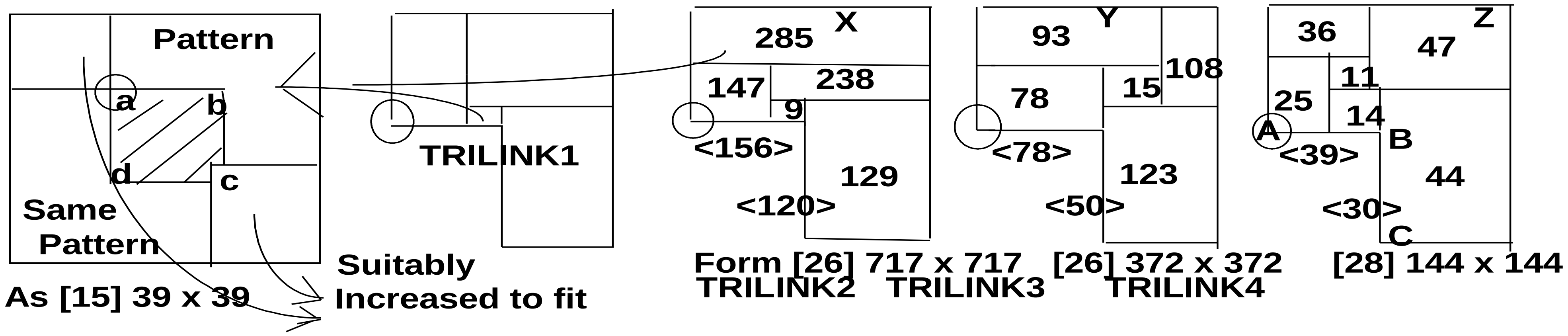
Under the original Trilink in M7.6 the Central Elements have to be multiplied by 1 2 or 4 according to the Solution, but with further additions (this Section) the increased group of Central Elements always need to be multiplied by 2 or 4 to avoid fractional Elements. Once again, four sets of different Branches can be found from each group, making a bewildering large quantity of Solutions!

N6.12. OTHER LINK PATTERNS SUBSTITUTES OF TRILINK. TRILINK1 2 3 … SERIES

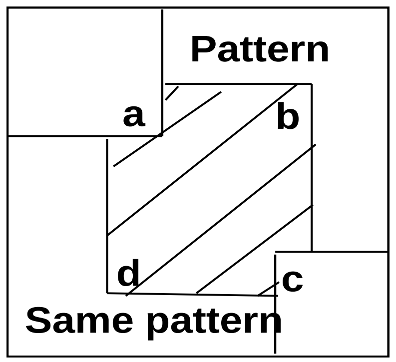
There are endless types of patterns or links which can be constructed from the majority of already known Squared-squares. Apart from the Trilink Pattern (Below 2), below 3 to 5 are the simplest of this set with 5 5 and 6 Elements respectively.

Some algebra is used to calculate these values, e.g. below 5 is taken from [15] 39 x 39 with the Element 9 taken out. We are looking for Elements totalling 39 (AB) and 30 - that is 39 - 9 - (BC). Sometimes fractional values mean the Origin must be suitably up-rated - but not in this example. The Solution is completed by adding two easily calculated corners and repeating the NW Elements in the SE.

From many Squares, four different solutions can be found from each pattern. The Extra links are coded X Y Z. Yet another prolific way of producing more Squared-squares!

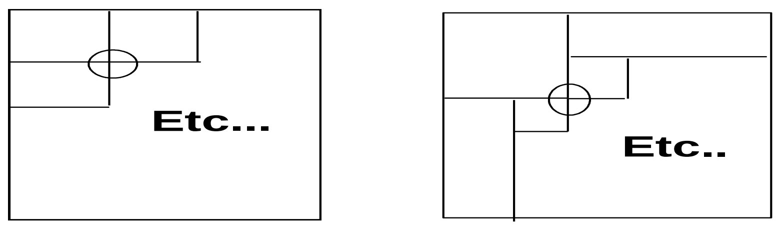


In these constructions note the presence of a Crossover in every case. This occurs where only one corner (c in above) has been amended. However all the Patterns also apply with two opposite corners amended and then no Crossovers occur. In fact TRILINK shown earlier applied to the pattern Below (TRILINK A), not that above as it gives an Invalid Solution (Below 2) until a Branch is added (Below 3). Thus there are in effect two groups of TRILLINKS although some give Invalid solutions containing a Repeater.



The TRILINK patterns from this follow those shown Above, except that no Crossovers ever occur.

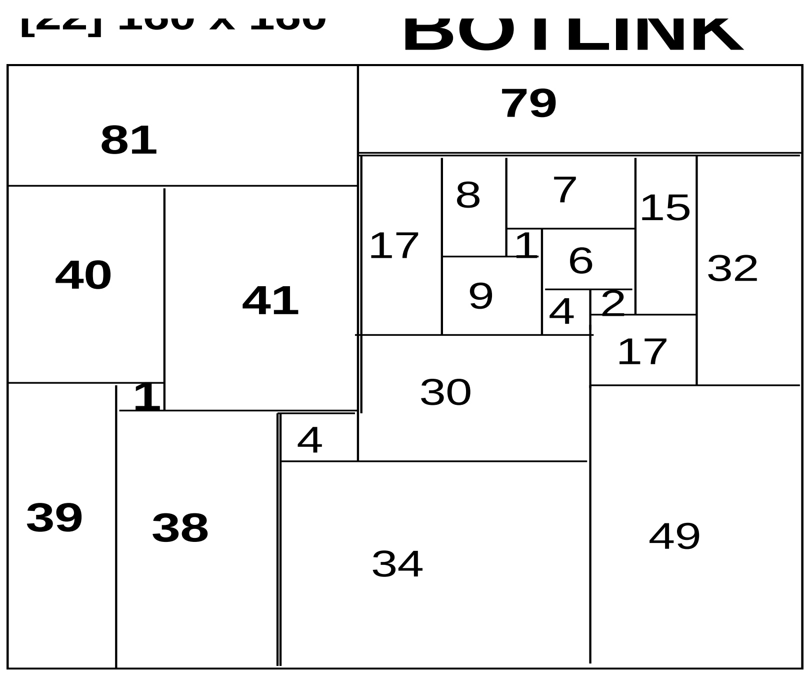
TRILINK A B C D

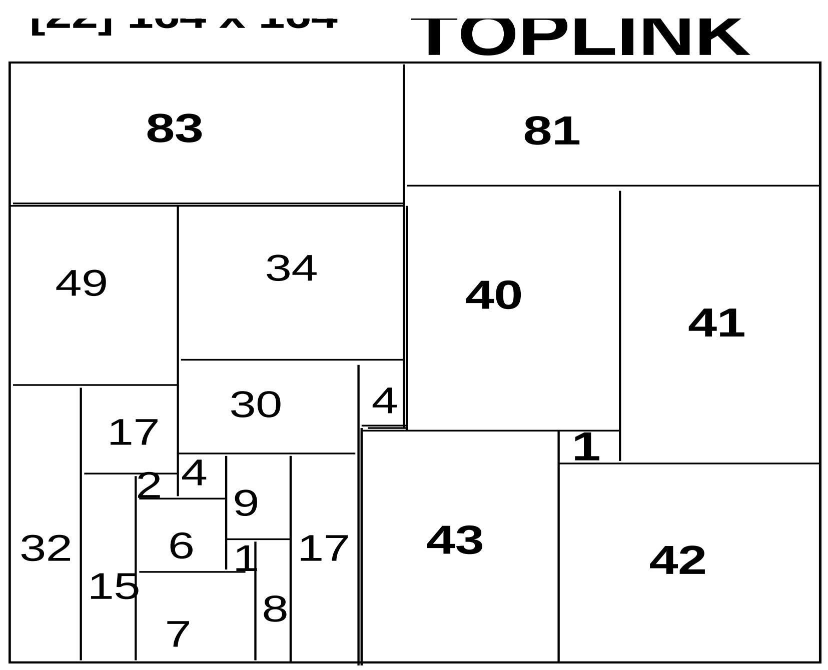


More needed to clarify Codes for these Links!

N6.13. TOP-LINK

A Link with a tiny resemblance to UNLINK - DOWN-LINK was seen in the following -

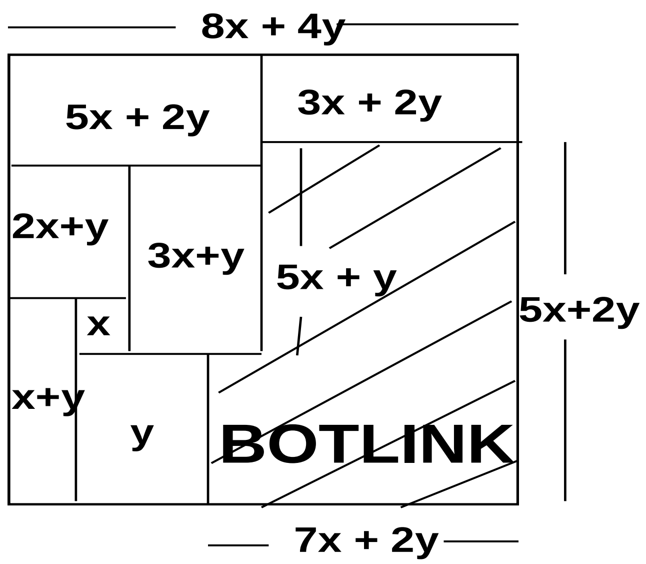


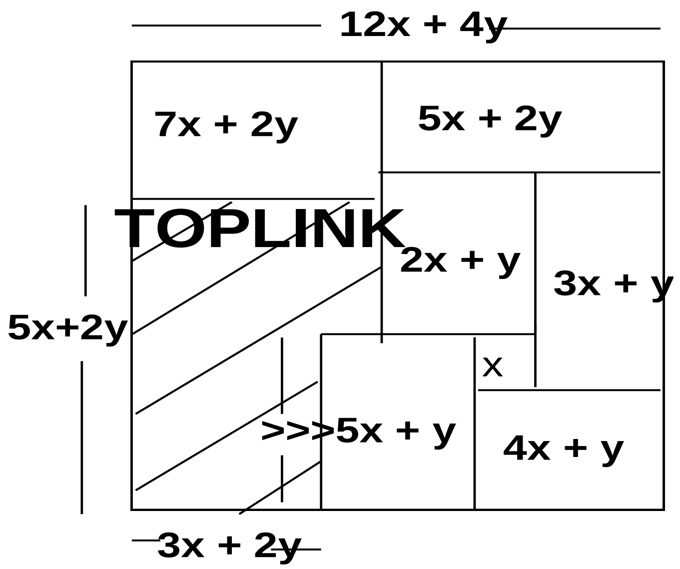


Note the series 41 40 39 38 turns into 40 41 42 43 in above 2, also that 79 (i.e. 81 minus 2) becomes 83 (i.e. 81 plus 2).
The remaining Elements in above 1 are turned upside-down in Above 2!
Though it fits notice that the slightly bigger square in above 2. BOTLINK and TOPLING are never Twins.
Although with Squares of this type pairs are always possible, one may have Adjacent Elements and be Invalid.
This occurs when the area of light Elements has a border of only one Element somewhere.

N6.14. ALGEBRAIC PROOF OF BOT-LINK TO TOP-LINK

This is easily seen by transposing the Algebra from below 1 to below 2 and seeing that the shaded areas remaining are identical -

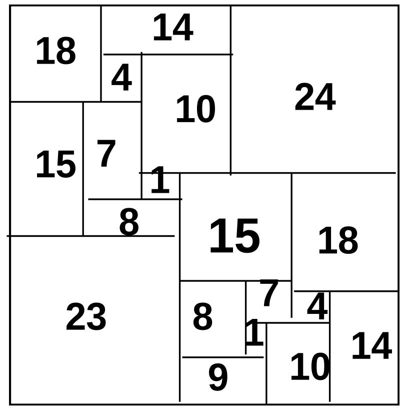




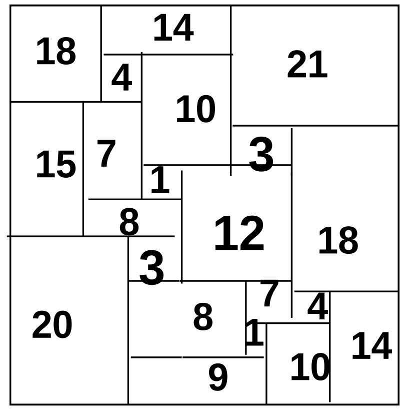
NB. The term TOP-LINK is used when the wider end of the shaded area is at the top; BOTTOM-LINK at the bottom.

N6.15. SLIP-LINK

It was shown earlier how Squares could easily be formed by putting two rectangles at right angles to each other (with one corner removed) and adding two extra corners.
But the following Link can always be applied though sometimes with Invalid results -



[19] 56 x 56



[21] 53 x 53

Note:- Top left & bottom right form the Rectangle [9] 33 x 32, but the Original rectangle may be a Repeater or some Invalid types and still give a good result.

SLIPLINK

A little investigation reveals that this Link always works and no algebra is required to prove it! In Above 1, Element 9 would complete the top left rectangle and Element 15 shows at top left of the bottom right angle Now 15 - 9 is 6 and half is 3 15 - 3 is 12 which replaces the 15 in Above 2 with two 3's added as shown. The subtraction often gives amounts like 8½ or 13½ in which case everything requires up-rating by two, and nearly all Elements are even. If no up-rating is needed then the Result - Order 2 greater - is of course a little smaller and thus carries a greater Reduction Index.

Yet again, a large number of Squares can be linked in this way. Good Origins will always give good Results, but the reverse is not always true e.g. Squares with Repeaters at two adjacent corners often arise.

N6.16. GENERAL FORMAT OF SQUARED-SQUARES

The general division between Squares being either ROOTS (+ STEMS) on the one hand and BRANCHES on the other first seemed the best format available to me - until many Links were discovered, and complicated this idea.

It was clear that a more general format was needed.
Also the presence of Compound and / or 1 or 2 Repeaters in what would have formed ROOTS was a nuisance!

N6.17. MAINSTREAM, STREAM AND EXTRA SQUARED-SQUARES

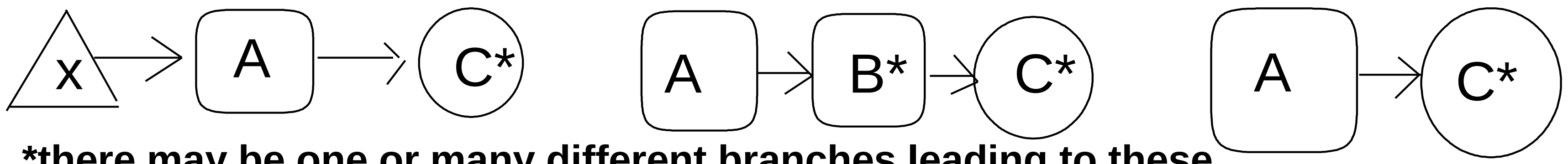
The intention is to allocate all Squared-squares into three categories A B and C, with known Links connecting them!
Look at this -
1. A Known FORMATION makes up the MAINSTREAM. The Formation may involve Squared-Rectangles or parts of them added together, but not Squared-squares. This is not a Square Solution, simply its construction.

There is no obvious FORMATION for some MAINSTREAMS.

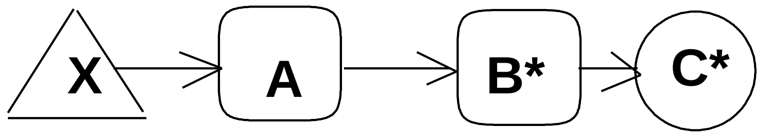
2. Any given Squared-Square is either a MAINSTREAM or connected to one, by one or several LINKS.

From any MAINSTREAM an infinite number of other Squares can be constructed (if no restriction is put on Orders or Dimensions, to be exact). These "other" Squares are mostly EXTRA with the rest SOURCE Solutions.
3. SOURCE Solutions flow only from a proportion of MAINSTREAMS. Otherwise, the EXTRA Solutions flow directly from the MAINSTREAM.
4. EXTRA Solutions always flow from MAINSTREAMS and sometimes also from SOURCES.

When any given Squared-Square is analysed, only MAINSTREAM and EXTRA Solutions A and C always apply. In others a FORMATION and or STREAMS may too, X and B. So the above can be replaced with one of these combinations -



*there may be one or many different branches leading to these.
Endless circles C each preceeded by arrows could also be drawn



N6.18. MAINSTREAM SQUARE FORMS

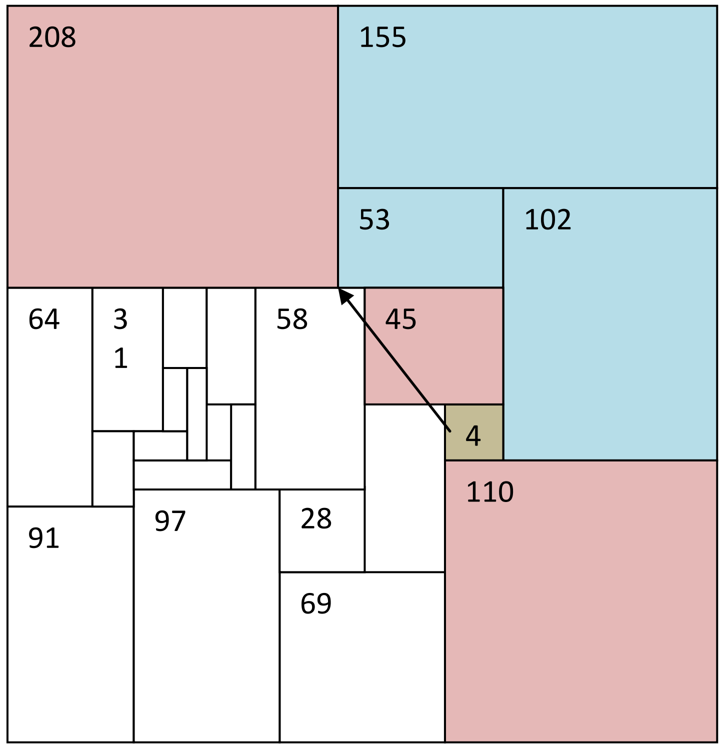
a. MAINSTREAM Squares divide first into ***Valid and Invalid***.
"Invalid" needs to be confined here to one of the following -
1. Having one Repeater Corner, or
2. Having two Repeaters in ***diagonally opposite*** corners.
(NB. Not adjacent opposite corners), or
3. Having a Smaller Rectangle situated in one, and only one, corner. (Called Compound).
There may be no Repeater, one or two. Any Repeaters must be within the Rectangle and/or the diagonally opposite corner.
b. MAINSTREAM Squares then divide into two main Groups of Sides Indexes. Those with a single "2" and those with no 2's. They cannot have sides of say S2223 2233 2323 at all, but examples within the two Groups are 2333 2334 2343 2444 2435...
and 3333 3334 3344 3435 3444 3445 4445.....
But in forming five Types 1 to 5, the Corner Elements and their status are also essential.
In order to record MAINSTREAM Squares it is also useful to fix rules as to which Elements appear in which corners, and these are shown below in the 2nd row.
P denotes corners which can be reused one at a time in Links to form new Squares, whereas C denotes Compulsory corners which must all be reused to obtain the first Valid Squares. In Type 5 for example, north-east corner A has to be extended followed by south-west corner C (with other Elements) to obtain the first Valid format, though A followed by C will equally suffice. In Type 1 any of four corners can initially be extended, with four lots of Branches following.

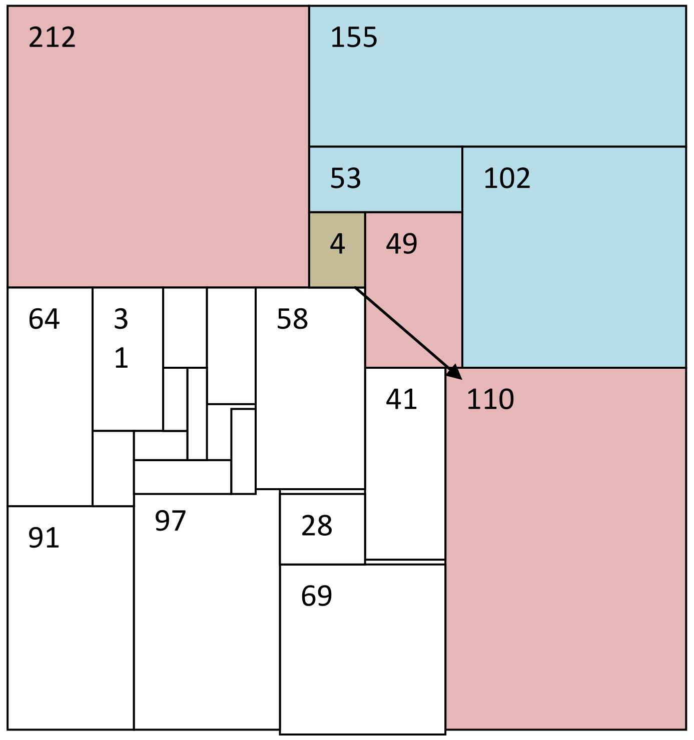
Type 4 can be used for 4 fold Squares. Only one corner can be extended (without duplicates occurring) ...

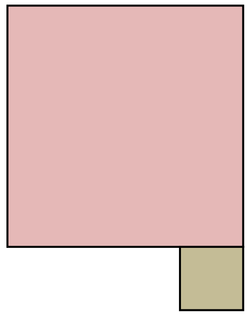
N7 SWITCH LINKS

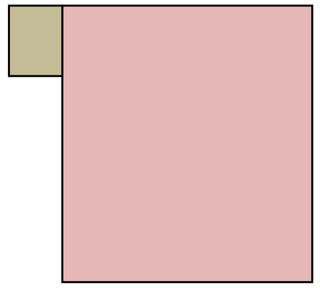
N7.1. SWITCH LINK

An interesting Link is shown below - which may give rise to either Imperfect or Perfect Solutions.
The Resulting Solution is a different size from the Original - the second being 4 smaller than the first - the value of the Element switched south-east as shown. Note that Blue Elements stay the same, as does 4 which SWICHES position. The White BODY area retains the same Elements.
The salmon Elements reduce by 4 (value of brown Element).







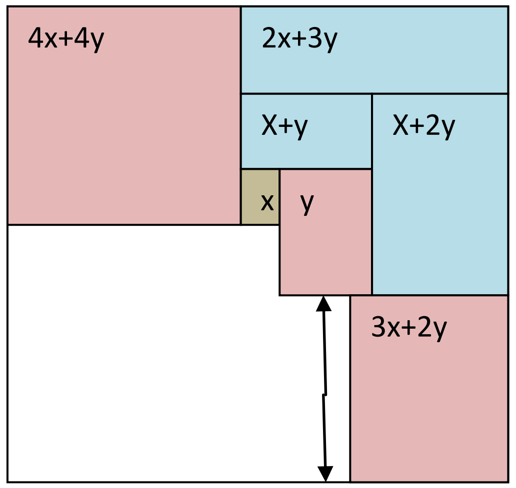


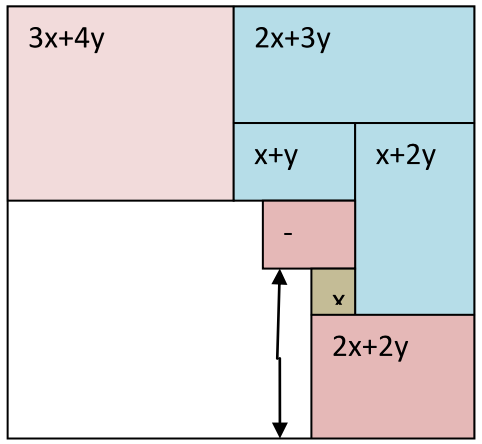
[25] 363 x 363 Perfect [25] 367 x 367 Perfect

Of course if a Solution of the 2nd type is known it can be converted to the 1st type. Decreasing salmon Elements.
If the brown Element is found to be negative in for <u>these patterns</u>, the patterns can be redrawn making it positive as shown in 3rd & 4th diagrams respectively and suitably adjusting the other coloured Elements.

N7.2. PROOF OF SWITCH LINK

Switch Link is easily proved by algebra. There must be more than one Element between the arrows shown or an adjacent Element occurs.
Note the switch of the Brown Element x which replaces a line 2x in length, not x as we might think! Though confusing, it all fits nicely.

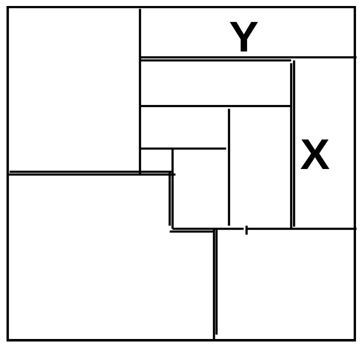


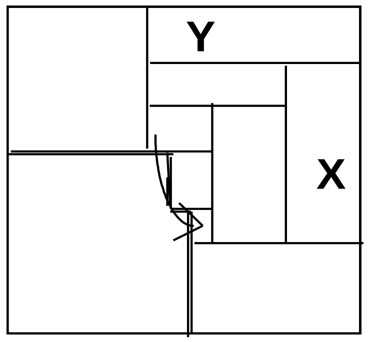


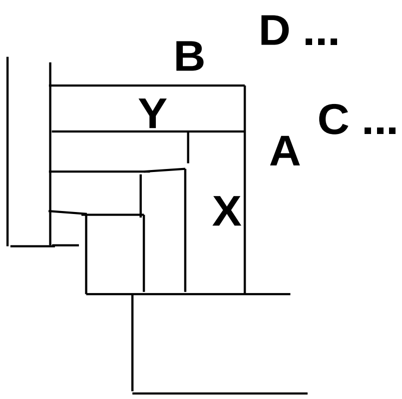
Upon investigation it will be seen (leaving the Reader to check) that the area (at SW) uses the same algebra. However the Square is 6x + 7y in size in the first and 5x + 7y in the second i.e. x smaller in the second.
Note that 2x + 2y is found in Above 1 by realising that the square dimension is 6x + 7y from the top Elements (being 6x + 7y less 2x + 3y less x +2y).

N7.3. EXTENDING THE SWITCH LINK SEVERAL TIMES

This is easiest shown as below where Elements X followed by Y have been added to both patterns. This does not give further solutions from the two already mentioned, but it does mean that if such as Solution becomes known then it is easy to find the other from it. I leave the reader to test this with simple arithmetic with hypothetical numbers, but assure the reader that both solutions are always Squares.
The operation can be repeated as many times as required by adding A (at X + Y) and B from it. Then C and D and so on.
It is important to stress that the Area (south-west) will alter in size each time, and will therefore have different Elements each time.



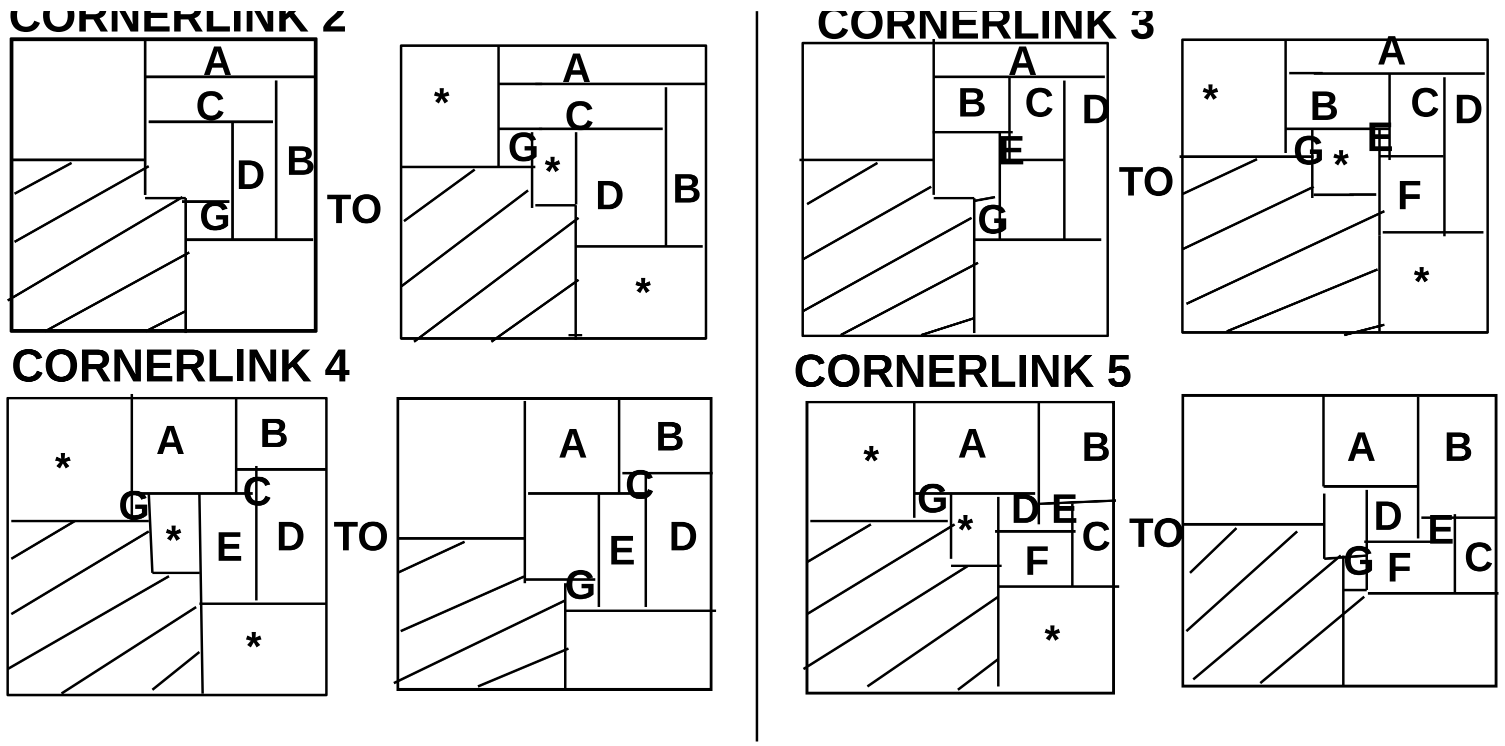




(and similar pattern as 2nd one)

N7.4. VARIOUS SUBSTITUTION PATTERNS

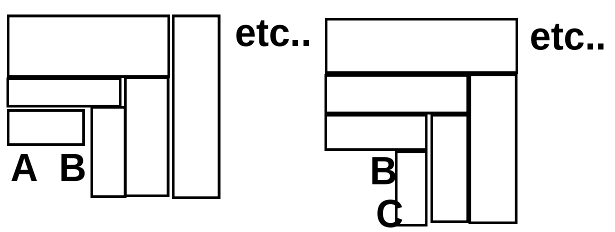
I have found several corner patterns which can be substituted for another with the shaded area staying the same. To use the Links it is necessary to have already a solution available of one of the types. Then the Link Solution can be easily found by arithmetic. The Resulting Solution will be valid but may change from Perfect to Imperfect or vice versa. For “CORNERLINK” please read “SWITCH

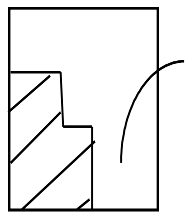


LINK”.

Where an asterisk * is shown the value reduces by the value of G from the other in the pair. This Solution is also smaller - by G in fact in each case. A to F indicate Elements which remain the same in each pair.

Look at A & B in Corner-link 2. If these are removed then there are still Substitute patterns which work in theory but one of them is Invalid (Non-Simple) and useless. The same is true for A & D when removed from Corner-link 3. No useful pairs are possible.



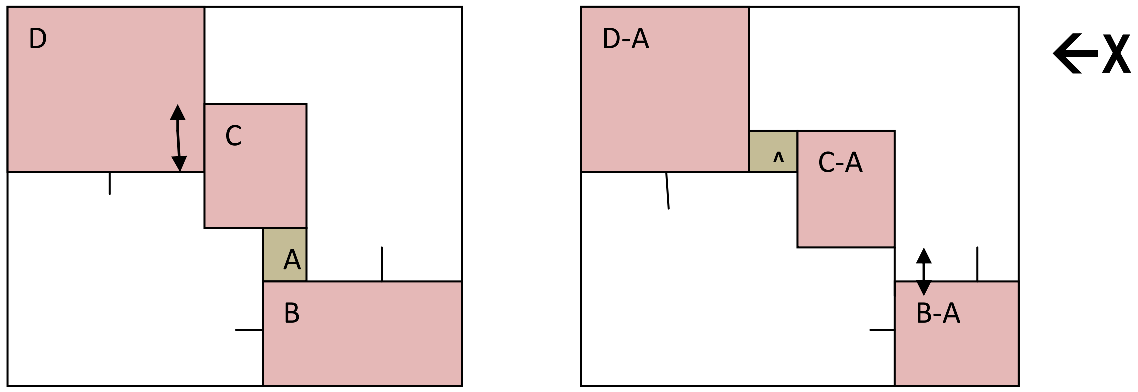


A PATTERN
WHERE SOLUTION
IS A SQUARED-SQUARE

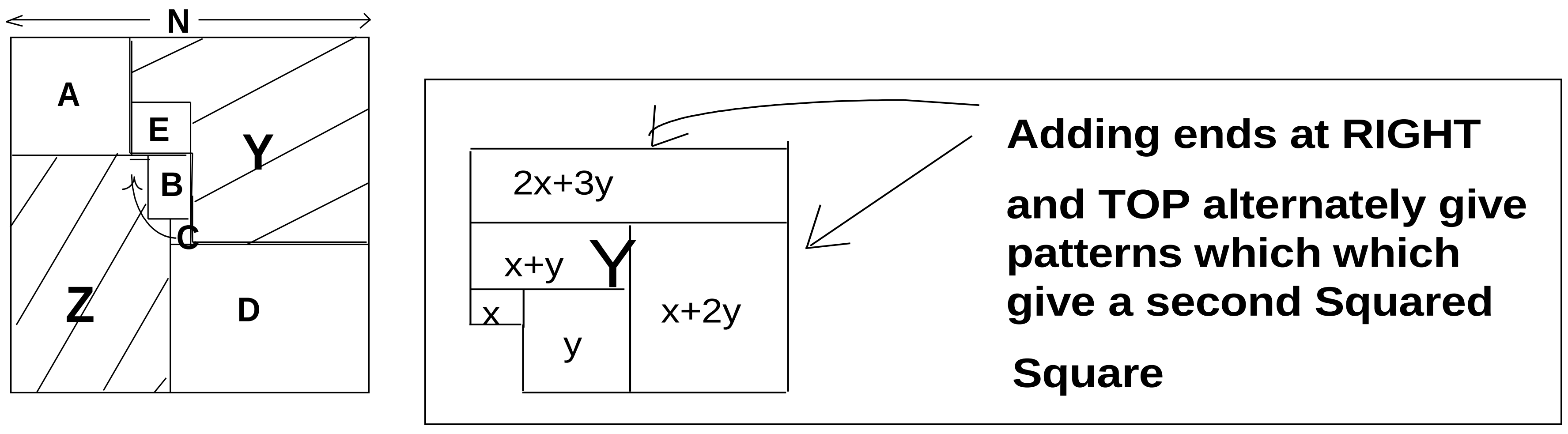
Now as before pairs of Elements can also be added which will totally alter the Elements in the shaded area but the Substitutions hold good. For example In Corner-link 2 adding an Element at A & B and another at A and A & B. Further pairs can be added ad infinitum. The same can be done in Corner-links 3 4 & 5. But in the case of Corner-links 4 & 5 the additions can alternative start at the top (A & B) and at the right (A & B plus BC) and repeated as many times as desired. Naturally, unless a Solution is already known of one type or the other, no Pairing solution can be found.

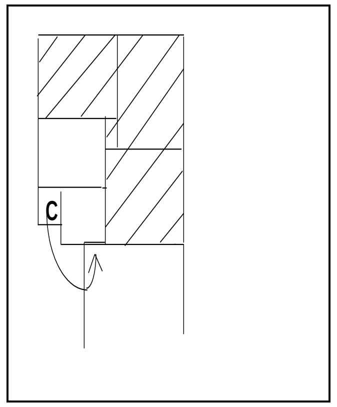
N7.5. GENERAL SWITCH-LINK PATTERN

Look at the pattern shown at the top & right of above 3. It is soon found that for many given patterns a corresponding pattern cannot be found. So what sort of patterns do have a Substitution Pattern - whether or not Valid?
Now look at below 1. Where the Squared-Square solution is of this type, which can be divided into two sectors each of which remain exactly the same and four Elements A B C & D. B C & D change value and A changes position but stays the same.
Now as the Squares can be seen to have dimensions (B + C + D - A) in the first and (B + C + D - 2A) in the second, this means that <u>whatever pattern is contained in X, a substitution is always possible</u> - except some Solutions may be found to be Compound or Adjacent. Note the dash marks indicating that 2 or more Elements must occur here to avoid Adjacent Elements.
For these patterns to be valid the arrowed lengths have to be 2x in length, <u>not</u> x.
The algebraic proof of this is easily verified.



Dimensions of first: D + C + B – A times D + C + B – A.
Dimensions of second: D + C + B – 2A times D + C + B – 2A
NOTE: In theory Perfect Solutions listed on the Internet with one of the above patterns can give rise to another Perfect Solution!
However many patterns at Y do not give a valid second Squared-Square so we now examine those which will work.





Taken from the first diagram in N7 it is found that by adding 3x + 5y at right, a second Squared-Square may be calculated from the first. If then 5x + 8y is added at the top, then again the pattern formed is suitable for a second Squared-Square. Then if 8x + 13y as added at the right the same applies and so ad-infinitum!

But the third diagram above also works with the Element C switched around as shown, and/or with additions at right & top alternately. I believe that if another 4 Elements are then added at the top right corner (i.e. similar to the shaded 4 Elements) that this pattern will also work.

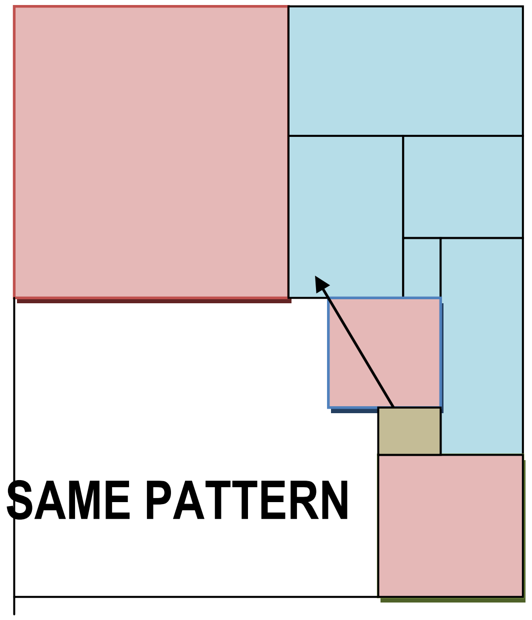
So when an actual Squared-Square of the required format is found, there is always a second easily found from it (though not always Valid).

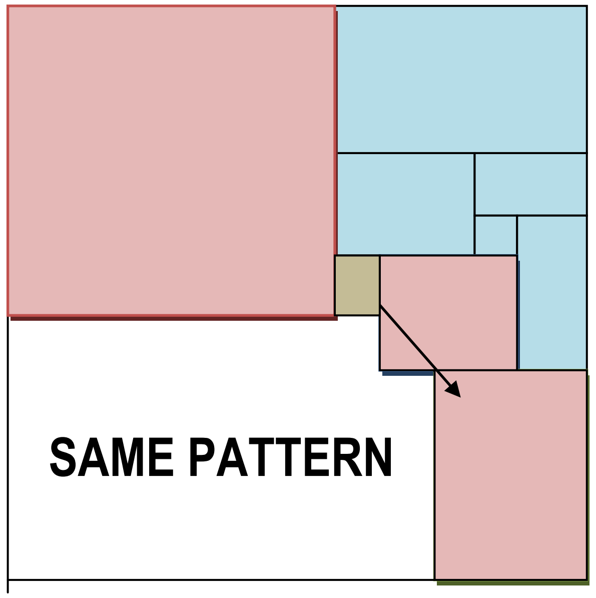
e.g. [25] 363 x 363 & 367 x 367 and [25] 378 x 378 & 385 x 385.

25 is the lowest Order possible for these Simple Perfect Squared-Squares. [26] 444 x 444 would give another solution [26] 429 x 429 but the latter contain adjacent Elements of 56 and so is unfortunately Invalid.

N7.6. SWITCH LINK PATTERN WITH FOUR ELEMENTS AT RIGHT

Another important find for the purpose of finding new Perfect Squares is from these patterns -





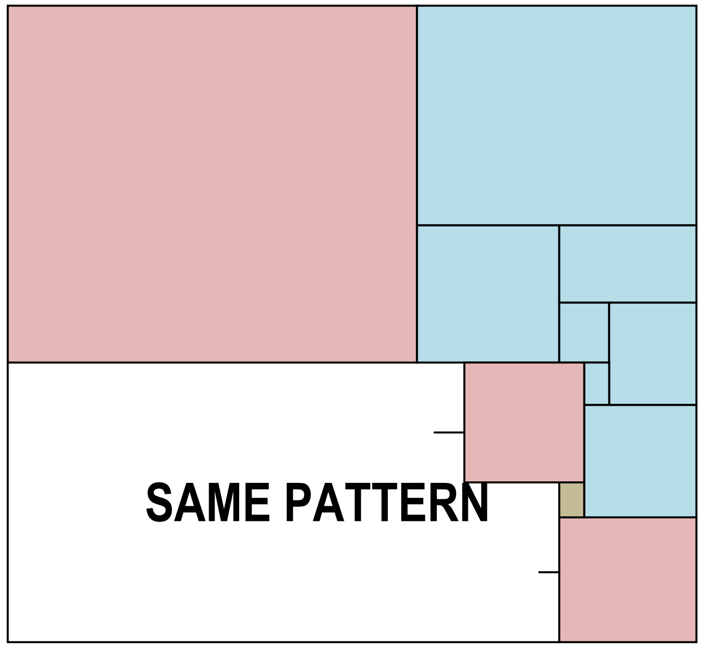
[67] 5798 x 5798 Elements top left to bottom right 3485 2313 1172 1141 31 1110 955 124 1234 TO (left) PERFECT

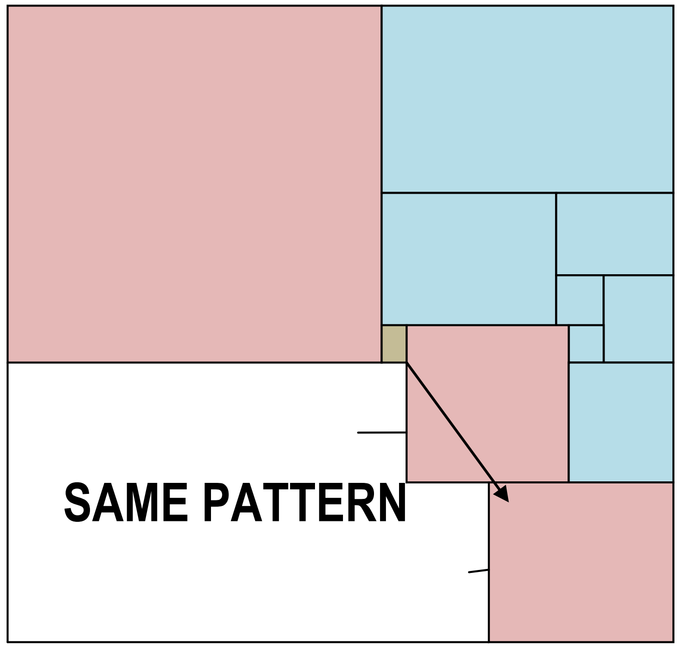
[67] 5922 x 5922 Elements top left to bottom right 3609 2318 1172 1141 31 1110 124 1079 1358 (right) PERFECT

The remaining areas in the south-west corner remain unchanged.
Note that the Blue Elements are also unchanged, but note the changed position of Brown Element 124.
Only three Elements have changed shown in salmon. It is important to check that their values do not appear in the south-east BODY – if so the Result is IMPERFECT. However the Top Left Element is never duplicated and the Bottom Right one is unlikely to appear. If duplication arises it is most likely with the Middle Salmon Element.
So in vetting the many Internet Perfect Solutions, wherever a Solution of the Left Hand format is found, another Solution is found providing that there are two or more Elements border the Green Element in the Patterns. Usually the result is Perfect.

N7.7. SWITCH LINK PATTERN WITH FIVE ELEMENTS AT RIGHT

The previous patterns had 3 and 4 Elements at the right hand side. We see now one with five Elements.
Outside the 'Body' area are 11 Elements on each side.
The salmon coloured Elements increase by the size of the Brown Element in the 2nd diagram.
The blue Elements stay the same in the same positions.
Though the Brown Element stays the same it SWITCHES north-east as before!
It is possible for the middle Element in the resulting solution to be repeated in the body.

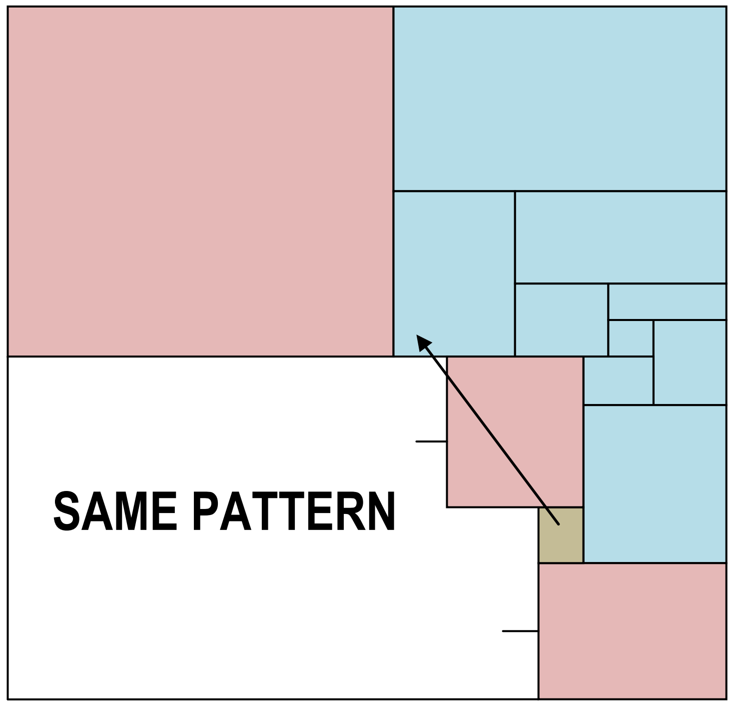


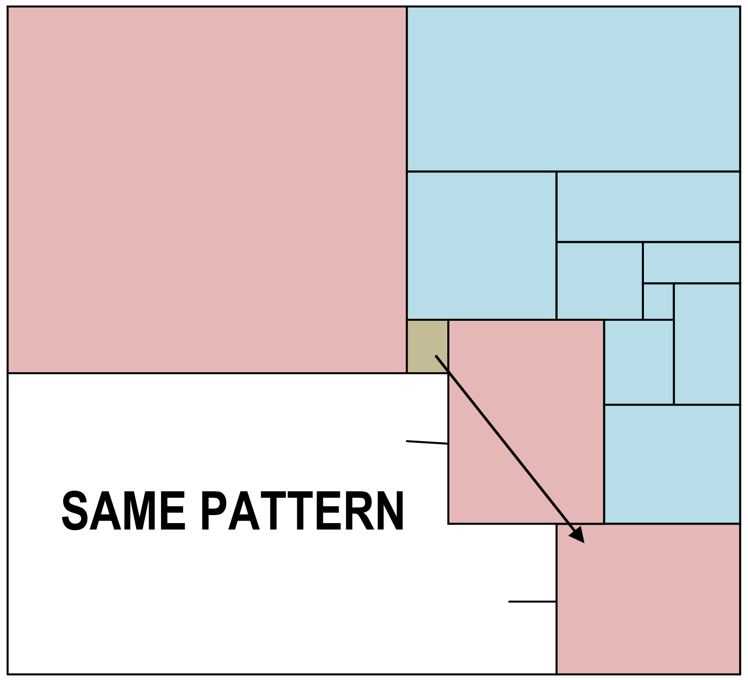


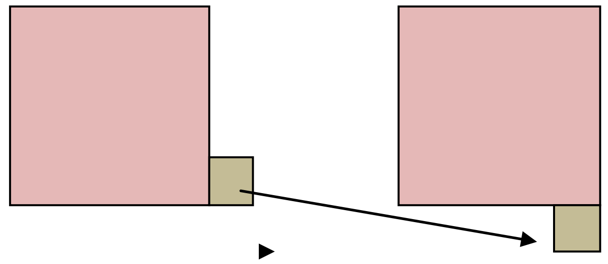
An example of the above is [68] 9772 x 9722 Perfect which converts to [68] 9946 x 9946 where the Code is 6149 3797, 2178 1619, 559 1066, 174, 2062, 501, 1561, 1909.

N7.8. CORNER LINK PATTERN WITH SIX ELEMENTS AT RIGHT

This one has six Elements on the right hand side, and 13 Elements surrounding the BODY. Once again the salmon Elements change value, the Blue ones stay unchanged. The Brown Element stays the same value but SWITCHES north east as before.
The Resulting Solution is larger if the Brown Element has a positive value, but smaller if it is a negative value. Note that changing it to positive means altering the pattern in practice.





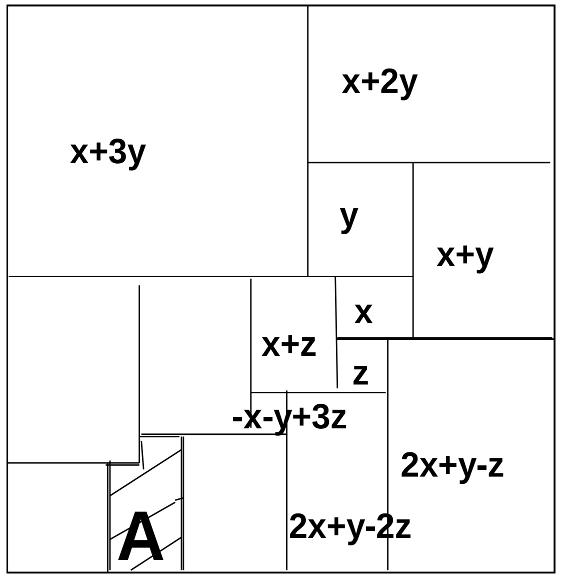


Negative Pattern, in practice, changes as shown to here

An example of the above is [57] 5094 x 5094 Perfect which converts to [57] 5413 x 5413 where the Code is 3456 1957, 1180 777, 403 374, 29 345, 319 977 316, 661, 1299.

N8. TEMPLATE SQUARES

N8.1. TRIAL & ERROR SQUARED-SQUARES USING TEMPLATE-SQUARES



A CONSTRUCTION *
USED FOR CREATING
SIMPLE PERFECT
SQUARED SQUARES

*ONE OF MANY

There have been thousands of spectacular Simple Perfect Squared-Squares recently created by using the above pattern which the writer has called "Template Squares"!

Amazingly the Square Solutions found are for many of the Orders between Order 81 and Order 132!

The basic construction, a kind of template, is relatively simple in that it is found by simple algebra in such a way that a Square construction will result, provided that z = 2x + y.

But in order to work, x ÷ y needs to be mainly between 0.734 and 0.739!

Having worked out a large number of alternate values for x and y it is fairly easy to discover the needed dimensions of Area A (Above). By running very advanced computer programs on at Trial & Error basis with a huge number of combinations of values for x & y it is then hoped that

(1) An exact pattern of Elements may be found, and

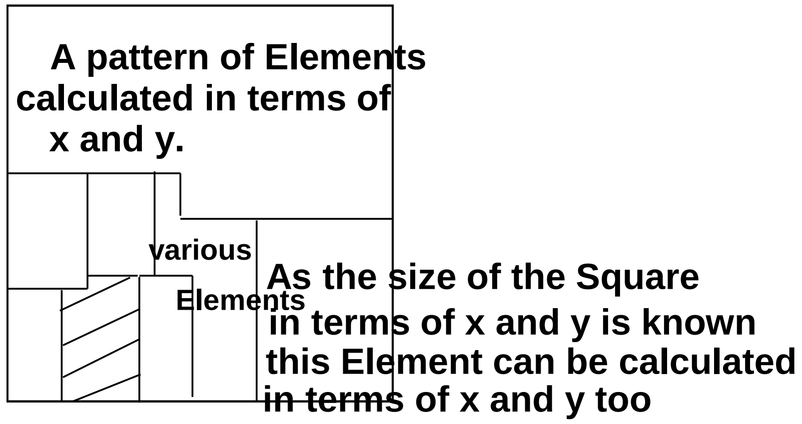
(2) That these Elements turn out to be all different.

Although the chances of finding a Perfect Solution is still very remote, it is a clever and canny way of using Trial & Error. The Area A is made long and thin so that the Elements are pretty small compared to the others outside the idea being that they are very unlikely to duplicate with any of the 14 Elements already there.

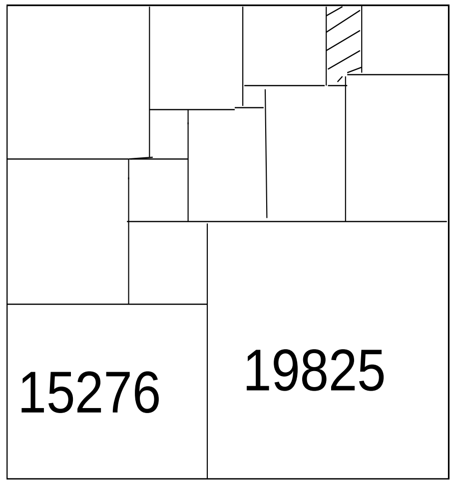
Where Solutions do exist they have Side Indexes of 2363 2373 2383 2393 . . . .

Looking at the Above Template again it is clear that every one of the 14 Elements can be calculated in coefficients of x and y only since we know z = 2x + y, and also that the dimension of the square is 2x + 5y (i.e. x + 3y + x + 2y). Thus Element 2x + y - z must be 2y (i.e. 2x + 5y less x + 2y less x + y).

Many patterns can be worked out with 10 or more Elements though many fail as exact square Elements cannot be drawn. The general format is shown below -



N8.2. SPECIAL TEMPLATE

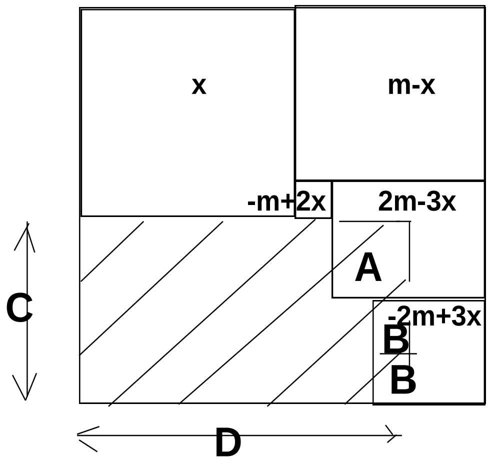


[72] 35051 x 35051

A program on the Internet shows hundreds of different Perfect Squared-Squares using this Template from [72] 35051 x 35051 onwards.
In Above the proportion of 15276 divided by 19825 is 0.7705422... with the SLIT (shaded area) being relatively wide.
in another solution ([87] 48204x 48204 where the relative Elements are 20381 and 27813 the proportion is 0.7327868..

N8.3. ANOTHER TEMPLATE

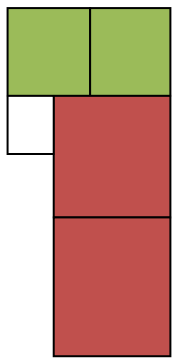
In 2014 someone found many SRSS's by the method shown below:



SPSS Solution with Triad at top and two more Elements (3m-3x and -2m+3x) as shown

In [52] 5689 x 5689 (with many others with a similar format), I wondered if the shaded area could be made into a Squared-Rectangle and it can! By replacing 2m-3x with A it was evident that B (twice) completed a rectangle with a Repeated End! Even more interesting is that the Rectangle is 2 by 1, i.e. 50% Elongation. C is exactly half D above!

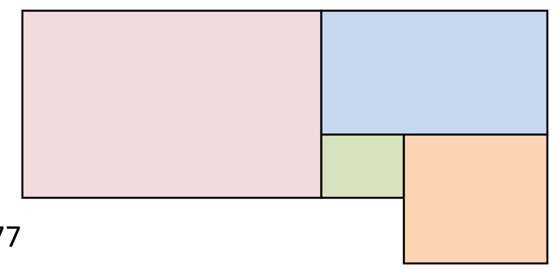
Put *simply*, all that needs to be done -
1. Is to discover a 2 by 1 Perfect Rectangle containing 3 Elements on the smaller side with two repeated ends.
2. Discard these 3 Elements and draw the five Elements as shown above.
3. This will complete a Square! The 5 Elements are unlikely to duplicate anything in the Rectangle - so a Perfect Simple Solution is obtained!
But finding such a Rectangle is very difficult!
There is a similar pattern requiring the replacement to look like this –

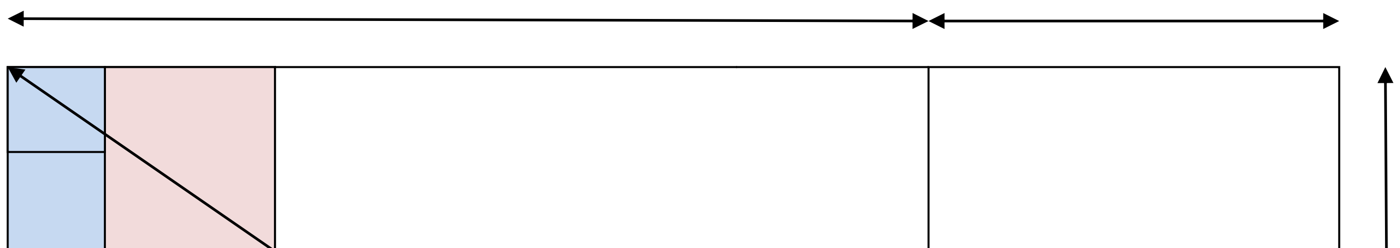


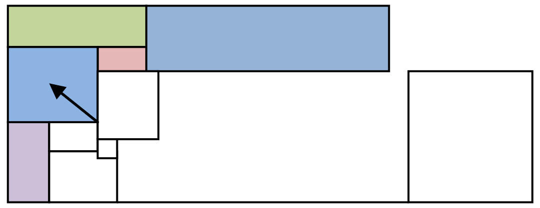
There are two pairs of adjacent Elements but the whole Rectangle is again of the Ratio 2:1. The Square [29] 357 x 357B is an example of this with a 2 by 1 Rectangle of 298 x 149.

N.8.4. A FURTHER TEMPLATE

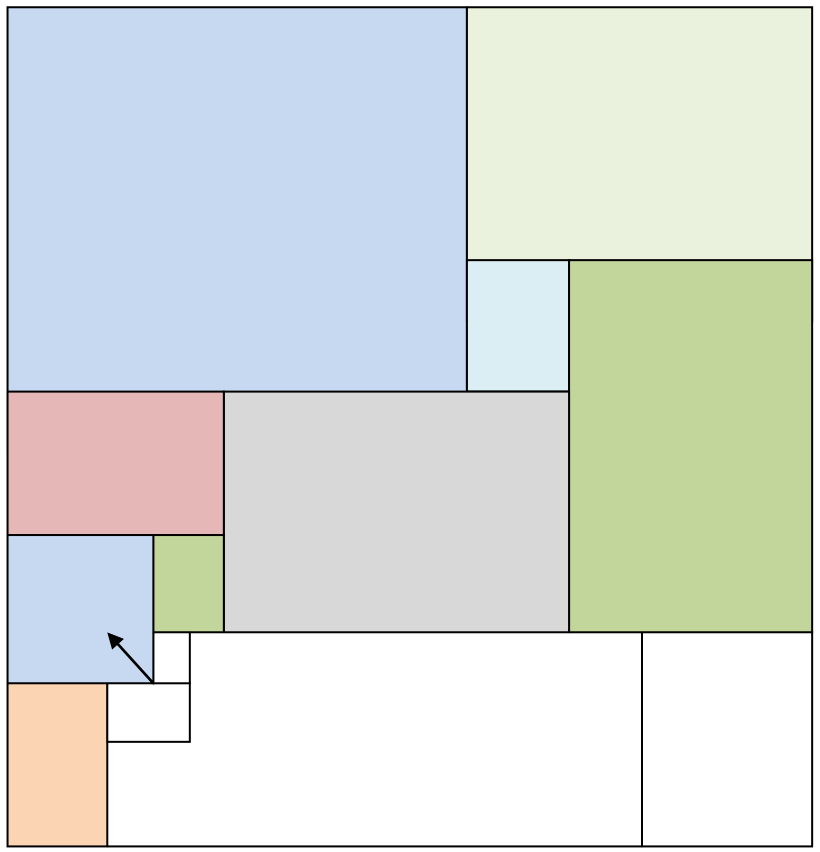
Below is a pattern (not to scale) for a Square, for which dummy values can easily be put.
A Square of Order [96] 10578 x 10578 happens to follow this pattern.

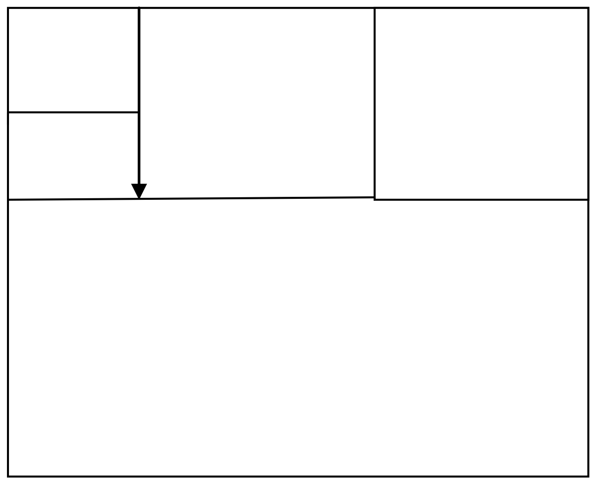






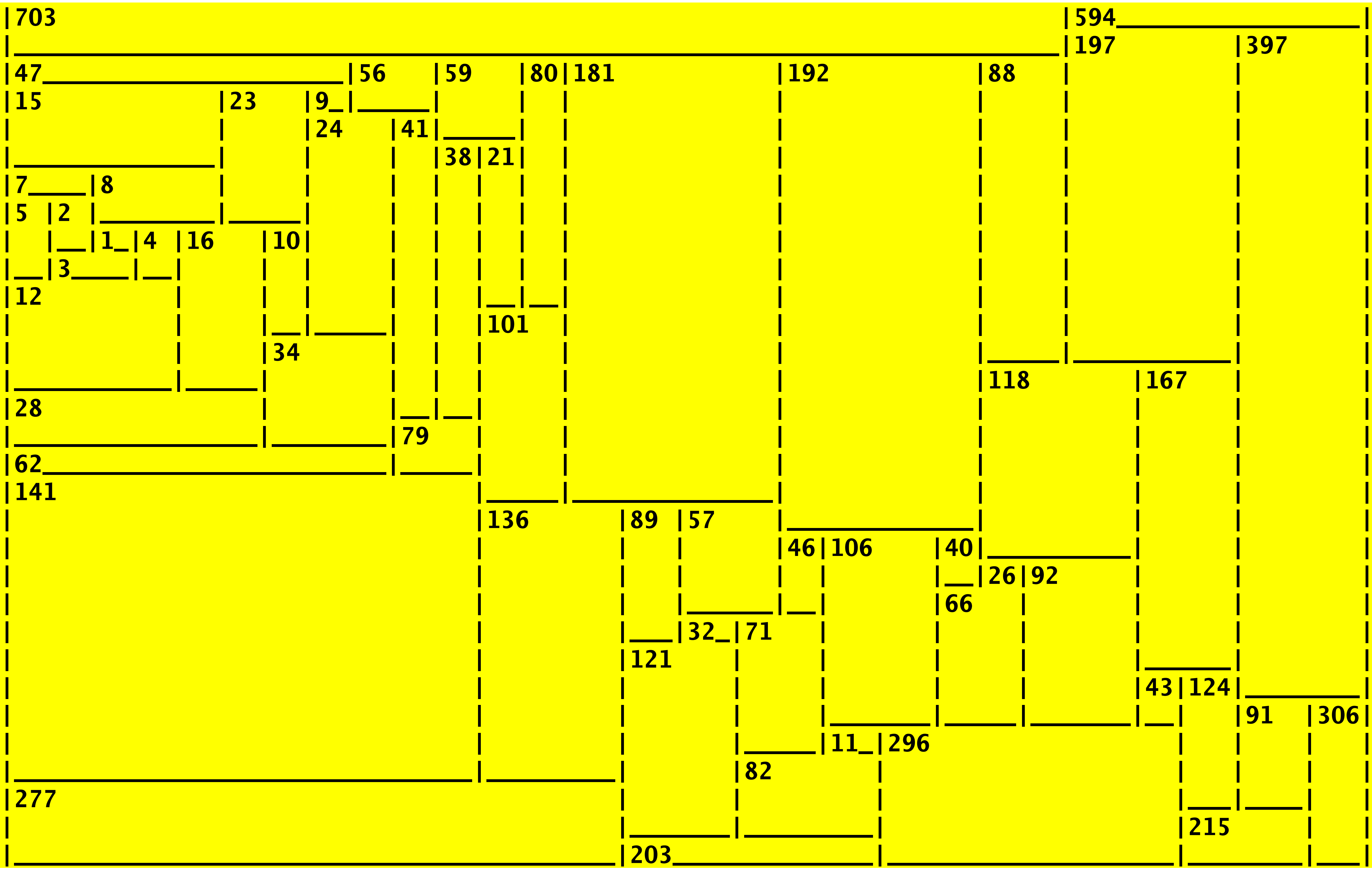
By looking at the Rectangle formed by the Arrow in the first diagram it is found that this can be replaced by three Elements as shown in the right diagram thus completing a Rectangle which always happens to be 7.5 times as long as it is deep (or a Ratio of 6.5:1 without a single Element on the right.
The Writer has checked by algebra the truth of the above.
In other words, if a rectangle containing repeated Element exactly 6.5 to 1 or 7.5 to 1 in length – it can then easily be converted into a Squared-Square (not always a Perfect one of course).
A pattern fairly similar requires a slightly different Rectangle at the top left corner – 5 units across by 7 down. However the resulting Rectangle is of ratio 7.5:1 or 6.5:1 as before





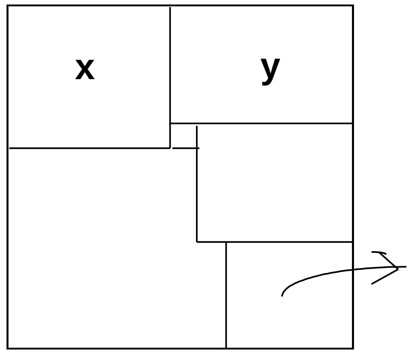
This time the Corner of the constructed Rectangle requires five Elements put together adjacently.

ORDER 58 1297 x 1297 Simple Perfect Squared-Square!
Contains Elements 1-2-3-4-5 7-8-9-10-11-12 15-16 21 23-24 26 28 (i.e. 13 of the first 16 numbers!).
Solution taken from the Internet, but shown differently here in Grid Format. Here is a staggering solution!



N8.5. TRIAL & ERROR SQUARED-SQUARES

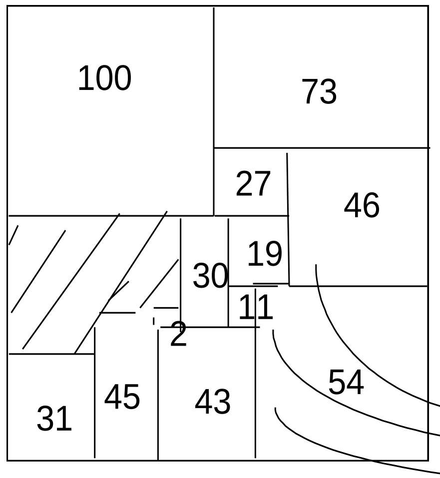
In observing known S-S Solutions the minimum number of Unknowns needed to create it is rarely 2 and often much higher. But by using the fact that both dimensions are the same in squares this can reduce it to 2 in certain circumstances.



SQUARE (x+y) x (x+y)

Not neccessary to call this z as can be shown in terms of x and y.

Where a Corner Element is added to complete a side where x and y suffice it can be shown in terms of x and y only.
Taking a hypothetical case suppose by trial and error attempt to 'fill' a Square as far as we can -



100 and 73 chosen for Square 173 x 173

DEMONSTRATION ONLY - NO ACTUAL SOLUTION LIKELY

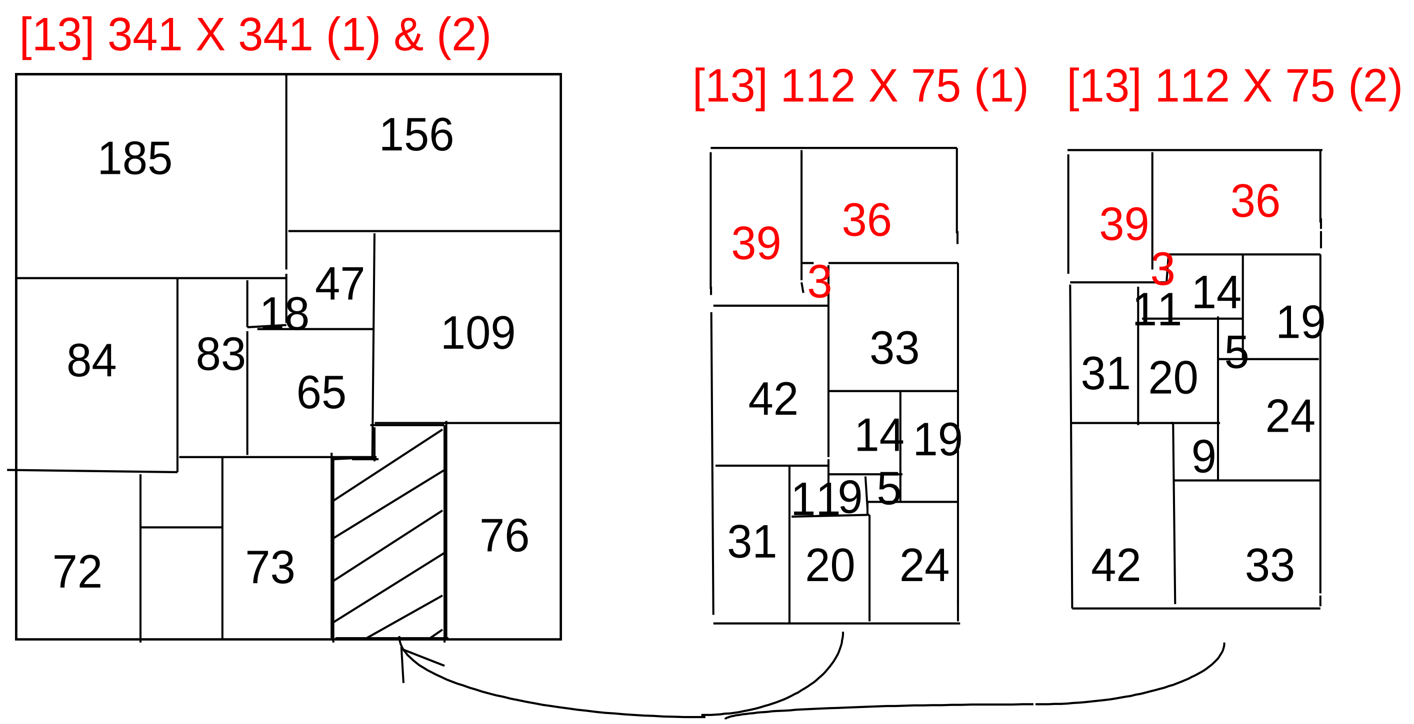
Elements added in order 27 46 54 (i.e. 173 less 73+46) 19 11 43 30 2 45 and 31 (i.e. 173 less 45+43+54) and so on...
Note that at every stage some Oblong with a corner chunk missing results.

See note below

Further Elements could be added to the above, each resulting in a smaller Oblong with a kink in one corner. Now if a huge database of the shapes and sizes of these is stored on a computer program, can any combination be found to fit the remaining gap? In practice, extremely unlikely! But testing after each Element in turn will increase the possibility. Still unlikely. So getting the Computer to test out a huge quantity of values for the top line (e.g. 101 73, 102 73, 101, 74 etc), we might if very lucky find a proper Squared-Square. After obtaining a number of these, we might even find one which is Perfect!
Attempting to find SS's by any Trial & Error system seems insane but by testing thousands or millions on a computer might bring some Solutions eventually. From <u>2014 was found a huge number of Solutions</u>, it appears by this empirical method, the end justifying the means!
The Writer was surprised that some solutions for known Perfect Squares can be established from 2 Elements alone being part of a Triad.
In other words every Element is readily connected from just these two! For this to happen simply by Trial and Error is extremely remote in practice - but who knows?
In the Above, if another 19 is added and two 27's below it are added a 2 by 1 Rectangle occurs, but the Writer considers that finding a Solution through discovering 2 by 1's of this type is probably more difficult than the method suggested above!

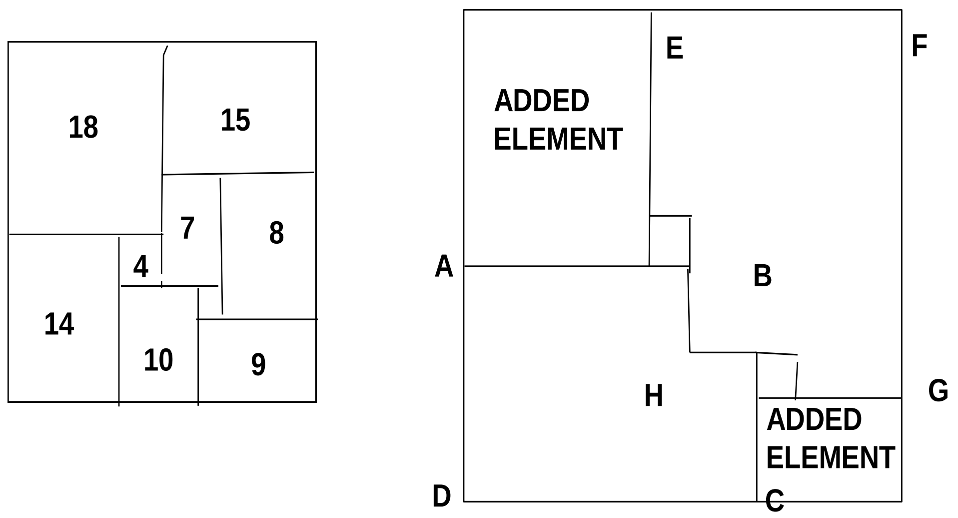
N8.5.1. A CURIOSITY IN [29] 341x 341 AND [29] 369 x 369

Observing the two twin Perfect Simple Squares of [29] 341 x 341 the writer noticed that many of the Elements of [13] 112 x 75 were contained in the area shown shaded and that adding three Elements 39 36 and 3 as shown completes that solution! The same was true of [29] 369 x 369



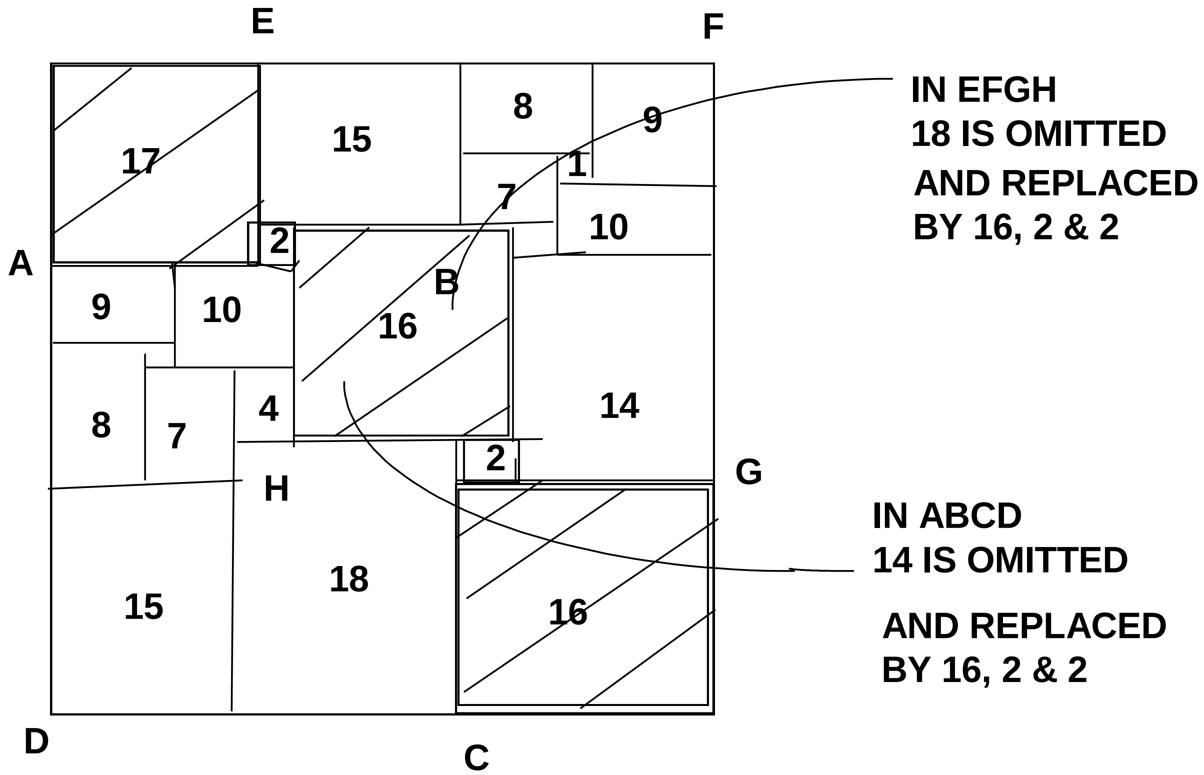
N9.1. SQUARES OBTAINED FROM SIMPLE RECTANGLES

Taking any given Symmetry 1 Rectangle it is relatively easy to construct six Imperfect Squared-Squares from it as follows:-
The resulting square is always Simple, not Compound.



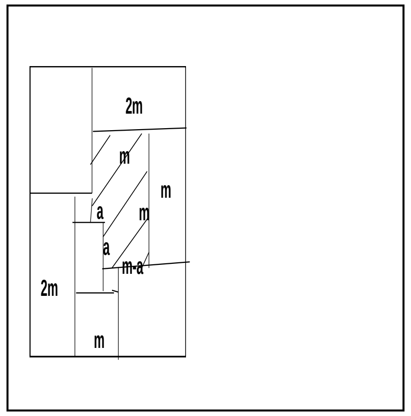
The trick is to include the Solution [9] 33 x 32 within the Square twice but to omit a different corner where the middle of the square occurs. Suppose for example, we omit 15 in the rectangle ABCD (at B), and 9 in the rectangle EFGH (at H), it is possible to construct an Imperfect but Simple Square providing EFGH is positioned vertically i.e. 32 x 33. This can be done by inserting two small Elements of the same size internally as shown (or similar pattern), and calculating two added corner Elements (which are not the same).

The small internal Elements if we omit 15 and 9 happen to be half the positive difference i.e. 6 div. 2 = 3.
It is clearer with an actual example-



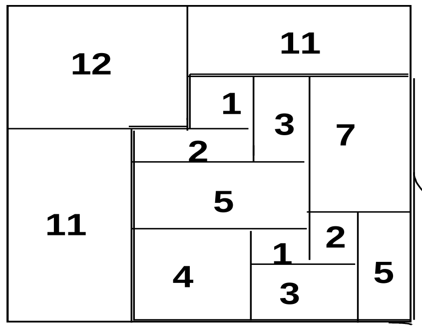
Here 18 and 14 have been omitted from the original Rectangles ABCD and EFGH where half the difference is 2 - the value of the two small added Elements shown.
The central Element is then found to be 16, i.e. 18 - 2 AND 14 + 2! The Corner Elements fit in at 16 and 17.
The result is [21] 49 x 49.
Six solutions are obtained by omitting 18 & 15, 18 & 14, 18 & 9, 15 & 14, 15 & 9 and 14 & 9.
In the case of 14 & 9 the small added Elements becomes 2½, for 18 & 15, 1½ and for 18 & 9, 4½! In such cases all Elements have to be doubled so no problem. Here the shaded Elements become all ODD whilst all others become EVEN. if Symmetry 2 or 3 Rectangles are used only ONE Imperfect Square can be found.
OTHER MORE COMPLICATED SQUARES CAN BE FOUND USING [9] 33 x 32 - TO BE CONTINUED.

N10. SQUARES TO OTHER SQUARES PATTERNS
N10.1. S2233. ORDER INCREASED BY 8

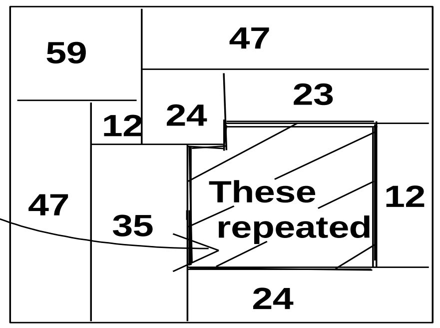


N10.2. S2234. ORDER INCREASED BY 8

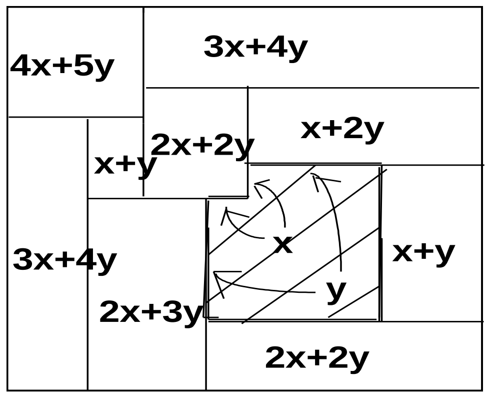
An unexpected link shown below can be applied to virtually any Squared-Square!
Using algebra (Below 3) it is calculated that this Pattern will always fit perfectly and that two pairs of Elements (shown as 12 and 24 24 or algebraically x + y and 2x + 2y one twice the other) are duplicated, always making the Result Imperfect.
But there are two results since the Block of Elements is always Reversible (i.e. can be switched diagonally “/ “).
So Twins exist for [19] 106 x 106!



[13] 23 x 23

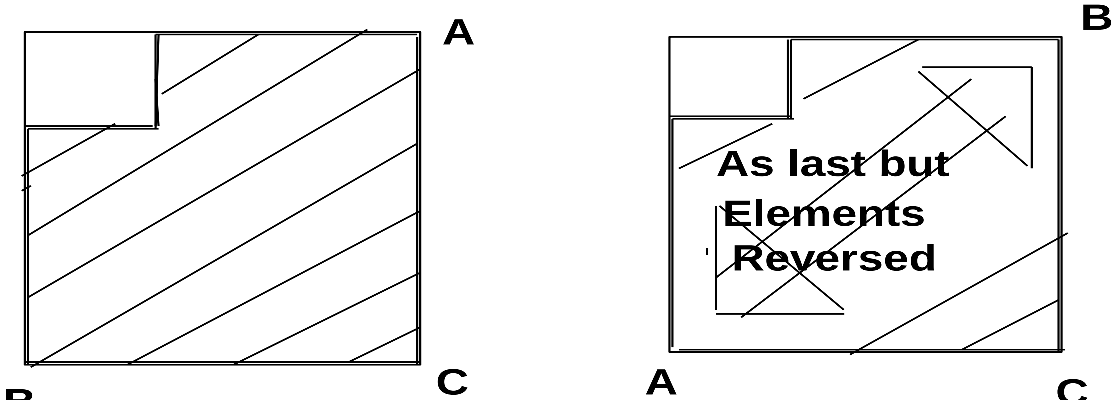


[19] 106 x 106
Order up by six

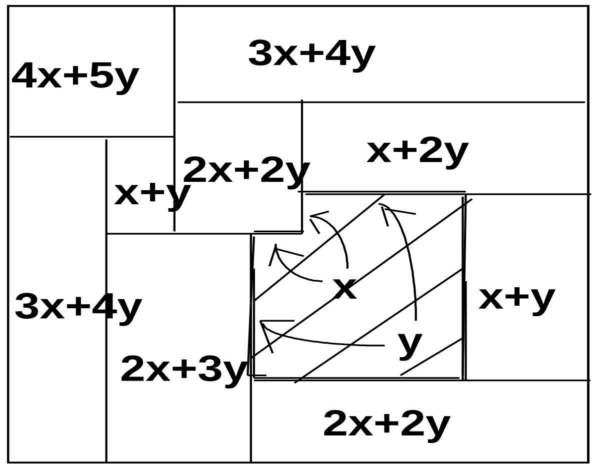


General Algebraic Form

Some points now need to be clarified -
1. The Original need not include a Branch (as in Above with 11 12 and 11). One corner Element may be omitted instead of all three. In fact any of the 4 corners can also be used. This is also still true of [13] 23 x 23 where all but Element 12 (or 11 or 11 or 5 on their own) can be repeated as per above 2, albeit with different Results and the Order up eight instead of six.
2. Eight new Squared-squares may be found from any given Squared-square! A better, less confusing and more general format than above follows, where the Orders for new Solutions rise by eight not six.
Thus the term EIGHTLINK is used. The Results have sides S2234. Here the
Original is [o] (x + y) X (x + y) and the Result [o+8] (7x + 9y) X (7x + 9y) (1) and (2) where x is the size of the removed corner Element.



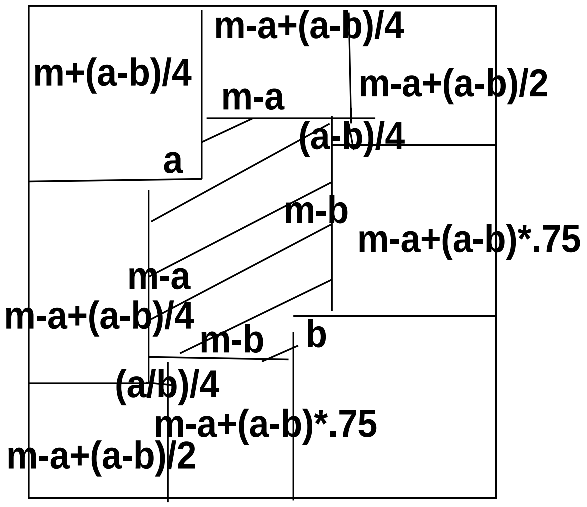
NB: The Corner Elements A, B, C can also be put at top left for a further 6 squares (3 x 2 twins)



EIGHTLINK

General Algebraic Form for Twin Solutions

N10.3. S3333. ORDER INCREASED BY 8



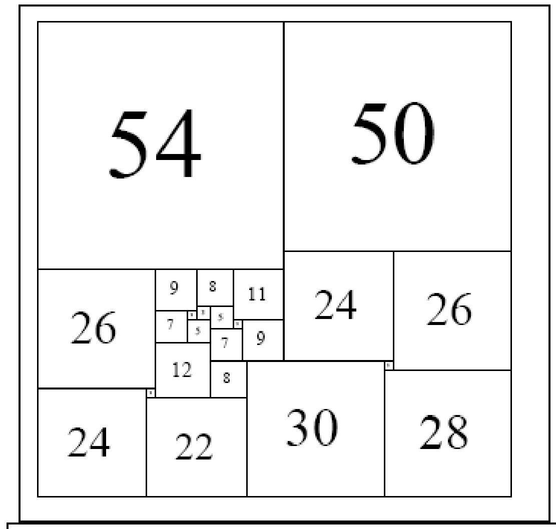
With Origin of [O] m x m
Result is [O+8] 3m +2.75a+b
X 3m +2.75a+b

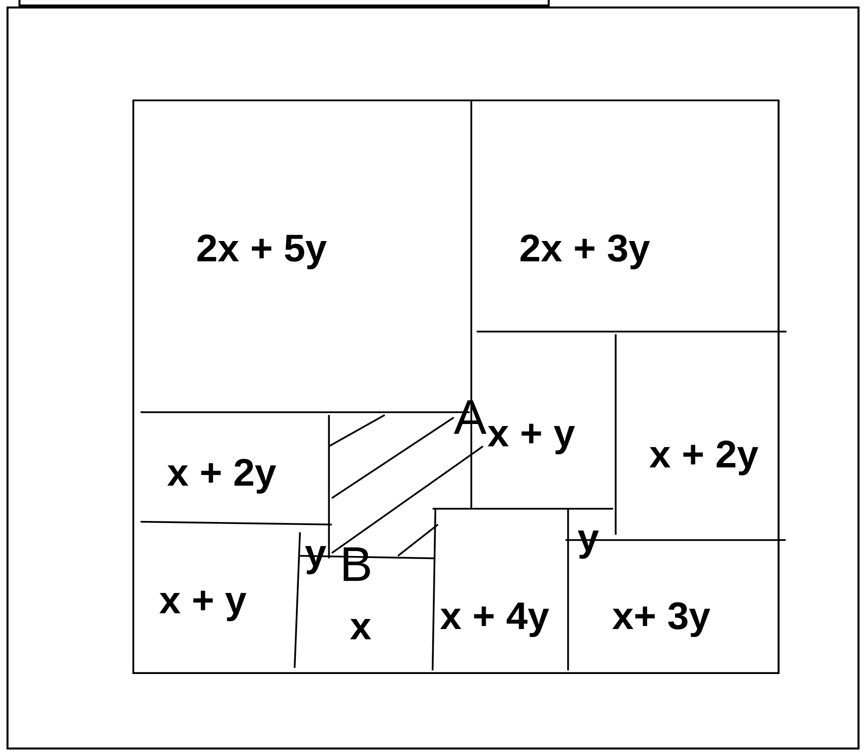
IF a-b is divisible by 4

The Result is an Imperfect Square with the outer 8 Elements have 3 pairs each the same, and 2 different.

N10.4. CONVERSION WHICH RESULTS IN A 2-3-4-3 SQUARE - ORDER INCREASED BY 10

Take any known Squared-Square say [14] 28 x 28. (In Below 1, this is that Square appearing 9 8 11 ... 12 8 with another 8 added). Removing the extra 8 and replacing with the pattern shown adds 10 to the Order and gives [24] 104 x 104
Such a Square has sides 2-3-4-3 and always Imperfect. Note y, x + y and x + 2y all appear twice.
y must be greater than 0 to prevent an Invalid Solution.
Another Twin Solution is found by turning A with B!



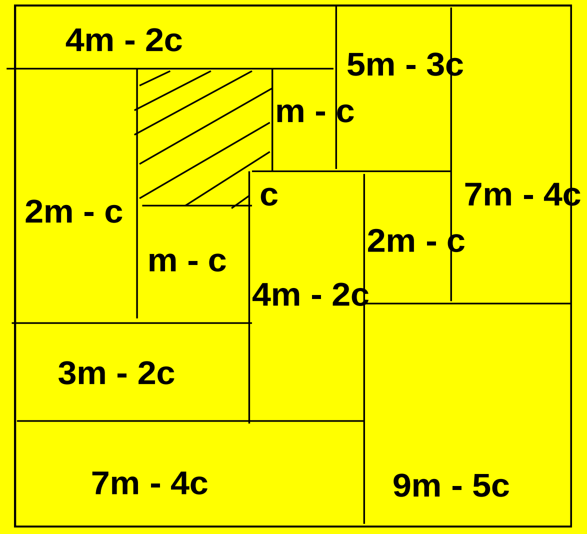


From above 2 it is seen that the Element which is removed is size 4y and must be divisible by 4, but If our Origin Square chosen corner is not, simply double or quadruple all Elements as needed.
If our chosen Square is Valid and Asymmetric (the Example above had a Repeater) then <u>8 Solutions</u> of an Order increased by 10 can be found forming four sets of twins! This is achieved by rotating and/or reflecting the Square. Again, it may be necessary to multiply every Element by 2, or 4 to construct the 2-3-4-3 Square.
Above we see that if the Origin Square is (x + 3y) X (x + 3y) then the Result Square is (4 x + 8 y) X (4 x+8 y).
If our Origin Square has a Square Index of 1, the Reduction Index being equal to the size of the Square, what is the SI of the Result Square?

N10.5. S2234. ORDER INCREASED BY 10

Starting with a Squared-Square with a corner removed, 11 other Elements can be added as Below, increasing the Order by 10. The Shaded area can represent almost any Squared-square, including those with a single Repeater at corner c, and / or a smaller Rectangle including corner c.
The only restrictions being-
1. There must not be a Repeater at any other corner.

2. There must be at least two Elements adjoining both vertically and horizontally to corner c, i.e. where the two m - c Elements come.
Finally, the shaded pattern can of course be reversed by swapping SW and NE corners. Twins always occur, both with the same range of Elements. The Resultant Sides are 2234 and comprise a Branch Link - a Repeater occurs if this is removed.

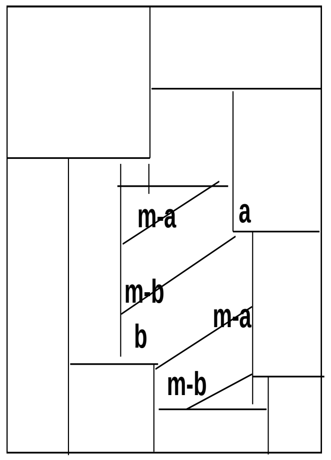


Original Square is Shaded Area plus Corner c

[o] m X n becomes [o+10] 16m - 9c X 16m - 9c

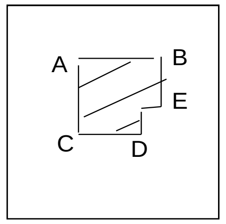
TENLINK

N10.6. S2244. ORDER INCREASED BY 10



N11.1. DO ANY PERFECT SQUARES CONTAIN A SMALLER SQUARE MINUS ONE CORNER ELEMENT?

As shown previously there are a number of patterns involving duplicated Elements where the Square contains an internal square less a corner Element. This means AB must equal AC and BE equal CD. Upon checking 1300 Perfect solutions I found no cases of this type.
I am assuming that no Square of this type will ever be Perfect. Not yet confirmed.



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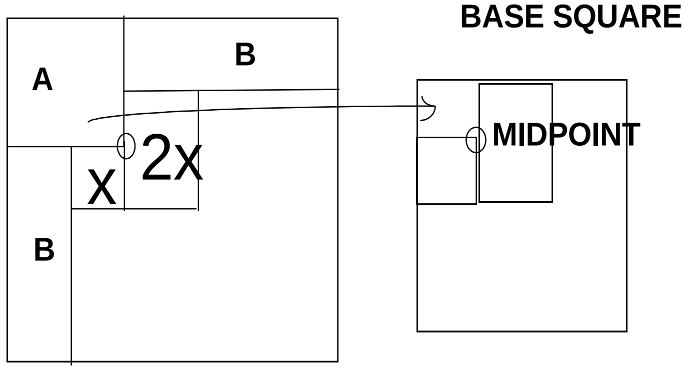
N11.2. SQUARED-SQUARES AND MID-POINTS

Although of minor importance it is shown here why many Imperfect Squared-squares contain a MIDPOINT. A MID-POINT occurs when one Element is adjacent to another twice its linear size: -



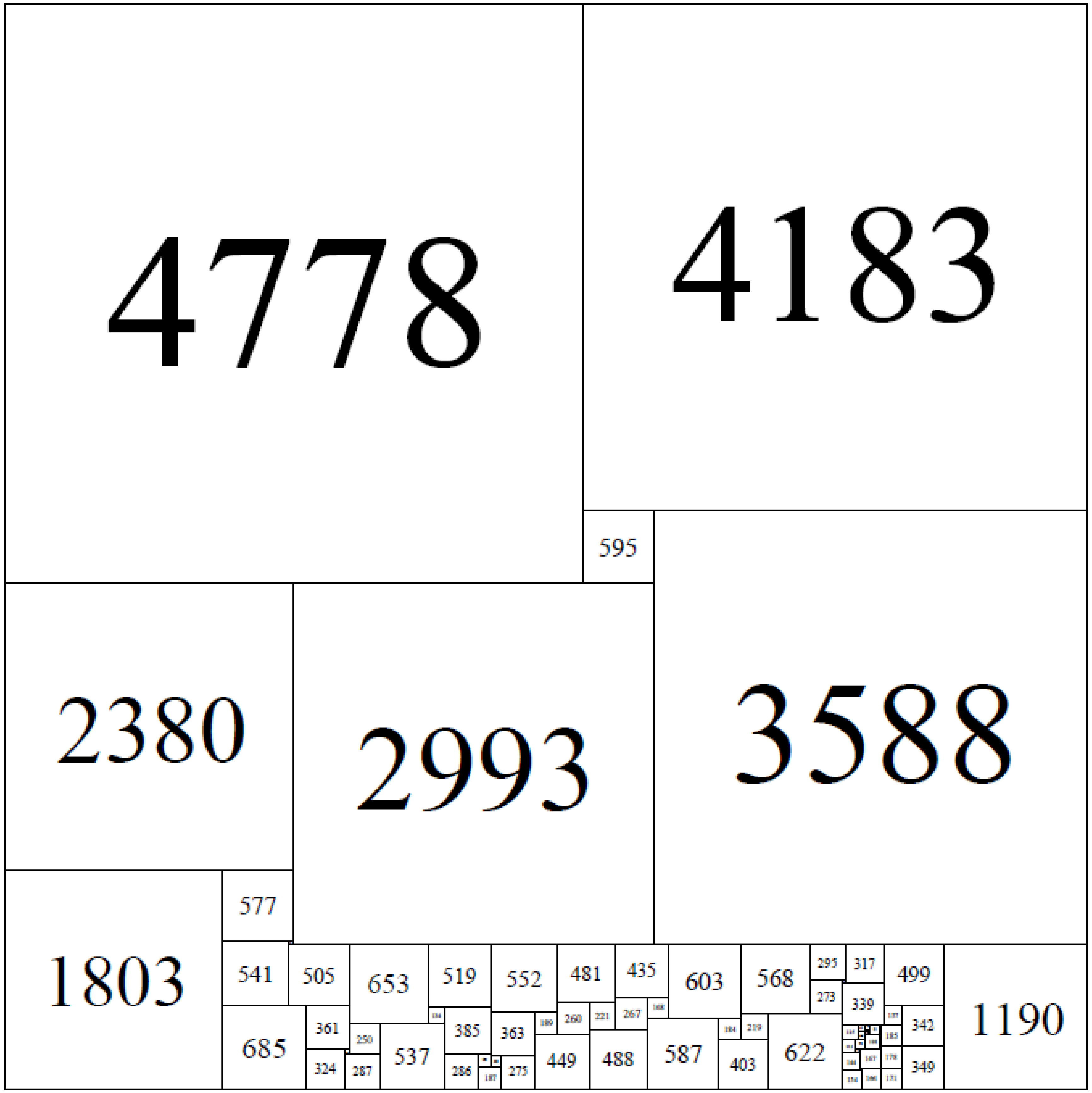
Whatever the relative proportions of A and B in above, it is evident that two adjacent Elements may be drawn, one twice the size of the other with a MIDPOINT occurring (shown by a ring).
It is also clear that the BASE SQUARE (when A & B are stripped off) will always be Invalid as it has a Repeater Element as shown.
So given any BASE SQUARE of this type and adding two more Elements, a MIDPOINT always occurs.

N12. OBSERVATIONS FROM THE SOLUTION BELOW

FROM [80] 8691 X 8691 PERFECT SIMPLE SQUARED-SQUARE.
Many solutions have been calculated by modern Computers using a kind of "Template" of which the simplest form is shown here.
It appears many pairs of numbers have been fed into a Computer to see if there is a known solution for a "Core" Pattern after calculating many random Elements.
4778 & 4183. Now 4778 – 4183 = 595, 4183 – 595 = 3588. Now for solution to be square bottom left has to be 8691-4183-3588 i.e. 1190 (twice 595 as it happens). 3588 – 595 = 2993 Then 2380 can be calculated and 8691 – 2380-4778 = 1803 (corner left). Close observation shows that a huge number of Elements can be readily found similarly leaving only a relatively small "Core" Pattern .
Look at the Element 339. Only the Elements below it cannot be calculated in this manner, but those at the bottom border can! This "Core" is not a rectangle but rather a rectangle with a "kink". The "kink only contains about 20 Elements despite the whole Solution being Order 80! If the computer has a vast bank of such "kinks" then a Solution might be found. To increase the possibility of finding a Solution the computer

can be asked to search many times (i.e. each time another Element is added. Of course the Computer can be asked to consider vast number of combinations of the two largest Elements, in the hope that a Solution might be eventually found . The last problem is to find a Solution which is actually Perfect (all Elements different).

To find Simple Perfect solutions by Trial & Error at first seems completely insane! But the theory is simple enough, and the construction always forced to be Square, and the Computer can be asked to test many millions of possibilities. We are unconcerned with what the Order size will be. The result has been however in an explosion of many high Order Simple Perfect Squares up to Order 236. The end justifies the means!!



80 : 8961 B_T 2014

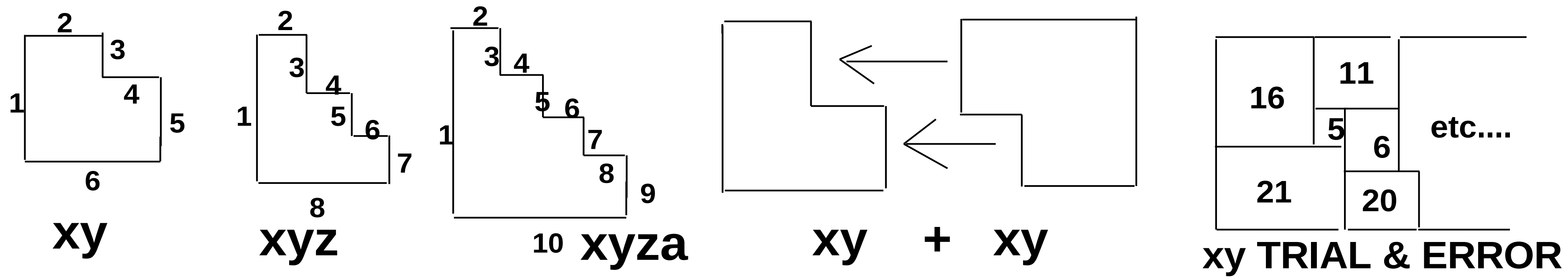
P. SQUARED-RECTANGLES & COMPUTER PROGRAMS

P1. COMPUTER SYSTEMS USED

This Book would have been impossible without extensive use of Computer Programs!
I have used several Types of Basic in the calculation of Squared-Rectangles including - Sinclair Spectrum BASIC initially and more recently Liberty Basic. But most programs have been done on GWBASIC.
In the Compiling of over 650,000 Valid Solutions from Orders 9 through to 36, BASIC programs both calculated the results and automatically saved (BSAVE) these results in memory in code form under a long string of BASIC programs.
These in turn were turned up on WORDPAD then copied to a LOTUS Spreadsheet where they were Parsed into Columns.
Only at this stage were the Solutions visible in intelligible form. They were then amalgamated into general LOTUS spreadsheets and duplicated entries then removed in bulk and the remaining information checked and numbered for Twins.

P2. GENERAL TYPES OF BASIC PROGRAMS

Programs calculating xy solutions only were later extended to include xyz and xyza solutions.
Although it would be possible to Calculate from Smith Rectangle designs this has so far been beyond my capability who has concentrated on the normal Algebraic methods which are miles easier to program.
1. Using Random Methods
Refer Below 1. A rectangle is built up by adding Elements to sides 1 or 2 or 3 or 4 or 5 or 6 in suitable random sequences selected by the Computer. This is OK but fails to produce all available solutions and there is no way of determining when all have been found using this method. Similarly in below 2 and 3 Elements are added to sides 1 to 8 (xyz) or 1 to 10 (xyza).
2. Adding Two xy Chunks
Shown in below 4 this is easily programmed once a tricky formula has been created. The Program is short compared to 1 but the Reduction Indexes are not found by this method. Although this seems to be a quick method of working xyza solutions, only xyz Solutions actually arise from this (and of course xy).
Recording these has its problems and reconstructing from the two series of Element numbers can be annoyingly tricky!



3. Trial and Error Systems

This has been mentioned previously and relies on good luck! The hope is that eventually the Solution attempted will terminate with a square end! Most attempts however to choose a couple of Element numbers (e.g. 16 and 21 above) will fail to give a Solution, but some do. I found a Solution of Order 1792 by this method!

Perfect Solutions are very rarely found, but the Reductions are more spectacular than in using any other methods!

There is no way of calculating the Reduction - actually it would often be a vast number in most cases.

4. Programs directed towards Links, Squares and Symmetric Solutions.

These have been used - some more successful than others.

5. Exclusive Systems.

Ideally, we need programs that systematically work out all Solutions possible by passing through a logical series. This is much more difficult to achieve.

ORDER 58 1297 x 1297 Simple Perfect Squared-Square!

Contains Elements 1-2-3-4-5 7-8-9-10-11-12 15-16 21 23-24 26 28 (i.e. 13 of the first 16 numbers!). Solution from the Internet, but shown here in Grid Format.

```
_______________________________________________________________________________________________
|703                                                                      |594_________________|
|_________________________________________________________________________|197        |397     |
|47_____________________|56   |59   |80|181           |192          |88   |           |        |
|15            |23   |9_|_____|     |  |              |             |     |           |        |
|              |     |24   |41|_____|  |              |             |     |           |        |
|______________|     |     |  |38|21|  |              |             |     |           |        |
|7____|8 | | | |     |     |  |  |  |  |              |             |
|5 |2 |________|_____|     |  |  |  |  |              |             |     |           |        |
|  |__|1_|4 |16   |10|     |  |  |  |  |              |             |     |           |        |
|__|3____|__|     |  |     |  |  |  |  |              |             |     |           |        |
|12         |     |  |     |  |  |__|__|              |             |     |           |        |
|           |     |__|_____|  |  |101  |              |             |     |           |        |
|           |     |34      |  |  |     |              |             |_____|___________|        |
|___________|_____|        |  |  |     |              |             |118       |167   |        |
|28               |        |__|__|     |              |             |          |      |        |
|_________________|________|79    |     |              |             |          |      |        |
|62________________________|_____|     |              |             |          |      |        |
|141                             |_____|______________|             |          |      |        |
|                                |136      |89 |57    |_____________|          |      |        |
|                                |         |   |      |46|106    |40|__________|      |        |
|                                |         |   |      |  |       |__|26|92     |      |        |
|                                |         |   |______|__|       |66   |       |      |        |
|                                |         |___|32_|71   |       |     |       |      |        |
|                                |         |121    |     |       |     |       |______|        |
|                                |         |       |     |       |     |       |43|124|________|
|                                |         |       |     |_______|_____|_______|__|   |91  |306|
|                                |         |       |_____|11_|296                 |   |    |   |
|________________________________|_________|       |82       |                    |   |    |   |
|277                                       |       |         |                    |___|___ |   |
|                                          |_______|_________|                    |215     |   |
|__________________________________________|203______________|____________________|________|___|
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Q. CYLINDRICAL SQUARED-RECTANGLES

Q1 DEFINITION OF SQUARED CYLINDERS

A Squared-Cylinder is simply a Cylinder divided into a number of Square Elements such that no gaps remain. The Elements may or may not be all different. The Result may be a Square i.e. the Unit distance of the circumference equals the depth of the Cylinder, otherwise a Rectangle.
NB This Book will not treat this subject to a high level, but will major on its links to known Squared-Rectangles & Squares.
Cylindrical Squared-Rectangles will be referred to as CSR's.

Q1.1. CSR's ARE VERY COMMON!

Take any Squared-Rectangle e.g.[9] 33 x 32. Clearly two Cylinders are found by twisting into a Cylinder either horizontally or vertical giving [9] 32C x 33 and [9] 33C x 32. ("C" denoting the circular part).
At once we see Cylinders must be more common than Squared-Rectangles! However this book deals with Cylinders NOT containing a SINGLE vertical line.
However ignoring these 2nd class Solutions we find they are still plentiful.

Q1.2. SOME CSR's NO COUNTERPART SQUARED-RECTANGLE

For example the spectacular [20] 79 x 79C Perfect Simple Cylinder shown below.

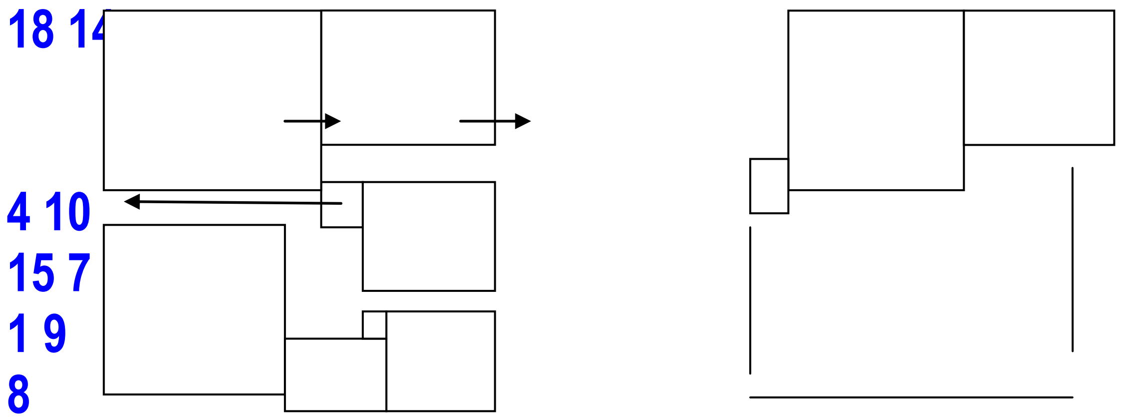
1. This has the lowest known Order for a Perfect Square Cylinder of only 20.
2. Note the Dimension of 79 is much smaller than the minimum 110 x 110 Perfect Squares for Orders 22 & 23!
3. Note that the largest Element is only 32 – incredibly small for 20 different values!
4. 19 is the smallest possible size for a PERFECT SQUARE CYLINDER and was only discovered in 2012! Yet the quantity of Cylinders is in fact infinite!

Q.1.2.1. SOME OTHER SPECTACULAR PERFECT SQUARED-SQUARES

There is a Perfect Cylinder for [20] 81 x 81

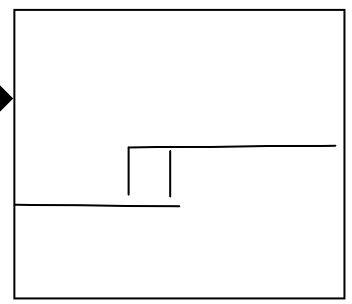
There is another for [20] 80 x 80. But by juggling around the Elements seven more are found – each with the same Elements appearing on the same horizontal level!
There is yet a further Solution [20] 80 x 80 and by juggling with the Elements a further eight distinct Solutions are readily found!
Thus there are 18 Solutions of Order 20 known with very reduced values – one for size 79 sixteen for 80 and one for 81 x 81 – all found in 2012
None of these have a Squared-Square counterpart however.

Q1.3. SOME SQUARED-CYLINDERS DO HAVE A COUNTERPART SQUARED-RECTANGLE



[9] 33 x 32 is easily converted into a Cylinder as shown above. Note how 4 is shunted left and 18 & 14 right closing the gap!
In fact any solution containing a Triad can be converted into a Cylinder.
In fact the solution does not need to have a Triad. *Any* solution with an Internal connecting Element between two horizontal lines as shown will suffice! 1. Remove the connecting Element. 2. <u>Shunt</u> the top pattern to the right and 3. Redraw the connecting Element to the left. Simple as that!

Any pattern here →



Any pattern here >

Q1.4. SLIDING OR SHUNTING OF INTERNAL ELEMENTS?

The term "SLIDE" is correctly used in the case of Squared-Rectangles where a horizontal (or vertical) line appears clearly with a break between. But when converted into a Cylinder the Slide ends join up and the Slide effectively disappears. So with CSR's the term "SHUNT" will be used, and the appropriate Internal Elements will be termed "Shunt Elements" and coded ABCD… etc in the natural order. In theory any combination might give a good CSR, but not always in practice.

Q1.5. SHUNT ELEMENT CODES & COMBINATIONS

1. **One A A**
2. **Two AB A B AB**
3. **Three ABC A B C AB AC BC ABC**
4. **Four ABCD A B C D AB AC AD BC BD CD ABCABDACDBCDABCD**

It should not be assumed that a good CSR is always obtained from every combination. Some duplications occur from Symmetric SR's. CSR's with a single dividing line may occur, etc.
Since the range of Elements is identical Perfect SR's give Perfect CSR's and Imperfect SR's give Imperfect CSR's.
In ENZ Solutions every Internal Element converts to a CSR (where better or worse)!

Q1.6. SOME SQUARED-RECTANGLES GIVE RISE TO MORE THAN ONE CSR SOLUTION

Suppose we have a Rectangle with *two* horizontal divisions instead of one shown above.

1. **One Cylinder is found by sliding the upper divide (i.e. ignoring the lower divide) &**
2. **Another CSR is found by sliding from the lower divide (i.e. ignoring the upper divide) &**
3. **A third Solution found by sliding from both!**

The Element sizes stay the same for all three. *Also each Element remains at the same horizontal level!*
But the Solutions differ.
Some Solutions will give rise to even more CSR's. [20] 80 x 80 has EIGHT variations but is not connected to any Squared-Square.

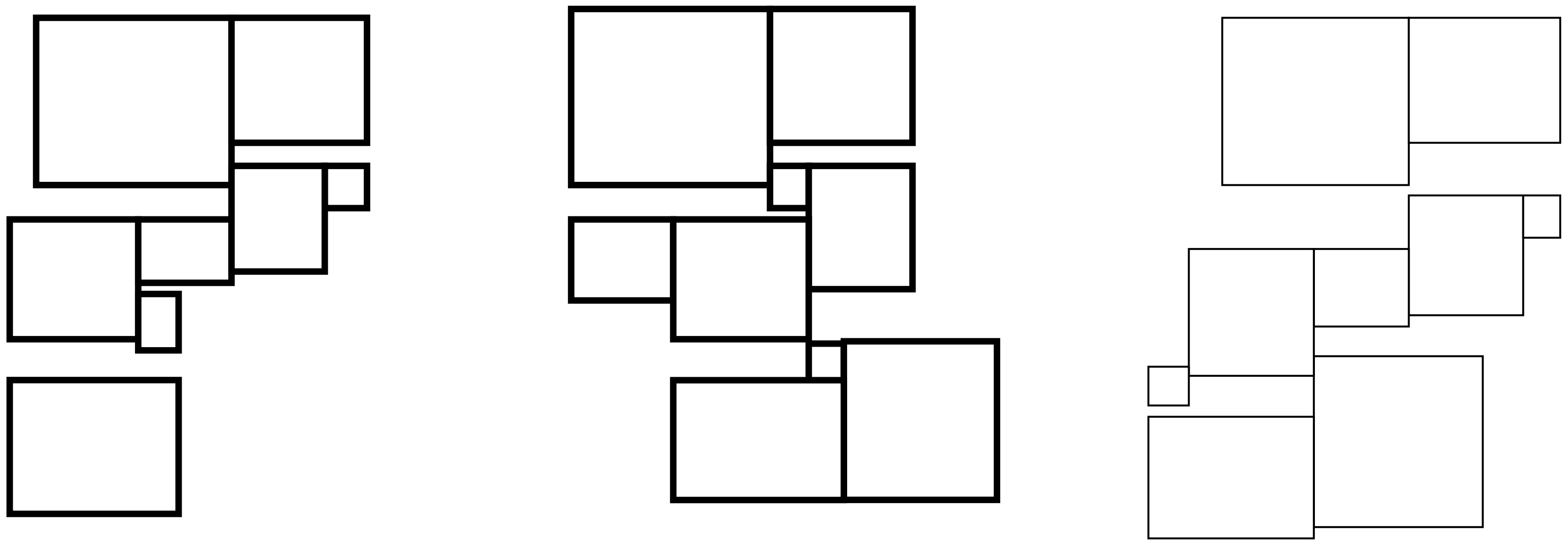
Q2.1. EXISTENCE OF EVERLASTING SERIES

It must be mentioned - in passing - that standard Squared-Rectangles can be added horizontally as many times as we wish and each form low class Squared Cylinders. E.g. [9] 33 x 32 can be repeated making Cylinders [18] 66 x 32, [27] 99 x 32 … ad infinitum. All solutions are very Imperfect.
But Squared Cylinder [9] 32C x 33 can likewise be extended to [18] 64 x 33, [27] 96 x 33 … ad infinitum. Of course all such solutions are highly Imperfect and of little interest.

Q2.2. OFTEN SYMMETRIC SR'S CAN BE TRANSLATED INTO NON-SYMMETRIC RECTANGULAR CYLINDERS

Look at Symmetric Solution [9] 15 x 11. By manipulating the internal Elements 1, 3 & 1 (which we will refer to as A B & C) we can find four Cylinders namely

1. "A" 11 x 15 NON-SYM 2. "B" 11 x 15 NON-SYM 3. "AB" 11 x 15 SYMMETRIC & 4. "ABC" 11 x 15 SYMMETRIC and "C" is equivalent to "A", and "BC" is equivalent to "AC" so no further solution.

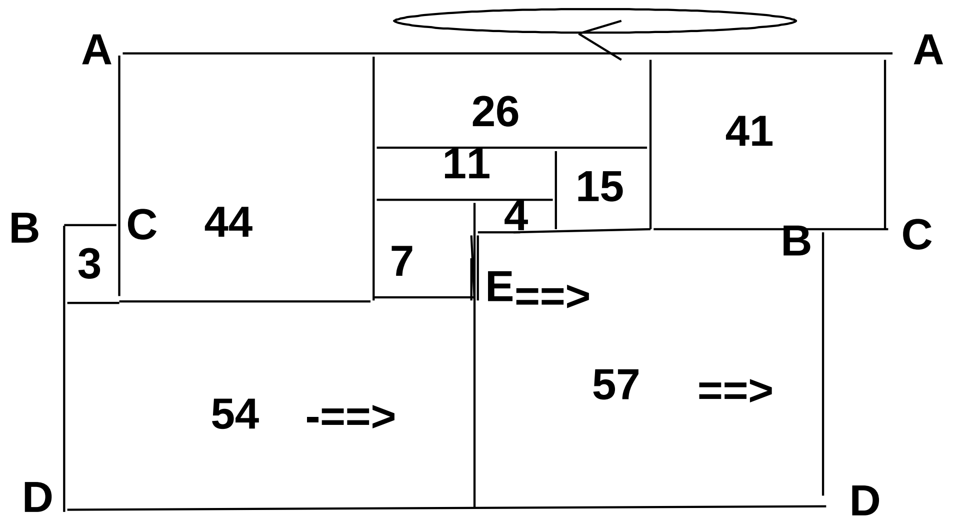


The two NON-SYM Imperfect Solutions of [9] 11C x 15 are shown above. 3rd diagram is Symmetric.

Q3.1. ORIENTATION OF SOLUTIONS

We shall regard all Cylinders as appearing Vertically.
The Circumference value is shown first and may be 1. Smaller 2. Larger or 3. The Same (i.e. Square) as the Height Value.
The quantity of Cylinders where the Circumference is less than the Height, is much more than the other way round.
This is because SHUNT ELEMENTS are much more frequently found vertically orientated than horizontally in Squared-Rectangles.

Q4. ANOTHER EXAMPLE OF A CSR (CYLINDRICAL SQUARED-RECTANGLE)



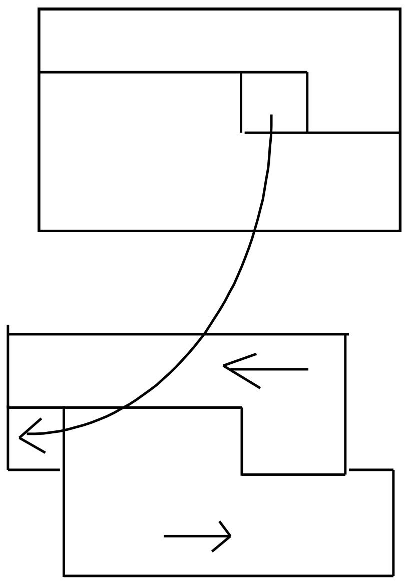
NOTE THAT THE LEFT AND RIGHT BORDERS ARE IDENTICAL

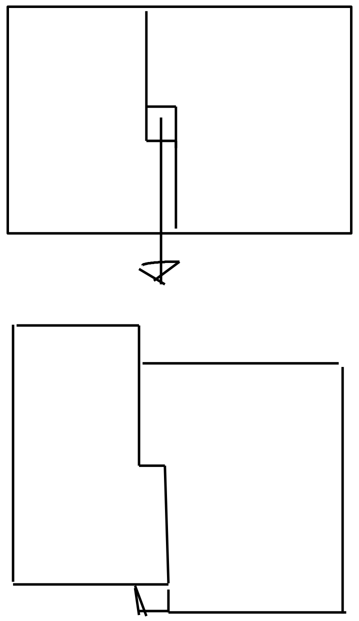
e.g AC = 41
BC = 3 & BD = 57

In Above, this LEFT-RIGHT Cylinder construction can be drawn on a piece of paper and swung round at points ABCD to form a cylinder. A little observation shows that Element 3 can be placed at line E - and Elements 54 and 57 suitably shunted to the right - giving [10] 111 x 98.

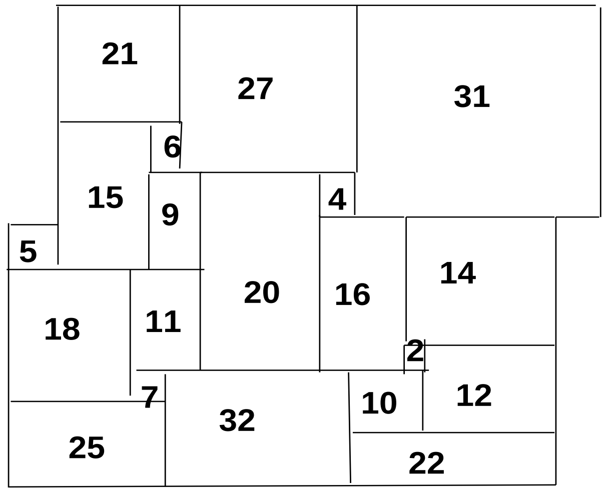
Q5. CSRs LINKED TO SQUARED-RECTANGLES

So we see here how an SR [10] 111 x 98 can so easily be transformed into a CSR by removing the Element 3, sliding it to the left and sliding the adjacent Elements of 54 & 57 to the left.
Likewise some SR's can be converted into UP-DOWN cylinders, but for simplicity LEFT-RIGHT patterns only will be shown.
In this example, we see that a Triad is changed into a Diad and the connecting Element (called the PLUS in this book) is redrawn elsewhere on one side.
It is not difficult to see that any and every SR solution containing a Triad can be made into a CSR, and since some have two Triads two different CSR's can be constructed from a single solution (assuming that solution is not Symmetric of course)!
Suppose the SR contains a Pentad Plus (Pentad+)? It soon becomes obvious that Cylinders can be easily formed from these too.
Looking again at the above it is equally true to regard the Element 3 as having been removed from the TOP half (and the Elements 44 26 41 ... all shunted to the right. This means all that is needed is any SR with a single Step with a PLUS in-between.





The CSR above taken from Internet shows sides with 2 steps & kinks. But note this 2-D pattern can be redrawn many ways - for instance if the section right of the thick line is switched to the left we have a pattern with only 1 step and kink.
Note also the Slide Lines found in this solution i.e. Lines at both A's and lines at both B's (vertical) and Lines at C's and at D's (horizontal).
There is no SR [16] 56 x 52 in existence and the above pattern cannot be manipulated to obtain it.



[20] 79 x 79

Elements
2 4 5 6 7 9 10 11
12 14 15 16 18 20
21 22 25 27 31 32

This taken from the Internet is truly amazing - A <u>CSR which is both Perfect and Simple and Square!</u>
1. Compare with SR [21] 112 x 112 the lowest Order possible for any Perfect Squared-square.
2. [22] 110 x 110 the smallest size possible for any Simple Perfect Squared-Square.
3. The Largest Element is 32, compared with 34 in [23] 110 x 110 or 50 in [21] 112 x 112).
4. In the range 1 to 34 the only numbers missing are 1 3 8 13 17 19 23 24 26 28 29 30 (twelve).

<u>Q.6. HUGE QTY OF SOLUTIONS FROM A SINGLE SR</u>

ENZ Solutions (See Section D7) contain Internal Elements every one being a SHUNT ELEMENT!
Now for n SHUNT ELEMENTS there are a possible 2 to the nth power minus 1 Squared Cylinders.
1 > 1 2 > 3 3 > 7 4 > 15 5 > 31 ….. 14 > 16383!
Now an ENZ solution found by the Writer is [34] 1873 x 484 has 14 Internal Elements which would appear to have *<u>16,383 Squared Cylinders</u>* – on at least this minus any duplications which may occur! However many are distinct each Solution contains the same Elements arranged differently but remaining on the same Horizontal levels!

<u>Q7. COMPOUND SR's AND CYLINDERS</u>

Many Compound Squared-Rectangles but not all, are no longer Compound when converted to Cylinders.

R. REMAINDER PARING & AREA RELATIONSHIPS

R1. REMAINDER PAIRS

R1. Gives general information regarding Remainder Pairs.
R2. Deals with Area Relationships being linked to Reciprocal Pairs.
R3. Deals specifically with Reciprocal and R4 with Identical Pairs.
R4. Deals with Area Relationships which has links with Reciprocal Pairs. This is followed by Solutions which have both properties explaining why.
An attempt has been made to present this tricky subject logically, but there are still areas on which I am puzzled!
The theory is interesting, but the following Sections are of little practical value.

R1.1. CAN INDIVIDUAL ELEMENTS BE PUT IN CONNECTING GROUPS?

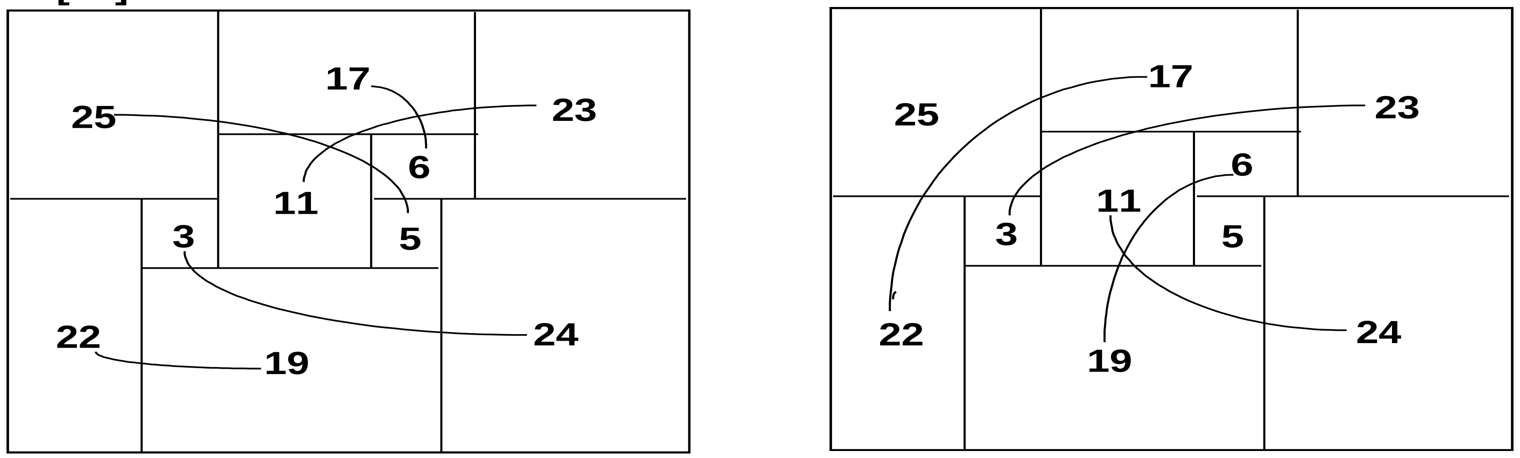
When Elements for any Squared-Rectangle are arranged in ascending numbers the series seems completely random.
Clearly there is no arithmetic progression like 1 3 5 7 9 11... or geometric progression like 1 3 9 27 81...
There appears only a jumble of random numbers without any apparent relationship.
However there are some hidden relationships which sometimes apply, but oddly, not always!

R1.2. GROUPING INTO PAIRS

The range of Element linear numbers offer no clues, but Elements are drawn two-dimensionally in Rectangles, and by considering their Unit Areas as Squared numbers rather than their linear size, interesting relationships sometimes apply.
For example Element 7 is examined as a unit area of 49 (7 squared).
It is sometimes possible to divide up the Elements into Pairs.
Having said this it is often not true for the entire Rectangle.
These Pairs divide into two independent types which are termed
RECIPROCAL PAIRS and IDENTICAL PAIRS in this Book.
Though the general idea of Section R1 is easily grasped, the subject is confused by (1) the several ways of finding Pairs and (2) what matching is defined as real Pairs, and what is not.

R1.3. EXAMPLES OF THE TWO PAIRING TYPES

[10] 65 x 47 SHOWING RECIPROCAL and IDENTICAL PAIRINGS



Observing that the longer side is 65 and squaring each Element and dividing each by 65 a string of remainders is found.
1. In the LEFT Diagram take 6 and 17 where 289 (17^2) - 65 x 4 = 29 and 36 (6^2) - 65 x 0 = 36.
Now 29 and 36 are RECIPROCAL, total 65!
Another way of looking at this is to note that <u>17^2 PLUS 6^2 is 325 a multiple of 65</u>.
The above is true for all other values on left hand side joined by a curved line.
2. In the RIGHT Diagram, take 6 and 19 which give remainders of 36 and 36 which are IDENTICAL.
Another way of looking at this is to note that <u>19^2 MINUS 6^2 is 325 a multiple of 65</u>
The above is true for other values on Right Hand Side joined by a curved line.
3. Sometimes one or more pairs fail to work - see 25 and 5 on Right Hand Side where 625 - 25 = 600 which is not a multiple of 65!

R1.4. REMAINDERS AFTER DIVISION BY EITHER SIDE OR SEMI-PERIMETER

To complicate matters the pairing phenomena can also occur in whole or in part when remainders are taken after division by
1. The Semi-perimeter e.g. 112 in above, or
2. The Larger Dimension (the horizontal side) e.g. 65 in above
3. The Smaller Dimension (the vertical side) e.g. 47 in above
Usually only one of these applies at a time, and discussion as to which one, and why is considered later. See R1.7.

R1.5. THE REMAINDER TYPES CODED

We shall code as follows- A denotes Identical pairs and B Reciprocal pairs. This is followed by a number denoting the amount of non-paring Elements 0,2,4,6 etc. AA or BA denotes no paring possible anywhere. The Example above codes as BO and A2.

R1.6. REMAINDER TYPE DEFINED BY SEMI-PERIMETER

Upon investigation it appears that the size of Semi-perimeter or of Larger Sizes tends to have a bearing on the type of Code. This being so, if the SP is multiplied by any positive integer the codes will not be altered.
Order 9 SP 130 B0.

Order 10 224 A2 and 209 A4.
Order 11 336 A2(3) 353 B0(1) 368 A2(3) 377 A0(1) 386 B0(1)
Order 12 585 A2 or A4 585 A2 593 AA 608 AA, A10 615 A6 633 AA 638 A2 ,
A4 644 A2 and 672 A2.

R1.7. IDENTICAL & RECIPROCAL PAIRS IN THE SAME SOLUTION

Though unusual, a few solutions have both types of Paring throughout. This is not completely true in [10] 65 x 47 above, but true of [11] 209 x 168 Below; Look at 73 and 44 in Remainders when the squares are ÷ by 377 are 51 and 51.
The rest of the Rectangle pairs as follows - 168 and 64, 92 and 60, 76 and 42, 16 and 11, 73 and 44, 53 and 31.
Another way to see Identical Pairs is to observe that the SUBTRACTION of the two unit areas is divisible by 377.
So 73 squared (5329) less 44^2 (1936) gives 3393 which is 9 x 377

R1.8. CONSIDERABLE VARIATION WITHIN RECTANGLES

After testing many rectangles it is soon found that some have no pairing whatever after division by either dimension or Semi-perimeter, [12] 338 x 295 being a case. Some have a few pairs with the remaining Elements not linking.
A few have pairing off of all Elements, and some have a dazzling display of pairing.
The Writing has only partly discovered, why some work, others work in part and the rest not at all.
Shown Below is [9] 69 x 61 with amazing pairing!

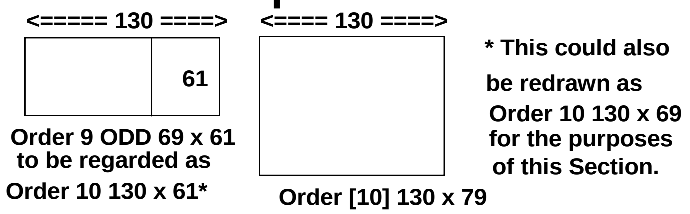
[9] 69 x 61	Remainder division 130	Identical pairs	Reciprocal pairs	Remainder division 61	Identical pairs	Remainder division 69	Identical pairs
61	81	a	a	3	-	64	-
36	126	b	b	15	c	54	d
33	49	c	a	52	b	54	d
28	4	d	b	52	b	25	c
25	105	e	c	15	c	4	b
16	126	b	d	4	a	49	a
9	81	a	e	20	-	12	-
7	49	c	e	49	-	49	a
5	25	e	c	25	-	25	c
2	4	d	d	4	a	4	b

R1.9. TOTAL PAIRING OFF FOR ODD ORDERS

Where the solution has an odd Order, we simply add an extra Element at the right (or left, top or bottom if preferred).
In the Above table [9] 69 x 61 is regarded as [10] 130 x 61. Note the Divisor Used is 130 being 69 + 61.
61 is the extra Element on one end to make the last match, but 69 could have used instead with the same result.

R1.10. THE TERM "SIDE"

As the term "Semi-perimeter" is confusing in the following Sections, the term SIDE in capitals is also used.
It is important to read the term "SIDE" as meaning
1. The Dimension of Even Order Rectangles. (The larger horizontal Dimension unless otherwise stated) or,
2. The Semi-perimeter of Odd Order Rectangles.



In above the SIDE is 130 in each case. For the purposes of this Section all Solutions have EVEN ORDERS (with a Single End Element where required), and the term "Semi-perimeter" is that applying to Even Orders.
Thus [9] 69 x 61 is regarded as [10] 130 x 61 with a SIDE of 130.

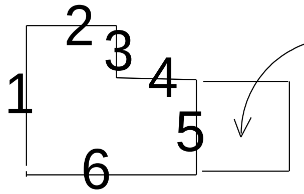
R1.11. GENERAL FORMAT FOR AN ELEMENT

Consider an individual Element E with an Area of E Squared, which we will express in terms of Modulus Side m.
The General Form will be E^2 = SIDE (Divisor) x Coefficient + Remainder which may be rewritten as E^2 - Divisor x Coefficient = Remainder
Examples 12 modulo 83 = 144 - 83 x 1 = remainder 61
2 modulo 83 = 4 - 83 x 0 = remainder 4
40 modulo 67 = 1600 - 67 x 23 = remainder 59

R2. AREA RELATIONSHIPS

R2.1. INTRODUCTION TO AREA RELATIONSHIPS

This Section considers unit areas of various pairs of Elements, by starting with two adjacent Elements, then adding pairs of others on given 'Sides'. The 'Sides' are coded 1 to 6 clockwise. The two adjacent Elements may have various shapes, but Side 1 will always be regarded on the left hand side as shown below, whatever the shape.

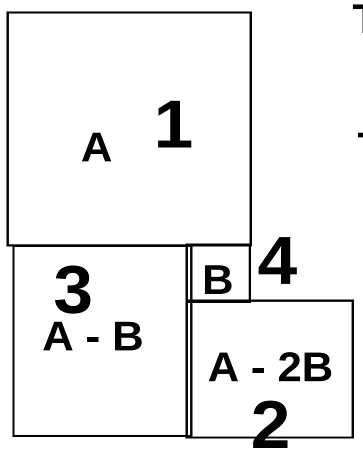


Elements are added to various sides - Side 5 illustrated

SHOWING SIDES CODES 1 TO 6

R2.2. AREA RELATIONSHIPS

In studying Reciprocal Pairs I suddenly discovered a relationship between four Elements within a Squared-Rectangle placed as Below -



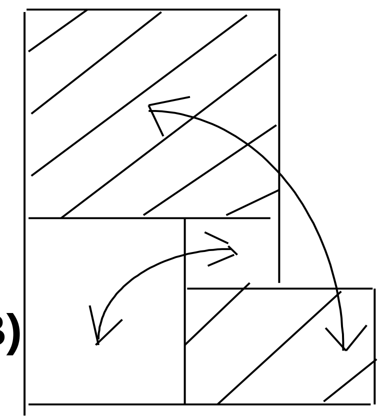
Total area of 1 & 2 is $a^2 + (a - 2b) * (a - 2b)$

$= 2A^2 - 4AB + 4B^2$

Total area of 3 & 4 calculates as

$= A^2 - 2AB + 2B^2$

so, AREAS 1 & 2 ARE EXACTLY TWICE AREAS 3 & 4 (for any values of A & B)

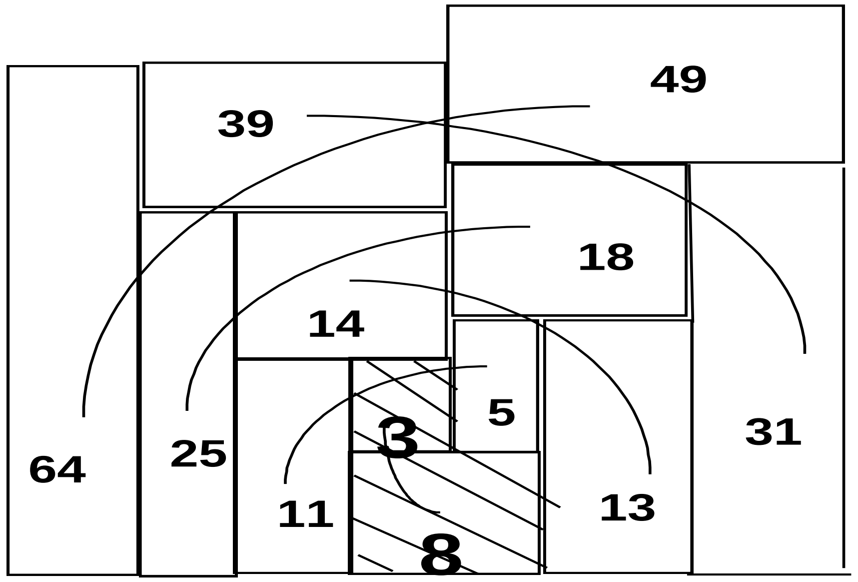


1. This pattern of Elements occurs very frequently in parts of Rectangles and sometimes several times in a Rectangle.
2. It is an important discovery since *some* Rectangles containing Reciprocal Pairs, pair up as above 2 where the shaded area is exactly twice that of the non shaded area.
3. It is easily seen that if one pair is a Reciprocal Pair, then the other must also be a Reciprocal Pair.
So in other Reciprocal Pair Rectangles it follows that those Elements which link up with these four Elements must all be somewhere else, and outside.
So... where Reciprocal Pairs exist, they will either be entirely INSIDE or entirely OUTSIDE such as block of four.
4. The block pattern above is shown occupying the leftmost side of the Rectangle, but could equally be at the right.
Also, there could be one at each end.
Or the whole pattern may be internal, or bordering any of the sides instead of the corners.
Simply it may occur anywhere within the Rectangle - also clockwise and / or anticlockwise.
5.. Next, observe that if there is also an Element 5 (as Above) then Areas 4 and 5 are also twice that of Areas 2 and 3!
Now clearly Areas 1 and 2, & Areas 3 and 2 cannot both be Reciprocal Pairs at the same time!

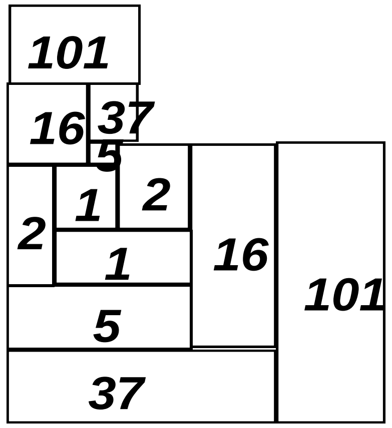
R2.3. RATIO AREAS BY SETS OF TWO ELEMENTS

Look at the following theory where a number of Elements have been calculated.
(NB. This is not a valid Rectangle nor can be made into one)

This commences with Elements 8 and 3 but any choice of two different Elements could be chosen instead:-



Now the Unit Area of the pair 8 and 3 is $8^2 + 3 * 3 = 73 = 73 \times 1$
11 and 5 have been added at sides 1 and 4. Now $11^2 + 5^2 = 146 = 73 \times 2$. This property was shown in C13.1.
Then 14 and 13 have been added at sides 2 and 5. Now $14^2 + 13^2 = 365 = 73 \times 5$
Then 25 and 18 have been added at sides 1 and 4. $25^2 + 18^2 = 73 \times 13$
Then 39 and 31 have been added at sides 2 and 5. The Area is found to be 73 x 34
Then 64 and 49 have been added at sides 1 and 4. The Area is found to be 73 x 89
The series could be continued ad infinitum by adding at 2 and 5 then 1 and 4 indefinitely.
The next two Areas are 233 and 610 times 73.
The coefficients of 73 are found to be the alternate values in the Fibonacci series 1 1 2 3 5 8 13 21 34 55 ...
Now whatever what Elements are chosen (e.g. 11 and 4 instead of 8 and 3) the Relative areas of the pairs of Elements remains the same - namely 1 : 2 : 5 : 13 : 34 : 89 : 233 : 610 .....
After the first pair of Elements (i.e. 8 and 3 above) the next pair can only come from sides 1 and 4 which we will give the Code A. But after this pairs can be added at either sides 2 and 5 which we will Code as B, or sides 3 and 6 which we will Code as C. Thus the series above is "-ABABABA..." with Relative Unit Areas of 1 : 2 : 5 : 13 : 34 : 89 : 233 : 610
Now consider "-AC" which will give Elements 8 and 3, 11 and 5 and 2 and 19 in the above. $2^2 + 19^2 = 385 = 73 \times 5$ again - but notice the different Elements from 13 and 14 above! The relative areas are 1 : 2 : 5 again.
There are endless combinations of A B and C as B can be followed by A or C; C can be followed by A or B and A can be followed by B or C.
The Relative Areas sometimes give duplicate Values (e.g. 5 and 5 above) but the patterns and the actual Element Values always change where this happens.
We shall now ignore the actual values of Elements and look at a different pattern -A followed by C and B alternately.



RELATIVE AREAS
FOR SERIES
-ACBCBCBCBC....

Numbers shown are Coefficients not Elements
e.g. The two labelled "101" are 101 times
the Unit Area of the two labelled "1" and
the two with "37" 37 times as large, and so on

.

This time we have Relative Unit Areas of 1 : 2 : 5 : 16 : 37 : 101 : 260 : 685 : 1789 ....

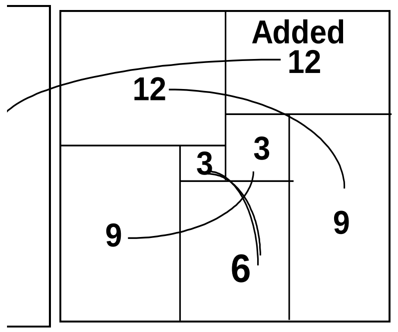
R2.4. COLLATING RELATIVE AREA RESULTS - 2 ELEMENTS

-A… is fixed at 1 : 2. -AB is 1 : 2 : 5 and -AC is 1 : 2 : different 5. After this codes take on many combinations.
-ABABABABA… gives 1 : 2 : 5 : 13 : 34 : 89 : 233 : 610... (same numerically as series ACACAC…) (minimum values)
-ABABAC gives 1 : 2 : 5 : 13 : 34 : 89 : 320
-ABABCA gives 1 : 2 : 5 : 13 : 34 : 121 : 320
-ABABCB gives 1 : 2 : 5 : 13 : 34 : 121 : 265
-ABACAB gives 1 : 2 : 5 : 13 : 45 : 100 : 353
-ABACAC gives 1 : 2 : 5 : 13 : 45 : 100 : 277
-ABACBA gives 1 : 2 : 5 : 13 : 45 : 121 : 353
-ABACBCBCB… gives 1 : 2 : 5 : 13 : 45 : 121 : 298 : 793 : 2061...
-ABCABA gives 1 : 2 : 5 : 16 : 45 : 130 : 313
-ABCABC gives 1 : 2 : 5 : 16 : 45 : 130 : 377 (maximum values at each stage)
-ABCACA gives 1 : 2 : 5 : 16 : 45 : 109 : 292
-ABCACB gives 1 : 2 : 5 : 16 : 45 : 109 : 377
-ABCBAB gives 1 : 2 : 5 : 16 : 37 : 130 : 289
-ABCBAC gives 1 : 2 : 5 : 16 : 37 : 130 : 353
-ABCBCA gives 1 : 2 : 5 : 16 : 37 : 101 : 353
-ABCBCBCBC… gives 1 : 2 : 5 : 16 : 37 : 101 : 260 : 685 : 1789…
-ACABAB gives 1 : 2 : 5 : 13 : 45 : 100 : 277
-ACABAC gives 1 : 2 : 5 : 13 : 45 : 100 : 353
-ACABCA gives 1 : 2 : 5 : 13 : 45 : 121 : 353
-ACABCB gives 1 : 2 : 5 : 13 : 45 : 121 : 298

-ACACAB gives 1 : 2 : 5 : 13 : 34 : 89 : 320
-ACACAC gives 1 : 2 : 5 : 13 : 34 : 89 : 233 each value is previous one times 3 less the one before that - so next value is 233 x 3 - 89 = 610
-ACACBA gives 1 : 2 : 5 : 13 : 34 : 101 : 320
-ACACBC gives 1 : 2 : 5 : 13 : 34 : 101 : 265
-ACBABA gives 1 : 2 : 5 : 16 : 45 : 109 : 292
-ACBABC gives 1 : 2 : 5 : 16 : 45 : 109 : 377
-ACBACA gives 1 : 2 : 5 : 16 : 45 : 130 : 313
-ACBACBA gives 1 : 2 : 5 : 16 : 45 : 130 : 377 : 1088
-ACBCAB gives 1 : 2 : 5 : 16 : 37 : 130 : 353
-ACBCAC gives 1 : 2 : 5 : 16 : 37 : 130 : 289
-ACBCBA gives 1 : 2 : 5 : 16 : 37 : 101 : 353
-ACBCBCBCB… gives 1 : 2 : 5 : 16 : 37 : 101 : 260 : 685 : 1789…
Side for Orders 4 6 8 10 12 14 16
Semi-perimeter for Orders .3 5 7 9 11 13 15
NB patterns for 7 upwards and the Elements vary from above.
There are many more. Note each is additions of two squares:
1st position 1 = 1 + 0 2nd position 2 = 1 + 1 3rd position 5 = 4 + 1
4th position 13 = 9 + 4 16 = 16 + 0
5th position 34 = 25 + 9 37 = 36 + 1 45 = 36 + 9
6th position 89 = 64 + 25 100 = 64 + 36 101 = 100 + 1 109 = 100 + 9 121 = 121 + 0 130 = 121 + 9
7th position 233 = 169 + 64 260 = 256 + 4 265 = 256 +9 277 = 196 + 81 289 = 289 + 0 292 = 256 + 36 298 = 289 + 9 313 = 169 + 144 320 = 256 + 64 353 = 289 + 64 377 = 361 + 16
8th position (some) 610 = 441 + 169 685 = 676 + 9 793= 729 + 64 1088 = 1024 + 64

R2.5. OBSERVATIONS FROM RESULTS

1. 130 377 1088 are typical sides or Semi-perimeters e.g. [10] 130 x 79 [12] 377 x various [14] 1088 x various.
2. All numbers above are the sum of two squared numbers e.g. 9 + 4 = 13, 100 + 1 = 1 01, 121 + 0 = 121 and therefore connect with Reciprocal Pairs theory.
3. Where a combination ends in a B, a Rectangle may be made by adding one Element at top right and using algebra, as below -



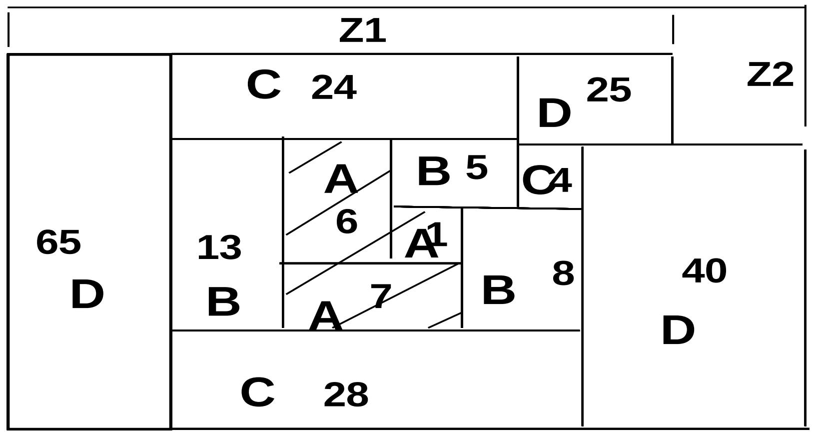
[7] 24 x 21 invalid
which reduces to
[7] 8 x 7 invalid

PATTERN -AB (plus two Elements, one not strictly needed)

Now in the above x and y are 3 and 6, whose unit area $3^2 + 6^2 = 45$ equals the SIDE (21 + 12 + 12).
Note that Reduced, $1^2 + 2^2$ is only 5 but the SIDE is 15 a multiple.
So full Dimensions have been shown.
Element 21 has been shown simply to link up Element 12.
A number of BO Solutions can be constructed in this way, but only those with two unknowns x and y.
Pattern -AB+ gives [7] 8 x 7 and Pattern -ABA+ gives [9] 6 x 5 invalid.
Some patterns give Singlends and some Duds.

R2.6. AREA RELATIONSHIP IN SETS OF 3 ELEMENTS

If three Elements are drawn as at A below (choosing any combination of numbers you choose) and 3 Elements are added at a time - at Sides 1 3 5 and 2 4 6 alternately a number of times, the total Unit Area of each group is related to Group A.
Area of A's is 1 + 36 + 49 = 86. Area of B's is found to be 258 which is 3 x 86. Area of C's is 1376 which is 16 x 86. Area of D's is 6450 which is 75 x 86.
The next Group total 361 x 86 and so on!



CHOOSE 3 ELEMENTS A - say 1 6 & 7
Add Elements at Sides 1 3 & 5 - B
Add Elements at Sides 2 4 & 6 - C
Add Elements at Sides 1 3 & 5 again - D
(alternately 2 4 6 & 1 3 5 and so on ....)

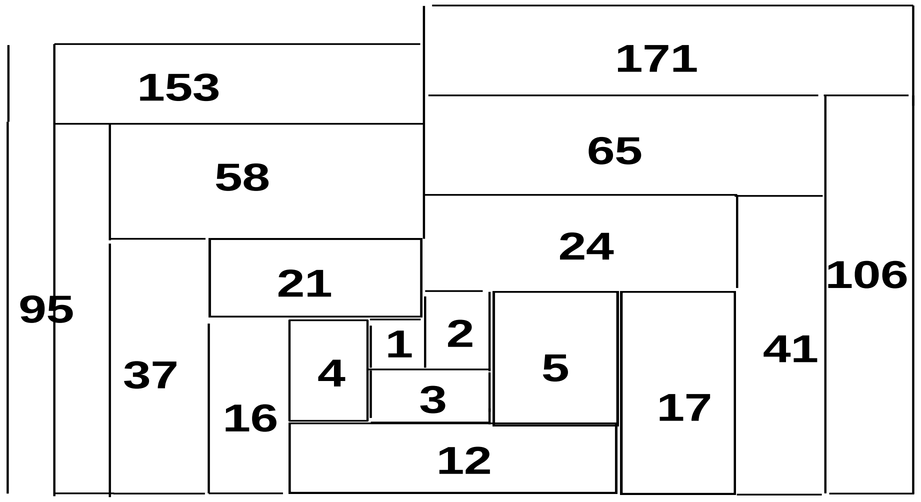
The reader can easily verify using other numbers for A that the Ratios remain constant at 1 : 3 : 16 : 75 : 361 ...
I am yet to discover the mathematical link for these numbers.
Unlike the previous construction with pairs of Elements there is only one construction for each Order possible.
At any stage two Elements can be added and a Rectangle recalculated with algebra e.g. Z1 and Z2 in above giving an Order 14 Solution.

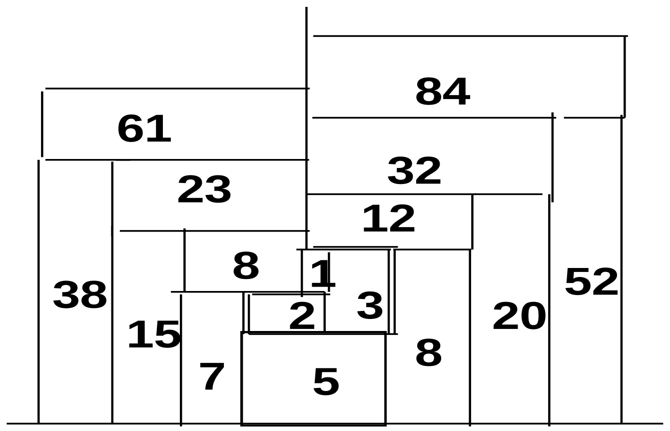
R2.7. THE SOLUTIONS OF ORDER 8, 11, 14, 17...

The single Solution for each third Order which must have sides of 2-2-2-3 commence with -
[5] 8 x 8 invalid (reduces to [5] 2 x 2 invalid R4).
[8] 40 x 35 invalid (reduces to [8] 8 x 7 invalid R5). The pattern actually gives 35 x 40.
[11] 190 x 171 invalid (reduces to [11] 10 x 9 invalid R19). The pattern actually gives 171 x 190
[14] 38 x 34 invalid (reduced)
[17] 48 x 43 invalid (reduced)
[20] x invalid (reduced)
[23] 134 x 110 invalid and so on. No excitement here! But note the Semi-perimeter of each full Dimensioned Solution - 16 75 361 ... which coincides with the Ratios above.

R2.8. ANOTHER ELEMENTS RELATIONSHIP

Look at the construction Below 1 where most of the Elements form symmetric pairs e.g. 1 and 2, 4 and 5, 16 and 17...
Note the following -
$2^2 - 1^2 = 3 = 3 \times 1$ 25 - 16 = 9 = 3 x 3
289 - 256 = 33 = 3 x 11 576 - 441 = 135 = 3 x 45 1681 - 1369 = 312 = 3 x 104 4225 - 3364 = 861 = 3 x 287
11236 - 9025 = 2211 = 3 x 737 29241 - 23409 = 5832 = 3 x 1944
All differences in squares are divisible by 3.
When “1 and 2” are substituted with any other values then all pairs are found to be divisible by the difference between the squares of the values. But the important thing is that the coefficients stay the same i.e. 1 3 11 45 104 287 737 1944 ...
The Elements 3 and 12 added at Side 6 do not form pairs.
Also different patterns can be drawn by adding odd Elements at positions 3 as well as 6.





In Above 2 the starting Elements are 3 and 2 where 3 x 3 - 2 x 2 = 5. The Elements not part of pairs are 1 and 5.
9 - 4 = 5 x 1 64 - 49 = 5 x 3 144 - 64 = 5 x 16 400 - 225 = 175 = 5 x 35 1024 - 529 = 495 = 5 x 99
2704 - 1444 = 1260 = 5 x 252 7056 - 3721 = 3335 = 5 x 667 and so on... with Coefficients of 1 3 16 35 99 252 667 ...

R3. RECIPROCAL PAIRS

R3.0. TEMPORARY SECTION – CONDITIONS FOR FULLY RECIPROCAL PAIRS – CODE “B0”

1. The Divisor has be the addition of two squared numbers (unequal positive integers). AND
2. The Divisor minus 1 and divided by 2 has to be an EVEN number, AND
3. All Prime Odd Factors of the Divisor, or combinations of them when diminished by 1 and divided by 2 must give an EVEN result. AND
4. The Divisor must exist as the Full Semi-Perimeter of some Even Order Solution and also be the Full Side of an even Order Solution.

For Example 377 = 29 x 13 = 361 + 16 (i.e. 19 squared + 4 squared)
377 is 29 x 13, and

(377 – 1) / 2 = 188 EVEN. (29 – 1) / 2 = 14 EVEN. (13 – 1) / 2 = 6 EVEN.
Lastly, amongst others, Solutions [12] 377 x 256 and [11] 209 x 168 (with 209 + 168 = 377) happen to exist.
377 gives Fully Reciprocal Pairs – Code B0.

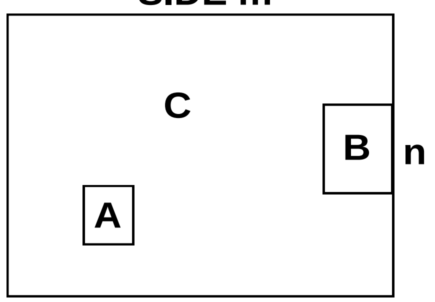
(a). If requirement 1 is true but any of requirements 2 3 or 4. fail, the Solution is usually type BA with NO RECIPROCAL PAIRS.
(b). If requirement 1. Is untrue – then some divisors with be the difference of two squared integers and the rest not. Some of the former may have a number of Identical Pairs, while the latter have no Identical Pairs either.
R3.0.1. FULLY RECIPROCAL PAIR SOLUTIONS [a]. TEMPORARY USEFUL INFO
These not only exist for its DIRECT DIVISOR [b] but also for ANY FACTOR [c] OF THE DIVISOR.
NOTES –
[a] Whether Full Dimensions, Reduced Dimensions or Reduced Dimensions Up-rated to Full.
[b] i.e. Longer side for Even Ordered Solutions, or Semi Perimeter (Reduced Semi Perimeter sometimes) for Odd Ordered Solutions.
[c] Some Divisors have no factors.

EXPLANATION BY EXAMPLES –
[9] 15 x 11 (Reduced from 75 x 55) Works for Divisors 26 13 & 2 as factors of the Divisor 26 = 15 + 11.
But NOT by 5 10 65 or 130 none of which are factors of 26, but are of 130.
[9] 75 x 55 (Up-rated to full size from [9] 15 x 11) Works for Divisors 130 65 26 13 10 5 and 2 – all of which are factors of 130 as 75 + 55.
[9] 69 x 61 (Full Size). Works for Divisors 130 65 26 13 10 5 and 2 – all of which are factors of 130 as 69 + 61.

R3.1. GENERAL FORMAT FOR RECIPROCAL PAIRS AND RECIPROCAL SETS

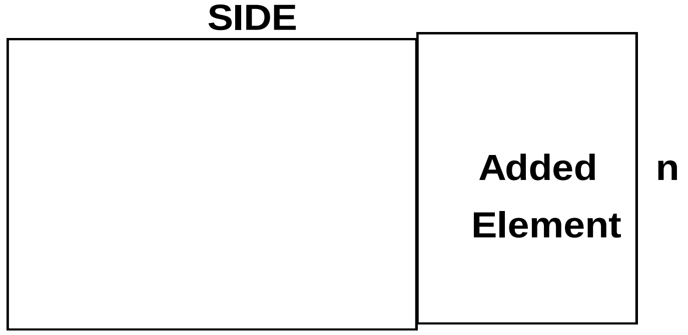
In a Rectangle m x n as the Sum of all Elements E1 to E(Order) is clearly m x n square units. This is a quantity obviously divisible by m and also by n.
If any combination(s) of Elements (however many, or however large or small) are found within the Rectangle to be divisible by m (say A and B Below) then the Remaining Area (say C Below) must also be divisible by m.
This is important since when all Reciprocal Pairs for Divisor m are excluded from the Solution (whether they exist or not, or to what extent), then the total of the remaining area must also be divisible by m.
This means that a B2 Solution cannot exist.
It is sometimes possible to subdivide such a remaining area into two or more chunks each being a RECIPROCAL SET. However B4 and B6 Solutions cannot be subdivided (since no more pairs exist).
But B8 Solutions sometimes divide into two RECIPROCAL SETS of 4.
So, all Solutions will Contain RECIPROCAL SETS AND / OR RECIPROCAL PAIRS with one pattern for divisor m and one for divisor n. However the number of Solutions with RECIPROCAL PAIRS throughout is much fewer.

SIDE m



EVEN ORDERS

SIDE



ODD ORDERS MADE EVEN

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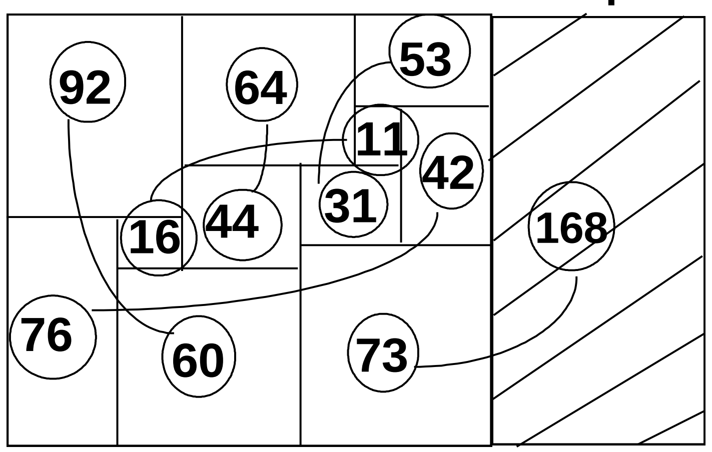
R3.2. GENERAL FORMAT FOR SIDES AND SEMI-PERIMETERS

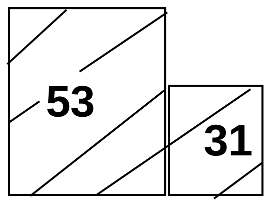
Actual Elements are ignored in this Section, and attention is reserved to the Areas of SIDES and the remainders found by dividing by them and Semi-perimeters.
1. Suppose 185 x 183 is tested which has a Semi-perimeter of 368 is chosen, the square of which is 135424
It is found that: 135424 = 185 x 732 + 4 remainder; and 135424 = 183 x 740 + 4 remainder.
Here remainders are the same, 4.
2. Many others can be tested with remainders found to be the same, but in many others such as 185 x 151 sp 336 (which squared is 112896).
112896 = 185 x 610 + 46 remainder. And 112896 = 151 x 747 + 99 remainder. Remainders differ!
3. The reason for the apparent discrepancy is now given. On testing several combinations it is found that Remainders which are the same happen to be the square of the difference of the sides i.e. (m - n) squared.
So 185-183 = 2 and 2 squared gives 4 as in 1. above. But 185 - 151 = 34 and 34^2 is 1156. The equation in 2. can be rewritten 112896 = 185 x 604 + 1156 and 112896 = 151 x 740 + 1156!
$(m + n)^2 = m \times coef1 + (m - n)^2$ ... $(m + n)^2 = n \times coef2 + (m - n)^2$.

or, $(m + n)^2 - (m - n)^2$ has to be divisible separately by both m and n.
$m^2 + 2mn + n^2 - (m^2 - 2mn + n^2) = 4mn$ which proves the point. Here the coefficient is 4 times the side e.g. 740 is 4 x 185 and 604 is 4 x 151 in above.
4. In looking at true remainders which naturally have to be less than the side m (or side n), if the difference between the sides squared is LESS than m and less than n, then the true remainders will be the same. If MORE, they will differ.
Codes - MR is remainder after semi-perimeter squared is divided by m and NR ditto with n.
5. Using 185 x 151 sp 336 again, other Remainder combinations exist. Now, $185^2 = 34225$ and $151^2 = 22801$
112896 = 336 x 336 + 0 34225 = 336 x 101 + 289 22801 = 336 x 67 + 289 289 = 17^2
112896 = 185 x 610 + 46 34225 = 185 x 185 + 0 22801 = 185 x 123 + 46 1156 = 34^2
112896 = 151 x 747 + 99 34225 = 151 x 226 + 99 22801 = 151 x 151 + 0 1156 = 34^2

R3.3. EXAMPLE OF RECIPROCAL PAIRS

Shown below is [11] 209 x 168 redrawn as [12] 377 x 168. (Note that Element 73 pairs equally well with the longer side 209, as 168).
The curved lines link the Reciprocal Pairs.





$53^2 + 31^2$
RECIPROCAL

[11] 209 x 168 amended to [12] 377 x 168

Take 11 and 16. Now 11 x 11 = 121 and remainder when ÷ by 377 is 121. 16^2 is 256 and remainder when ÷ by 377 is 256. Now, 121 + 256 is 377 our Divider!
Try doing this with any pair and you will find two differing numbers which total 377. Now look at 53 and 31 (Above 3) where the two remainders total 377. Another way of looking at Reciprocal Pairs is to realize that the ADDITION of the two unit areas is divisible by 377. So 53^2 (2809) + 31^2 (961) gives 3770, a multiple of 377

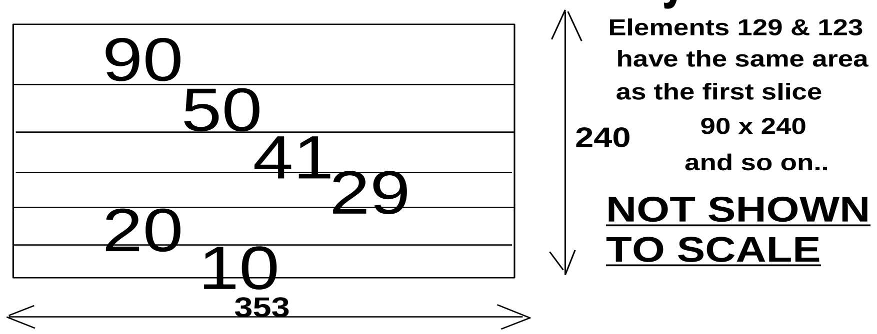
R3.4. USING SP AND DIVISORS OF SPS

Where a rectangle of full size happens to have Reciprocal Pairs for the full size Semi-Perimeter, and that Semi-Perimeter is a factorial number, it is found that RP's exist whatever factor is used as the base. The pairs always follow the same positions.
e.g. In [9] 69 x 61 with full size SP of 130. RP's are not only possible for 130 but also using numbers 65, 26, 13, 5 and 2 instead, and pair off according to the same patterns as for 130.

But for [9] 33 x 32 (RI 2) RP's work for 65 (reduced SP) but not in the case of
[9] 66 x 64 (full size) with 130. Using 13 and 5 RP's still exist but not for 26 or 2 due to the factor of 2.

R3.5. RECIPROCAL PAIRS DIVIDE INTO AREA ROWS

Rectangles where pairs of Elements are divisible by the SIDE when added (but not subtracted) as happens to apply in [12] 353 x 240 the Elements pair as follows.
129 x 129 + 123 x 123 = 353 x 90
73 x 73 + 111 x 111 = 353 x 50
117 x 117 + 28 x 28 = 353 x 41
101 x 101 + 6 x 6 = 353 x 29
66 x 66 + 52 x 52 = 353 x 20
59 x 59 + 7 x 7 = 353 x 10.
In total = 353 x 240. This curiosity is shown below -



R3.5.1. THE RECIPROCAL COEFFICIENTS SERIES

Is there any connecting link in the numbers 10, 20, 29, 41, 50 and 90 above?
Well a small clue may be that -
90 = 81 + 9. 50 = 49 + 1 (or 25 x 25) 41 = 16 +25. 29 = 25 +4. 20 = 16 +4. 10 = 9 + 1. Each coefficient happens to be the ***sum of two squared integers***!
Other examples also reveal this is true!
13 16 26 32 65 & 80 in [12] 353 x 232
5 10 25 34 61 & 65 in [12] 353 x 207
9 10 17 20 32 & 89 in [11] 209 x 177 when regarded as [12] 207 x 386
Note that with 9, 9 = 0 +9.
Numbers like 3, 6 and 21 which cannot be expressed as addition of two squares never appear! Some can appear in more than one way e.g. 25 = 0 + 25 or 9 + 16 & 50 (see above).
In the range 1 to 50 roughly half are addition of two squares and the other half not.

As is shown later some solutions give alternative Fibonacci Numbers such as 1, 2, 5, 13, 34, 89 etc. and it is interesting that all these happen to be the sum of two squares (e.g. 89 = 64 +25), while many of the in-between Fibonacci Numbers are not (e.g. 21 & 55).
I have yet to discover any further link between the numbers.

R3.6. RECIPROCAL PAIR RECTANGLES CAN BE DIVIDED INTO SETS OF RECTANGLES

Like Identical Pairs with its curious set of rather unrelated rectangles based on
$x^2 - y^2$ which of course is (x + y) x (x - y), Reciprocal Pairs have remainders totaling $x^2 + y^2$ this expression having no algebraic factors (though factors often do exist).

R3.7. RECIPROCAL PAIRS & LARGER DIMENSIONS, PRIME OR TWICE PRIMES

B0 cases exist in Order 12 where the Larger Dimension is 353 which happen to be prime.
This is also true for 193 (or 386 = twice 193).
Fine so far! But in the case of 313 and 307 which are also primes, the Pairs fail completely!
377 is product of primes 13 and 29. In fact there is annoying inconsistency, as revealed by these results -
1. [14] 197 x 159 s2334 [14] 197 x 155 s2333 [14] 197 x 157 ALL WORK. ***But [18] 197 x 190 s3334 3 pairs only.***
2. [12] 193 x 126 s2324 [12] 193 x 129 s2324 [12] 386 x 277 s2323 [12] 386 x 247 s2324 [12] 386 x 207 ALL WORK
but not [18] 193 x 137 s2335 and [18] 193 x 152 2335 and [18] 193 x 176 s3333..
3. [12] 353 x 207 2325 [12] 353 x 232 s2334 [12] 353 x 240 s2324 [12] 353 x 255 s2333 [12] 353 x 262 s2323 [12] 353 x 280 s 2333 and [12] 353 x 285 s2333 ALL WORK
4. [12] 377 x 231 s2324 and [12] 377 x 231 s2324 BOTH WORK
5. [12] 313 x 247 s2225 [12] 313 x 280 s2234 and [12] 313 x 295 s2233 - none work!
6. [12] 307 x 278 s2223 does not work.

R3.8. SOME RULES CONNECTING RECIPROCAL PAIRS RECTANGLES

The position is made a little clearer when it is found that Full Sides (not Reduced Sides) must be taken into account and that the factors of these Full Side are also crucial.
For divisor 65, Element 1 is found to connect by remainders with four values 8 or 18 or 47 or 57 i.e. $1^2 + 57^2$ is divisible by 65 and so on.
Actually this relationship is true of an infinite number of integers - negative or positive - if multiples of 65 - 65, 130, 195 … is added to any of these - 8 18 47 57 e.g. 73 83 112 122 and so on.. the condition remains.
In practice we need only know of the first two 8 and 18 to work out the next two 65 - 8 = 57 and 65 - 18 = 47 and then add on multiples of 65 to these four as required.
Given a divisor p there are a *maximum* of 2 values between 1 and half p and a further 2 between half p and p.
It is difficult to find absolute rules, but the following seem correct -

1. For any given Element number there are actually always two connecting numbers e.g. with SIDE 197 (prime) Element 1* pairs with 14, but also theoretically with 183. Now 14 + 183 equals the SIDE 197, and it is found that the numbers associated with 2 and with 3 also total 197 - in fact a total of 197 is always found to apply whatever starting number is used.
Now half of 197 is 98.5 and since one of these numbers is always lower than 98.5 it follows that the other is always above 98.5. Actually the real Reciprocal is always the lower and never the higher one.
For instance in [14] 197 x 157 which pairs in reciprocals throughout the highest Element is 88.
Thankfully this means that we can ignore the higher numbers entirely and regard the lower ones as the only matching ones. (Note: * To be exact, 196 (being 197 - 1) as well as 1 strictly connects with those of 14 and 183. i.e. 1 and 196 and 14 and 183 both pairs summing the Dimension used - in this case 197. But really the only important Elements which link are 1 and 14)
2. In view of 1. It follows that for a SR to have completely Reciprocal pairs, the largest Element can never exceed half the Dimension side in question.
3. In view of 2. It follows that there cannot be just two Elements along the top of the Rectangle which means solutions of sides 222 (anything) and 22 (anything) can never have fully Reciprocal Pairs! However this statement is only true of Even Order solutions, since in the case of Odd Orders (e.g. 11 or 13) Reciprocal Pairs rely on a component Element being added at left or right so making at least 3 Elements along the top.
4. In view of 3 this explains why many of the Rectangles listed above are not Reciprocal Pairs type, but what about [18] 197 x 190 s3334 and [18] 193 x 176 s3333?
5. Worse are [18] 193 x 137 s2335 and [18] 193 x 152 s2335 which shows that s23** rectangles with a longer prime SIDE do not automatically have Reciprocal Pairs.
In Above the oblique lines show the Reciprocal Pairs of a certain Rectangle. In this example, 39 and 71 and 16 and 55 are Reciprocal Pairs so note that the other pairs all form within a separate area (to the right).
Now $16^2 + 55^2 = 3281$ and $71^2 + 39^2 = 6562$; twice as much. Now the factors are x 193 and 34 x 193.
Since 193 is prime, this indicates that the rectangle must be 193 x 126 in size. In fact it is [12] 193 x 126!
Theoretically, it ought to be possible to tamper with various figures and obtain the actual dimensions of a Perfect Rectangle though without finding what the Order or the Reduction Index is! But there are problems with this, not the least being that so many Rectangles do not have the Reciprocal Pairs property in the first place!
And often the four Elements will pair up with others elsewhere, in which case the other Dimension cannot be found.

R3.9. THERE MAY BE MORE THAN ONE SET OF PAIRING POSITIONS

In considering [9] 33 x 32 plus a Singlend of 32 (to make Order 10 i.e. 5 sets of 2) the Writer found at least 6 ways of pairing the Rectangle - as follows - Semi-perimeter of 33 + 32 = 65.
1. 9 & 7 (9 squared + 7 squared = 2x65) 8 & 1 (1x65) 14 & 18 (8x65) 4 & 32 (16x65) 15 & 10 (5x65)
2. 7 & 4 (1) 8 & 1 (1) 14 & 18 (8) 32 & 9 (17) 15 & 10 (5)
3. 18 & 1 (5) 32 & 9 (17) 8 & 14 (4) 7 & 4 (1) 15 & 10 (5)

4. 9 & 7 (2) 8 & 14 (4) 32 & 4 (16) 18 & 1 (5) 15 & 10 (5)
5. 32 & 9 (17) 18 & 1 (5) 14 & 8 (4) 7 & 4 (1) 15 & 10 (5)
6. 32 & 9 (17) 8 & 1 (1) 14 & 18 (8) 7 & 4 (1) 15 & 10 (5)
Alternative pairing occurs when there is also some Identical Pairs existing. Although some solutions have fixed pairing (i.e. no alternative pairings), others do not. So the association between pairs is a bit variable.

R3.10. RECIPROCALS SETS

Up to now we have considered only Reciprocal Pairs. But could it be that Reciprocal Elements might apply in sets other than pairs. Yes! When [14] 97 x 81 is examined which contains Elements of 2 2 3 5 7 8 11 13 15 17 28 37 44 53 in numerical order, it is annoying to find that where Reciprocal Pairs are concerned only 5 and 13, 2 and 44, 2 and 53 apply, leaving eight Elements not pairing.

Now since all Elements total 97 x 81 or 7857 and those which are pairs are divisible by 97, it is clear that the total area of the 8 remaining Elements are also divisible by 97. But can these 8 be divided into bits each divisible by 97? In this case yes.
8 11 17 37 give a total area of 19 x 97, and 3 7 15 28 an area of 11 x 97.
5 13 - area 2 x 97 2 44 - area 20 x 97 2 53 - area 29 x 97. Whole area 81 x 97.

But if the remainders after division by 97 are considered separately, these always are multiples of 97.
Repeating the values above in terms of remainders we obtain -
64 + 24 + 95 + 11 = 2 x 97 9 + 49 + 31 + 8 = 1 x 97 *
25 + 72 = 1 x 97 4 + 93 = 1 x 97 4 + 93 (again) = 1 x 97 In total 6 x 97.
(* NB not 2 x 97 as I expected - the coefficient of 97 is not always half the quantity of Elements!).
Now it becomes evident that this Reciprocal Idea exists for any and every Solution, as in the worst case the total of all Elements areas is always a multiple of the larger SIDE (and a multiple of the smaller side), even if the Elements cannot be broken down into smaller groups..

R3.11. MORE ON RECIPROCAL PAIRS

Where fully RECIPROCAL PAIRS exist, I have discovered
THE SIDE IS THE SUM OF TWO INTEGERS SQUARED
One of these Squared Integers can be zero, as SIDE may be Squared Numbers such as 64 or 121.
SIDE here often refers to the larger Dimension, but occasionally refers to the smaller one.
The SIDE can be primes (of type 2x + 1 where x is an integer) or factorial.
In the range 50 to 100 these are the additions of two squared numbers -
50 52 53 58 61 65 68 73 80 82 85 89 90 97 100
Excluded are 51 54 55 56 57 59 60 62 63 64 66 67 69 70 71 72 74 75 76 77 78 79 81 83 84 86 87 88 81 92 93 94 95 96 98 and 99 including various primes. Note that these primes are not divisible by 8 when diminished by 1.
As numbers get larger most are the products of two squared integers.

Many Rectangles fail to have the Reciprocal Pairs Property.

R3.12. FURTHER INVESTIGATION - ORDER 13

1. The following Values are found to give Full Reciprocal Pairs - type B0

937 = prime, 962 = 2 x 481, 985 = 5 x 197, 1033 = prime, 1049 = prime, 1060 = 2^2 x 5 x 53, 1066 = 2 x 13 x 41, 1082 = 2 x 541 and 1145 = 5 x 229

1025 works for divisor 1025 (= 5 x 5 x 41) but not for divisor 205 (R5) in the case of [13] 106 x 99, 108 x 97, 109 x 96, 111 x 94, 112 x 93 (1) and (2) and 117 x 88. To get pairs for these either (1) up-rate to full Dimensions and use Divisor 1025 or (2) leave unchanged and use divisor 41! The effect is the same.

1156 (= 2 x 2 x 17 x 17) also works but only if the Divisor used is 289 and not 578 or 1156, as in [13] 145 x 144, 161 x 128, 293 x 285, 305 x 273 and 319 x 259.

2. The following Values contain prime factors NOT of the type 4x + 1 and so as expected no BO cases occur -

928 943 992 1008 1015 1058 1065 1102 1103 1115 1122 and 1166.

3. This leaves the following with factors of 2 and some primes of type 4x + 1 - but fail to give Reciprocal Pairs! Why??

928 = 2 x 2 x 2 x 2 x 2 x 29 = 28 squared + 12 squared. Works with 29 and 58 however.

986 = 2 x 17 x 29 = 961 + 25 = $31^2 + 5^2$. Have tried 593 x 393 17 29 493 58 do not work!

There are elements 34 and 174 which conflict with factors however. May be the reason.

1040 = 2 x 2 x 2 x 2 x 5 x 13 = 1024 + 16 = $32^2 + 4^2$.

1073 = 29 x 37 = 1069 + 4 = $33^2 + 2^2$.

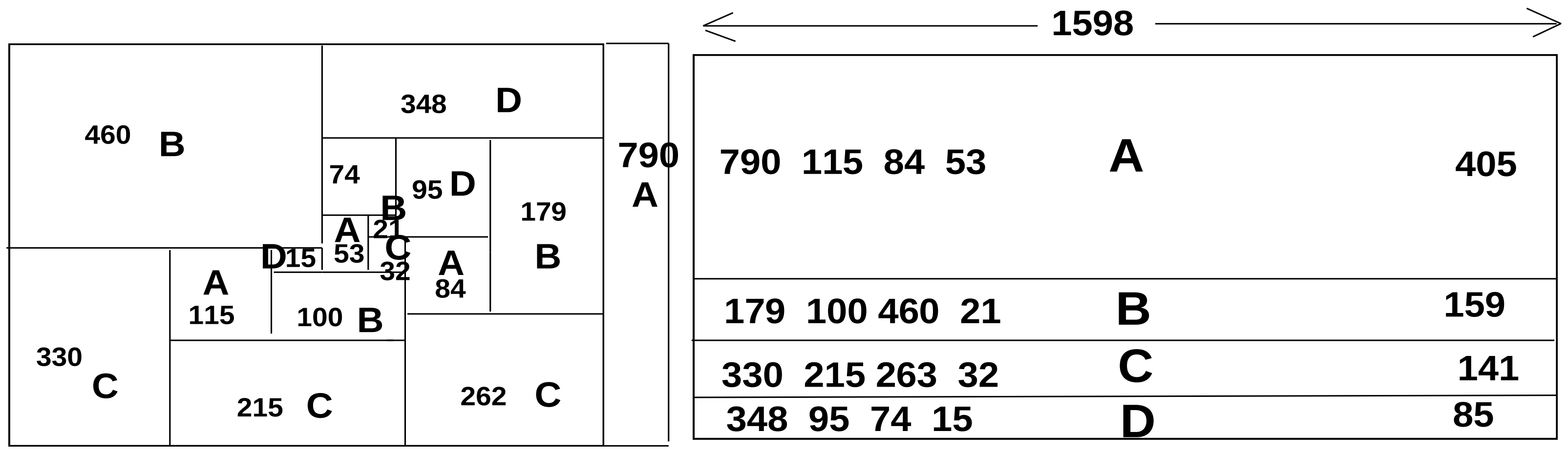
1088 = 2 x 2 x 2 x 2 x 2 x 2 x 2 x 17 = 1024 + 64 = $32^2 + 8^2$.

NOTE: when the odd part of the factor minus 1 is considered, the factors of 2 are 4 or 16 i.e. even factors of 2 for those that don't work. For those which do work we have factors of 8 or 32 i.e. odd factors of 2, but annoyingly also 532 540 and 1024 which have 4 or 1024!

***

R3.13. SLICING A RECTANGLE WITH RECIPROCAL SETS

The Solution Below is interesting as it contains no Reciprocal Pairs whatever, but happens to have four Reciprocal Sets of four! e.g. The squares of Set A 790 115 84 53 total 405 x 1598, B 159 x 1598, C 141 x 1598 and D 85 x 1598, 1598 being the Semi-Perimeter of this Order 15 Rectangle (or if you like the larger side when 790 added at right making the Order effectively 16). This means the Solution can be shown as four slices as in below 2 corresponding to the exact areas!

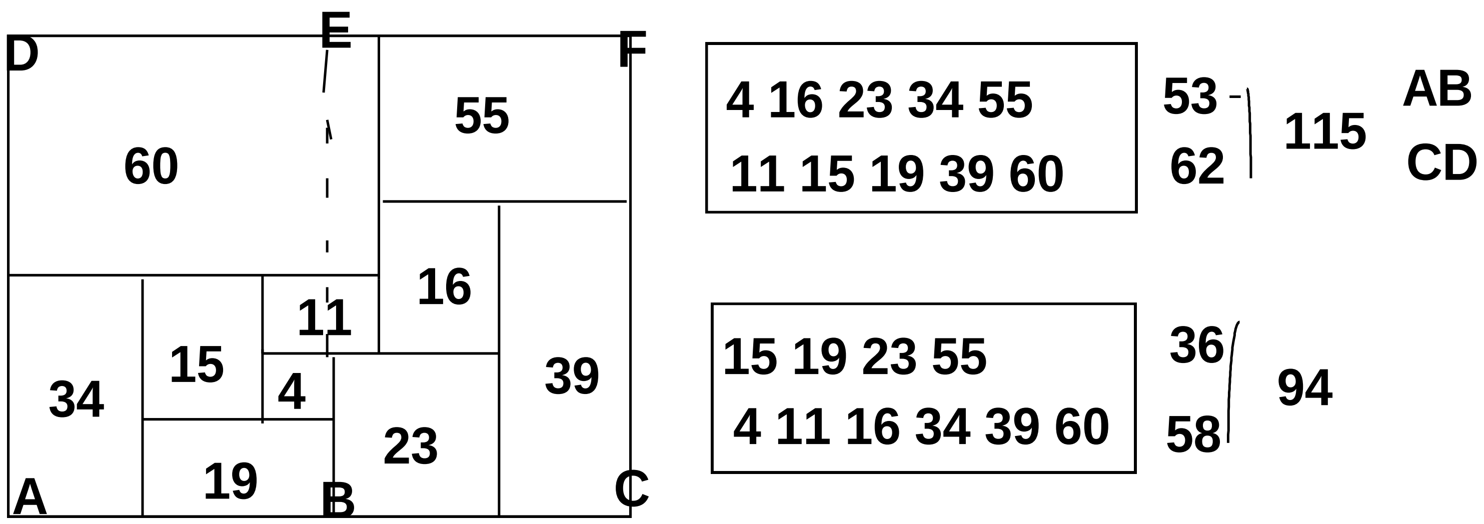


[15] 808 x 790 (plus Element 790 at right to make Order 16 - an even number.

Other solutions may be less tidy. I found a Solution of Order 13 (14 when end Element added) where there is a Set of 4 plus a Set of 10 only.

In the case of [10] 115 x 94 Below it is interesting to find groups of Elements whose total area equals 53 x 94 (coinciding with ABED) and 62 x 94 (coinciding with BCFE) as 53 is the distance AB and 62 the distance BC. But there are also 4 Elements with total area 115 x 36 and 6 Elements with 115 x 58, the 36 & 58 having no significance, and I believe any apparent relationship with distances within the Rectangle is simply coincidence. In this Rectangle no Reciprocal ***Pairs*** exist for the Divisors 94 and 115.

(Note: The Area 53 x 94 below is made up of the total areas for 4 16 23 34 and 55 and so on).

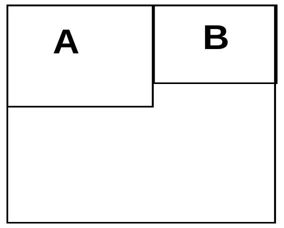


[10] 115 x 94

Checking on a number of Rectangles, so far most divide up into at least two areas, but not always on both the horizontal & vertical Divisors together.

R3.14. ADJACENT ELEMENTS

Where two adjacent Elements occur along a Side, and that Side is the Divisor under consideration then it can be shown that the two Elements must be Identical Pairs.



A and B form Identical Pairs when
Divisor m (= A+B) is considered

Let us look at whether the Rectangle above could have Reciprocal Pairs throughout for Side m (i.e. A + B). If so then there would have to be a Pairing with A, and a pairing with B also. But Rem. (A^2 ÷ (A + B)) is the same as Rem. (B^2 ÷ (A + B), implying that the Reciprocal Pairings may have to be the same. We need to consider -

1. If A = B then they form a Reciprocal pair themselves, but of course such solutions are Invalid.
2. If there is only a single Reciprocal Link for A, and ditto B then this link is duplicated meaning that such Rectangle must either be Imperfect assuming it does have Reciprocal Pairs throughout, or Perfect and not fully have Reciprocal Pairs, or possibly no RP's at all.
3. If there is more than one Reciprocal Link for A and B (say called C & D) then theoretically A might link with C and B with D, thus making a Perfect Solution still a possibility - so far no such Solution has been seen however.

Could it be that no Rectangle with two Elements across ever have RP's? (However note that for some Odd Order Rectangles with SP Divisors, RP's are possible, as in making the Order even we are introducing a third Element across the top).

R3.15. ANY SQUARED-SQUARES WITH RECIPROCAL ELEMENTS

So far the Writer has failed to find a single example even where the Divisor seems to be suitable for it. Does it mean they never exist? If so, why?

R3.16. FINDINGS FROM A BASIC COMPUTER PROGRAM

1. Prime Divisors of type 2x + 1 (but not 4x + 1). (Odd numbers).
No combinations of numbers can ever give a single Reciprocal Pair!

2. Factorial Divisors of type 2x + 1 (but not 4x + 1). (Odd numbers). These give isolated pairs but since they all have a common factor of 3 or 5 or 7 or another prime only, it is clear that such a Rectangle cannot possibly have fully RP's, so
2x + 1 Divisors - NO RP'S.

3. Odd Divisors having a single or number of factors all of the type 4x + 1 may *possibly* have RP'S though there are exceptions.

4. Divisors mentioned in 3. multiplied by two also may possibly have RP's though there are exceptions.

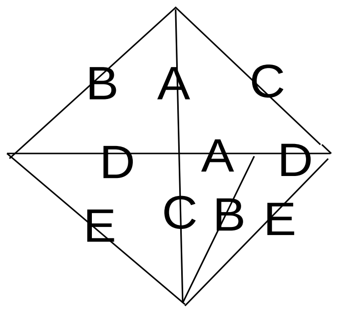
5. Divisors containing more than one factor of 2 never give fully RP's.

6. For Prime Divisors (type 4x + 1 only), there are in practice at least one pairing for all individual numbers 1, 2, 3, 4 up to the Divisor! Some values may also have alternative RP values (so if for example 1 and 31 match but also 1 & 54 and 1 & 71 no assumption can be made in advance as to which is applicable in a particular case!).

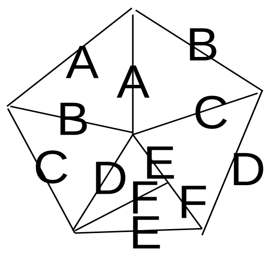
R3.17. NETWORKS AND RECIPROCAL PAIRS

Reciprocal Pairs are linked with Complexities and Complexities with Networks.
Networks do not show Poles as Poles can be chosen from any two outer points.
Where Poles are chosen one line in-between, then a Solution of the Order below + a component square is found (unless found to be Invalid).

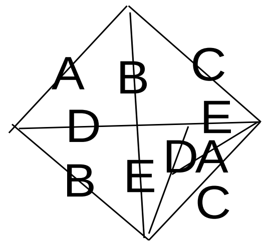
In Below 1, the Smith Diagram for [10] 130 x 79 has been shown for which Reciprocal Pairs exist. These are shown A & A, B & B C & C etc.
All very well, but if different Poles are used what then? Well that solution also has Reciprocal Pairs.
But does the pairing up change? Conveniently it is found to be the same!
This remains true for Order 9 Solutions with <u>Semi-Perimeter</u> of 130 as well.



[10] Complexity 130

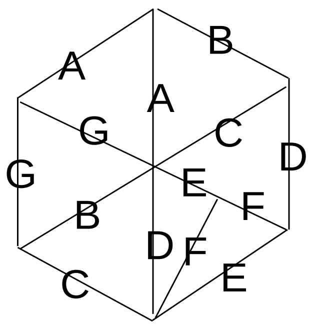


[12] Complexity 353

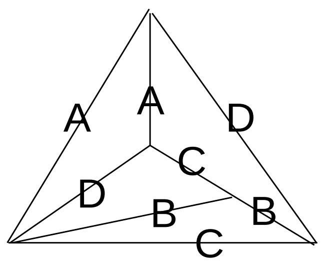


[12] Complexity 377

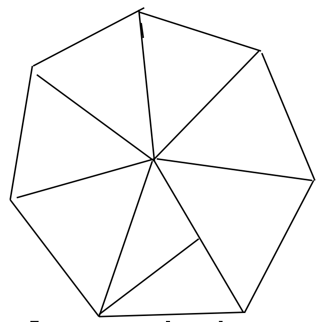
Likewise with [12] 353 x various, the Reciprocal Pairing are found as shown as in above 2. (NB 353 has two Complexities and two different Diagrams but the same applies to the other).
Interesting questions now arise -
1. Is it possible to ascertain what patterns will produce R.P's?
2. Having found a Network, it is possible to ascertain the fixed individual Pairings?



[14] Complexity 937



[8] Complexity 45


[16] Complexity 2466

Notice the similarity between the 130 & 353 Complexities which are 4 and 5 sided?
Equivalent patterns for 6 and 3 sided reveal Complexities of 937 and 45, and sure enough they have reciprocal pairs as indicated. Although the 7 sided form also has Reciprocal Pairs some are RP's for 137 (1/18 of 2466).
We have a series of (8), 45, 130, 353, 937 and 2466 and a connecting formula can be found! 353 = 3 x 130 - 45 + 8. 937 = 3 x 353 - 130 + 8. 2466 = 3 x 937 -353 + 8!

We take the last known value (2466) multiply by 3, deduct the previous value (937) and add 8 = 6469 for Order 18 (and 17) this time a prime.

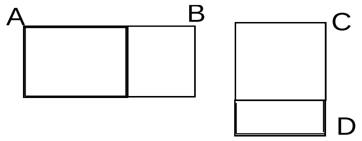
One of the Solutions for this is [18] 6469 x 3715 - +2292-204+1973:.231+1742: .924-1511:+1423.869:.337-587:-554.315:.87-250:-239.76:-163 and the pairing checked as follows 76 & 163, 87 & 239, 231 & 1973, 250 & 315, 337 & 554, 587 & 869, 924 & 1423, 1511 & 2292, 1742 & 2204 for Divisor 6469.

CHECK 6468 NOT DIV BY 8 BUT BY 4 - DID NOT EXPECT THIS TO WORK!!

R3.18. CALCULATING SOME RECIPROCAL SOLUTIONS FROM ALGEBRA

The Writer has found the following:

If a Rectangle is calculated by algebra in the following fashion the result has Reciprocal Pairs:

1. Start with two Elements the second being smaller thus []¬. Call them x and y.
2. Add Elements at Sides 3 and 6.
3. Add Elements at Sides 1 & 4 or 2 & 5
4. Add various pairs of Elements at 1 & 4, 2 & 5 or 3 & 6 providing that the pair chosen is not a repeat of the last pair.
5. Add a suitable final pair which will complete a rectangle and calculate out. This will result in a Rectangle + an extra Element added at left or right (or possibly top or 

bottom) as shown. This means we obtain an Odd Ordered Solution (9, 11, 13 etc...) if the extra Element is ignored.
6. The Divisor is the Semi-perimeter (AB or CD in above). Our total unit area of the first two Elements (x & y) is found to be this Semi-Perimeter Divisor! The next two, the SP x 2. The next two, the SP x 5. The next two, either SP x 13 or SP x 16. Then various coefficients increasing in size (but with increasing variations with larger Orders). The total of all the Coefficients is of course the depth of the Rectangle (side n).

R3.19. CALCULATED RECIPROCAL SOLUTIONS - FINDINGS

It is soon found that not all Odd Order Reciprocal Type solutions have this 1, 2, 5, 13 (or 16) etc. relationship. But what about Even Ordered Reciprocal Solutions?

Well no Even Order can have ***this type of series***, since the final 1 & 4, 2 & 5 or 3 & 6 always gives a Singlend Solution as shown Above, so effectively an Odd Order Solution plus a square on one side.

In this we are not saying that Even Reciprocal Pairs don't exist, nor that (if suitable) the Electrical Network found cannot be translated to Even Orders. But the series of ascending Coefficients in Even Orders can never be the tidy Fibonacci series of 1,2,5,13,34,89 etc. or similar.

For example: [12] 193 x 126 has Reciprocal Pairs of series 5:17:20:25:34 (total 126)

[12] 193 x 129 has series 5:18:20:25:25:36 (total 129) and

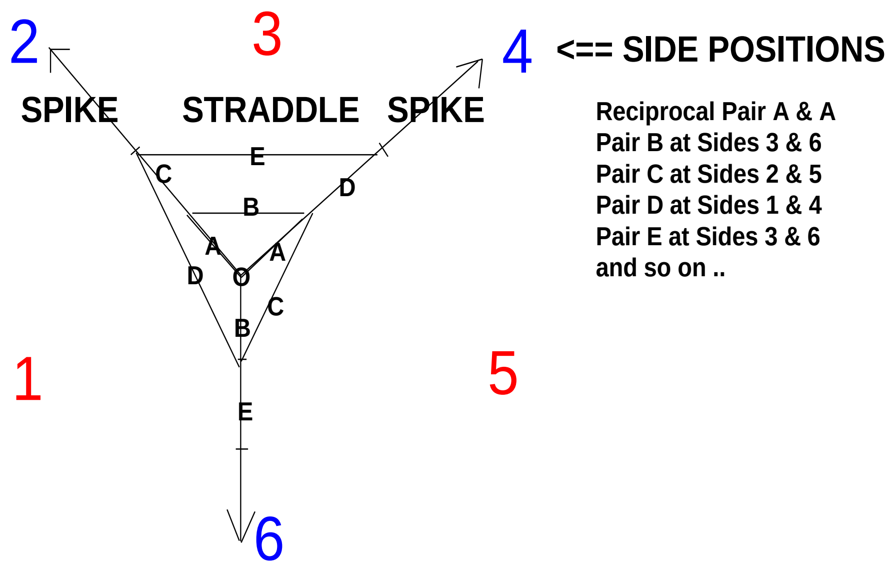
[12] 193 x 143 has series 2:4:26:26:32:53 (total 143)

These examples show that it is possible for coefficients to be duplicated (e.g. 25 & 26). There also is a tendency for some of the Coefficients to have similar factors e.g. 5 17 18. Curious!

Nor is it true that all Odd Ordered Rectangles can be calculated by the above rules. For example [13] 560 x 377 SP 937 with added Square gives a series of
5:10:25:34:65:68:170. In this case note that those which can be divided by 5 give 1, 2, 5, 13 and 34! (Fibonacci series!) and 34 and 68 ÷ by 34 give 1 & 2 (Fibonacci series!).
Nor is it true that Reciprocal Solutions always have two Unknowns x and y only. They frequently have 3 or more.

R3.20. xy RECIPROCAL SOLUTIONS AND Y SMITH DIAGRAMS

Concerning our "Tidy Series" of Odd Order Solutions, it is interesting to observe these from the angle of "Y" type Smith Diagrams. This works as follows.
To add an Element on Sides 2 4 or 6 (blue) we simply extend the spike line out by a fixed distance. To add an Element at 3 we straddle the most outer points from 2 to 4.
To add an Element at Side 5 we straddle the most outer points from 4 to 6. To add an Element at Side 1 we straddle points from 6 to 2. One example of many has been shown -

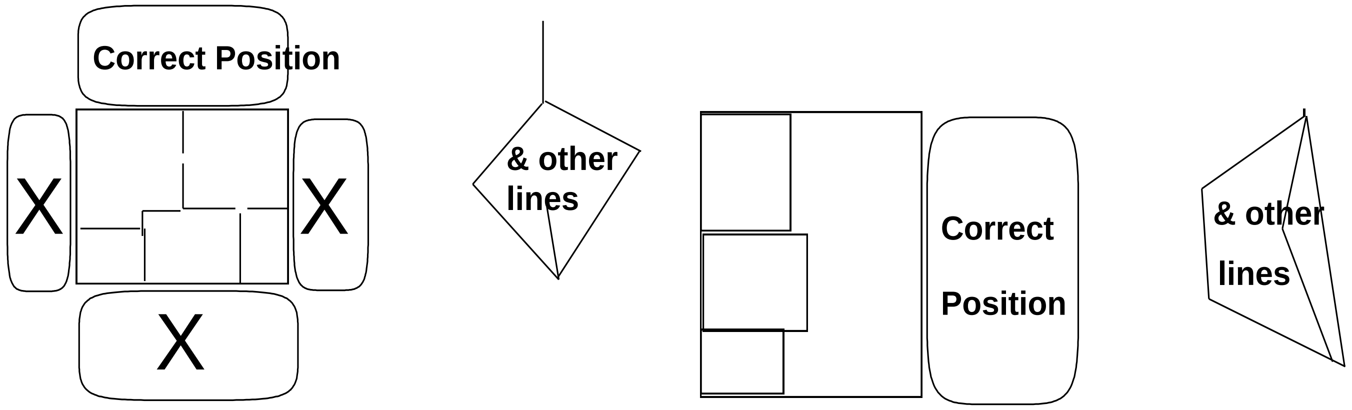


Once calculated A (1 & 2) always put as shown, our Elements for unknowns x & y, are Reciprocal Pairs for 1 times the Semi-Perimeter. B (1 & 2) twice the SP. C (1 & 2) are 5 times the SP. Note that at the stage of each pair 1. The Elements are on opposite sides and 2. One is on a spike and the other on a straddle. 3. The final spike gives rise to the extra Singlend Element.
(Note - Although there is always two Smith Diagrams for any give Rectangle, the other one is not relevant here).

R3.20.1. SMITH DIAGRAMS AND xy RECIPROCAL PAIRS - ANOMALY!

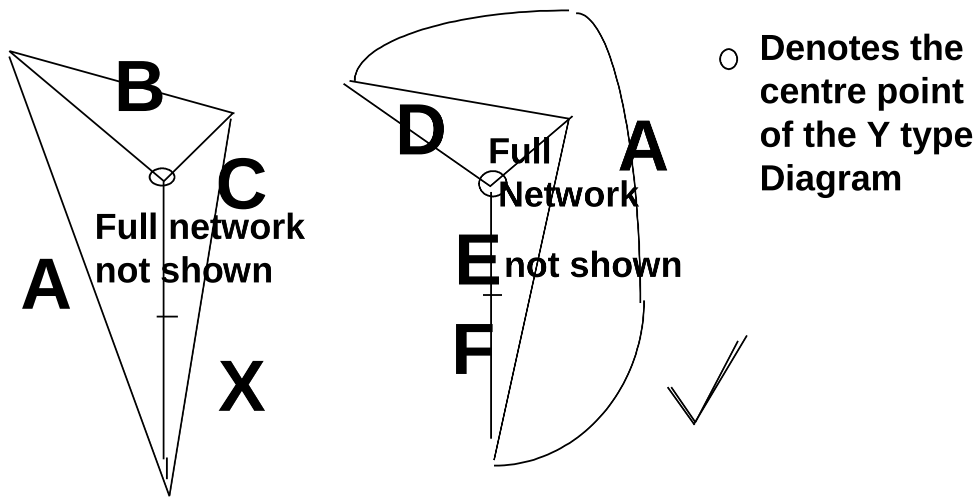
Ideally we wish to produce SD's Networks for Odd Orders which will be suitable for Even Order solutions (next Order up). But using the system above unaltered we are left, annoyingly with a Network with only three external Elements e.g. 1 Element on the right & 2 on the left, or vice versa! (Any valid SR of course has 4 or more Elements at the right & left ends in total).

For example, if our initial Odd Ordered solution has sides S2223 then to make a valid Network the extra Element can go on the top and nowhere else.



Now if we have calculated the Rectangle as above 1 (using the longer side m as our base), the resulting Network will take the form of Above 2 - unacceptable! But if instead we find the Network for the smaller side n with an extra Element at the right (i.e. the Odd solution is turned on its side with the extra Element on the side opposite the three Elements) all will be well. If we regard the first two Elements *vertically* & add the next two then proceed with additions of 1 & 4 or 2 & 5 or 3 & 6 there is no problem until we get to the last 4 Elements. If we call these Elements A B C D with D the End Element, we will discover that the Reciprocal pairs are not A & B and C & D as would seem logic but A & D and B & C!

Furthermore the convenient Y type diagrams, become messed up by the final End Element.



With the Rectangle worked logically according to the last Section we obtain a Network something like above 1. Then we realize that the Element A is placed on the opposite side to that required as AB & C are only three Elements. But when Element A is placed on the other side (Above 2) the Y Type Network looks clumsy, but the pattern itself now correct! Note there are now 4 Elements left and right in total ADEF which is correct.

How inconvenient!

R3.20.2. CHOOSING WHERE THE ADDED END ELEMENT MAY GO

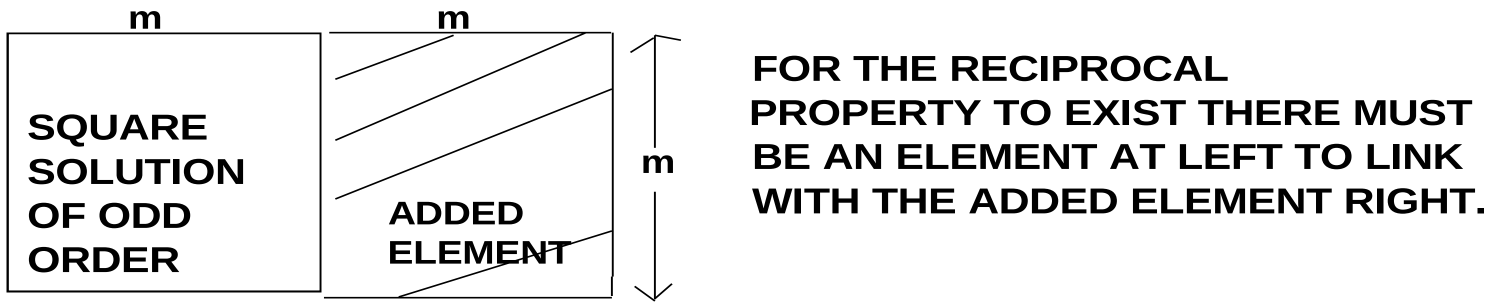
If our starting solution is 2233 then our End Element will be where only 2 Elements exist. E.g. [9] 33 x 32 S2223 it will be on other left, or at the top (in theory). In the second case we must work from [9] 32 x 33 with the End Element appearing now at the left (or right (in practice). The resulting Singlend solutions are [10] 65 x 32 and [10] 65 x 33 but valid ones from the Network can also be found.
(Note: In theory the End Element can be placed in any of the 4 sides where the Solution with sides 3333 or greater. This section relies on xy solutions, but in practice however no xy solutions exist with sides 3333 or greater).

R3.21. TABLE OF RECIPROCAL BASIC SOLUTIONS

Calculated on a BASIC program these are shown below. Only Odd Orders are shown but information for Even Orders is indicated in this table.
For instance Order 10 solutions which *may be* Reciprocal, *may be* 89 100 101 109 121 or 130. I say *may be* as some solutions will not be valid, not having sufficient Elements along the sides.
The alternative Coefficients shown in red require explanation. Take [9] 69 x 61 which has Coefficients of 1 2 5 16 and 37 or 45. 37 arise when the unit square is added on the side 61, and 45 when it is added to the side 69. (The extra Element needed since 9 is odd and we are considering pairs). The reason why there is no Order 8 valid solutions with side 37 is that the Extra Element is on the right or left and opposite are 2 Elements, but there need to be above 2 for valid solutions to exist.
However this is not a problem with the larger Alternative Coefficients where Reciprocal type solutions always exist with this as the longer side of Even Orders.

R3.22. RECIPROCAL PAIRS AND SQUARED-SQUARES

I have yet to find any. Obviously in the case of Odd Orders we must have an even Divisor i.e. divisible by 2. In the case of Even Orders the Divisor may be Odd or Even.
Suppose we have a Reciprocal Solution for a square of an ODD Order. This makes the Divisor equal to 2m (i.e. semi-perimeter), an even number. Now the extra Element on the right has to link with some Element within the solution.
We know that 2m has to be the sum of two integers squared. So [13] 23 x 23 cannot work since 46 is not the addition of two squares.



If we call the Element E then E squared + m squared has to = 2m x some coefficient. Now 2m x coefficient is divisible by 2m. So E2 + m2 needs to be divisible by 2m also. But as m is already divisible by m, E2 needs to be divisible by m too. This cuts down the values E might be.

Suppose m is prime, say 37 making 2m = 74. Now 37 squared + E squared has to be divisible by 74 so E can only be odd. But E squared has to be divisible by 37 so would have to be 0 or 37 which means m cannot ever be prime!

Now suppose m = 25 and E = 5. Then 625 + 25 = 650 = 50 x 13, 13 being 4 + 9, i.e. two squares.

Suppose m = 25 and E = 15. Then 625 + 225 = 850 = 50 x 17, 17 being 1 + 16, i.e. two squares.

These examples are theoretically possible for Reciprocal Pairs though not in reality.

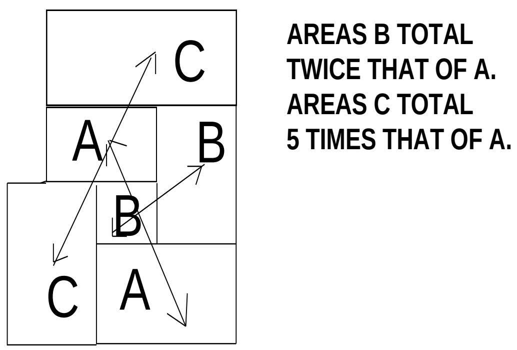
It transpires that m has to be a squared number, and m has to be divisible by E if RP's to be a possibility. Also either m and E must either be both even, or both odd. Its minimum value is the square-root of m.

Either RP's are rare or don't exist at all. To be continued!

If m = 81 then E might be 9 (square root), 81 squared + 9 squared = 162 x 41 (41 = 16 +25), or even 27, 81 squared + 27 squared = 162 x 45 (45 = 9 + 36).

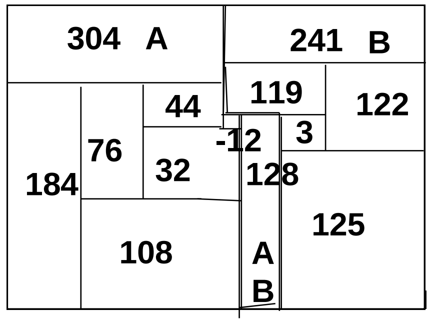
R3.23. DIFFERENT SERIES WITHIN A SINGLE SOLUTION

The writer observed that with a Pentad ending, three areas with the proportions of 1, 2 and 5 exist if an extra Element is included as shown below -



Now suppose we introduce a pattern to the left of this with a series which has Areas which would have Reciprocal Pairs if the sequence was continued correctly.

Here is [13] 545 x 488 which does happen to have Reciprocal Pairs throughout -



R3.24. TABULATING THREE LOTS OF AREAS

3	32	1,033	1	3	122	14,893	1				
12	128	2,064	2	119	125	29,786	2				
44	125	17,561	17	128	241	74,465	5	32	76	6,800	1
76	122	20,660	20					44	108	13,600	2
119	108	25,825	25					12	184	34,800	5
184	241	91,937	89					188	304	108,800	16
488	304	330,860	320								
EL	EL	AREA	COEF	RIGHT		AREA		LEFT		AREA	

The Unit Areas which connect form three different series and three different Divisors. As Reciprocal Pairs the Divisor is 1033 throughout as shown on the left. Right hand Elements are shown in the middle and the Divisor is 14,893!
Left hand Elements are shown on the right and the Divisor is 6800!
If this is not curious enough, note that 128 pairs with both 304 on the left and 241 on the right. (A with A, B with B).
In the main Reciprocal sequence 128 pairs with 12!

Note the Coefficients 1 2 17 20 25 89 320 appear to be haphazard, not following the 1 2 5 13 (or 16)... sequence of some Reciprocal Series seen earlier.

The 6800 and 14,493 are unlikely to be related to 1033, and may be unimportant.

R4. IDENTICAL PAIRS

R4.1. AN EXAMPLE OF IDENTICAL PAIRS

In the Rectangle [12] 193 x 143 below where the Semi-perimeter is 336, *most* of the Elements reveal IDENTICAL PAIRS.

Each Element squared is considered in turn, and the Remainder after division by SP 336 then calculated.
In this table the Remainder of $76^2 \div 336$ is 64. But the Remainder of $20^2 \div 336$ is also 64.

In this example all the other Elements pair up in the same way - except 24 and 73.

Element 143	Element ²	Remainder after division by 336 = SP	Links
76	5,776	64	e
73	5,329	289	- but links with Element 143
70	4,900	196	d
67	4,489	121	c
53	2,809	121	c
47	2,209	193	a
24	576	240	-
23	529	193	a
20	400	64	e
19	361	25	b
14	196	196	d
5	15	25	b

R4.2. GENERAL NATURE OF INDIVIDUAL IDENTICAL PAIRS

(For the purposes of this Section forget about Squared-Rectangles as this is only an arithmetical exercise!)

Calling any particular divisor as D and considering a range of numbers from 1 up to half of D then each of these numbers when squared and divided by D gives a remainder.

Where two have the same remainder then they form an Individual Identical Pair.

If we consider a whole series of Identical Pairs possible for a given Divisor D within the range 1 to half of D then four distinct types of series emerge.

Only No 3 and 4 are helpful:-

1. No series of pairs possible. (Note: If range is increased from 1 to D there are always some Pairs however)
2. Not a proper series as the only Pairs are all divisible by a common factor of Divisor D only. (Might be multiples of 3 only). As a result there are no pairs for numbers 1 or 2 or for many higher numbers.
3. There are pairs for 1 and 2 but certain factors of the Divisor D are totally absent, e.g. there may be nothing divisible by 3, or by 5, or by 3 or 5. .
4. Pairs exist in quantity, including some for 1 and 2, and also for any factor of D times 1, 2 ,3 ...but both parts of the pair are a multiple of that same factor. For example if 7 is a factor there 14 might Pair with 35 - i.e. each divisible by 7.

Now knowing what the prime factors of D are, there are fixed rules to determine which group applies to any given value D.

Without explaining why, the Rules are as follows:-

Group 1. Primes or Primes x two.
Group 2. A single Prime to the power 2 or greater. Or a single Prime to power 2 or greater, times two.
Group 3. Any single prime times 3, or any single prime times six.
Group 4. All other Divisors

Examples
D = 59 (prime) is Group 1 with no Identical Pairs within the range of numbers 1 to 29 (that is, half of 59).
D = 162 = 3 x 3 x 3 x 3 x 2 is in Group 2. The only pairs possible consist of both parts divisible by 3 throughout.
D = 87 = 3 x 29, is in Group 3. In this nothing pairs with 3 6 9 12 15… or with 29 58 87 ...
D = 90 = 2 x 3 x 3 x 5, is in Group 4. This time there are pairs for 3 6 9 12.. and 5 10 15... but both parts are divisible by 3 - or by 5 respectively.
The Factor 2
Ignoring Groups 1 and 2, pairs always exist for factors of 2 except where some other factor of D under Group 3 eliminates it. Each part of the pair is then always divisible by 2 which effectively mean odds pair with odds and even numbers with evens throughout.
As in Reciprocal Pairs, the study is complicated by which type of Divisor is considered - it may be 1. The Semi-Perimeter. 2. The Vertical SIDE. or 3. The Horizontal SIDE relating to the Reduced sizes.

R4.3.

Take, as an example, the Divisor number 65 and Element value 1.
The numbers to which Element 1 connects are basically 1 14 51 and 64
Unlike Reciprocal Pairs these can be applied in combinations of any two, as shown here.
$14^2 - 1^2 = 195 = 65 \times 3$
$64^2 - 51^2 = 1495 = 65 \times 23$
$51^2 - 14^2 = 2405 = 65 \times 37$
$51^2 - 1^2 = 2600 = 65 \times 40$
$64^2 - 14^2 = 3900 = 65 \times 60$
$64^2 - 1^2 = 4095 = 65 \times 63$
Also note that 1 and 1, 14 and 14, 51 and 51 and 64 and 64 all = 0 = 65 x 0
For certain Divisors and Element values, there will not be any values connecting (apart from the same Element value of course)
For the purposes of study we need to be very clear as to whether we are going to define and limit each Pair to its two lowest values i.e. below half the divisor amount, that is 1 relating to 14 alone in the above - if this is not done, there is not a fixed size Element corresponding to each original Element.

R4.4. EXAMPLES OF A0 PAIRS - TABLED

Many A2 Solutions exist but relatively few A0 as shown by the following table up to Order 12.
Note the remarkable occurrence of SIDE 377 which has Reciprocal Pairs also. The SIDE must be the difference between two squares and though 26 defies this rule, 52 104 and the full side of 208 are the difference between two squares.
In [12] 40 x 32 we would expect one of the sides 40 or 32 to have the A0 property rather than Semi-perimeter 72, but they don't. However, 16 - half of 32 does (but 16 is not a factor of 72).

A0		Sp = semi-perimeter is = larger side ss = smaller side		
Pairs		* not diff between two squares but 52 or 104 is		
Order	Dimensions	SIDE DIVISOR (not Prime)	SIDES	xyz…
10	105 x 104	105 is = 3 x 5 x 7 = 15 x 7 = $11^2 - 4^2$	s2233	Xy
11	194 x 183	377 sp = 29 x 13 = $21^2 - 8^2$	s2233	Xy
11	199 x 178	377 sp = 29 x 13 = $21^2 - 8^2$	s2223	Xy
11	209 x 168	377 sp = 29 x 13 = $21^2 - 8^2$	s2333	Xy
12 imp	40 x 32 (320 x	72 sp = 2^2 x 2 x 3 x 3 = $9^2 - 3^2$ also 16 ss = $4^2 - 0^2$	s2225	Xy
12	46 x 26 (368 x	26 ss = 2 x 13* Full 208 ss = $28^2 - 24^2$	s2325	Xy
12	81 x 80 (324 x	80 ss = 2^2 x 2^2 x 5 = $9^2 - 1^2$ or Full 320 ss = 16 x 20 = $18^2 - 2^2$	s2233	Xyz
12	353 x 240	240 ss = 2 x 2 x 2 x 2 x 3 x 5 = 20 x 12 = $16^2 - 4^2$	s2324	Xy
12	377 x 231	377 ls = 29 x 13 = $21^2 - 8^2$	s2324	Xy
12	377 x 256	377 ls = 29 x 13 = $21^2 - 8^2$	s2324	Xy

R4.5. WHERE SIDE IS A PRIME NO IDENTICAL PAIRS PROPERTY

Where identical pairs arise it is seen that the larger Element squared minus the smaller one squared is divisible by the SIDE, and $x^2 - y^2$ is divisible by the Semi-perimeter (whereas $x^2 + y^2$ is divisible by the SIDE in reciprocal pairs).

Now $x^2 - y^2 = (x + y)(x - y)$. If this was divisible by a prime SP then x + y or x - y would have to be too, but these are much smaller than the SP. Now suppose the SP is prime x 2 (2p).

If $x^2 - y^2$ is divisible by 2p it is clear that x - y would have to be divisible by 2 and x + y by p or vice versa.
x + y could be large enough to be divisible by half the Semi-perimeter in extreme circumstances, and theoretically could approach but not obtain two-thirds of the SP as illustrated.
Identical pairs are not likely to occur often in practice, although their occurrence cannot be ruled out.
Now suppose the SP is a product of two primes p1 p2 as in the case of 65. (x + y) (x - y) could be divisible by 65 if x + y was divisible by 13 and x - y by 5, or vice versa.

R4.6. COMPULSORY IDENTICAL PAIRS

If we are considering the Horizontal Side as the Divisor, and there happen to be just two Elements bordering that side, the two Elements are always found to be Identical Pairs. This is easily proved - if our Divisor is say 65 and we consider Element 31 then 31 and 34 pair up - since $34^2 - 31^2 = 65 \times 4$. $a^2 - b^2 = (a + b) \times (a - b)$ and a + b is the horizontal side.

The equivalent is also true with the Vertical Side as the Divisor.

In many solutions no further Identical Pairs are possible and since the two top Elements are a Compulsory Identical Pair, the property is of no real interest.

R4.7. WHERE IDENTICAL PAIRS EXIST WHOLLY OR PARTIALLY

In the case of ODD ORDERS 1, the Semi-perimeter is the difference between two squared integers and
In the case of EVEN ORDERS 2, the Side is the difference between two squared integers.

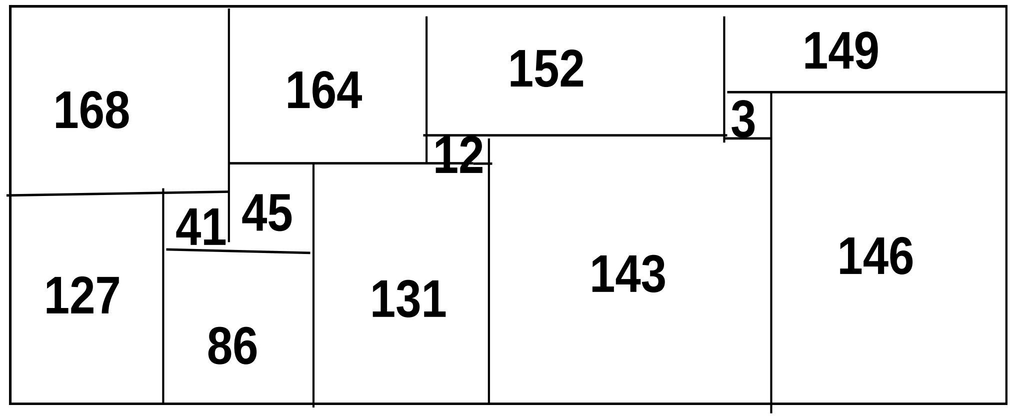
As previously stated such a Side (or Semi-perimeter) must always be a factorial number and never a prime number.

We might well expect Solutions to have Identical Pairs throughout. But strangely although some do - most leave two Elements which do not pair up.

At present it appears that Solutions with both Reciprocal and Identical Pairs, that the Solution has Identical Pairs completely but this needs verifying.

R4.8. SQUARED NUMBER MYSTERY

Consider the following solution-



The writer first looked at [10]130 x 79 which contained Identical Pairs throughout. This is the above minus one Triad. Then wondering if the Order [13] 633 x 295 Solution followed suite with Identical Pairs for 928 (Semi-perimeter) found it does not at all! But the following was observed instead:-

168 squared 18224 rem. ÷ 928 384 4096 = 64^2
164 26896 912 4624 = 62^2
152 23104 832 6400 = 80^2

149	22201	857 8281 = 91^2
146	21316	400 = 20^2
143	20449	33 961 = 31^2
131	17161	457 = 70^2
127	16129	353 2309 = 47^2
86	7396	900 = 30^2
45	2025	169 = 13^2
41	1681	753 1681 = 41^2
12	144	144 = 12^2
3	9	9 = 3^2

Although in 4th column various multiples of 928 have had to be added for the number to be a squared number it is still remarkable!

Looking at [10] 130 x 79 and the side of 130 -

45 rem ÷ 130	75	2025 = 45^2
44	116	1156 = 34^2
41	121	121 = 11^2
38	14	144 = 12^2
35	55	1225 = 35^2
34	116	1156 = 34^2
23	9	9 = 3^2
12	14	144 = 12^2
11	121	121 = 11^2
3	9	9 = 3^2

Some of the numbers have required doctoring to meet a squared number. Note the duplicated 3, 11 12 34 whilst 34 and 45 differ and appear once!

R4.9. A FEATURE IN SOME EVEN ORDER RECTANGLES WITH SP DIVISOR

The Writer was puzzled at quite a number of Even Ordered rectangles possessing Identical Pairs for all but two Elements!

This occurs when the divisor is the Semi-Perimeter. Upon investigation he found that by including an additional Element on the horizontal side (though the vertical side can be used instead) that there was then a match with one of the two Elements, leaving just one unaccounted for! The odd man out Element was then found to have a factor relationship with the SP Divisor!

An Example makes it clearer.

[12] 160 x 159 with SP 319 has Identical Pairs as follows - Rem. 4 for Elements 2 and 31, 81 for 9 and 20, 168 for 38 and 49, 256 for 71 and 16, 5 for 40 and 18. But 72 gives remainder 80 and 88 gives remainder 88. Now if we take the remainder of 159 squared from 319 we obtain 80! So 72 and 159 become an Identical Pair!

This leaves Element 88 out in the cold! Weird!

Now our Divisor of 319 happens to be 319 = 11 x 29. In this case none of the Elements happen to be multiples of either of these numbers except 88 (11 x 8)! However some solutions contain Elements which do have common factors with the Semi Perimeter, yet form Identical Pairs.

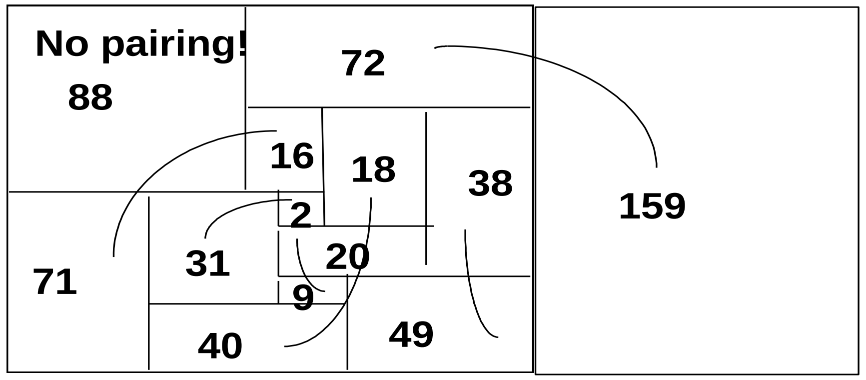
Testing other similar cases with only 2 Elements_apparently unmatched, it is found that one always matches with an added Element on the side, leaving just one which always has some common factor with the Semi Perimeter.

There are many Even Order Rectangles where the SP Divisor fails to give a single pair.

Others annoyingly give 4, 6, 8, 10... Elements which fail to match. I have not yet discovered why!

E.g. using [12] 160 x 159 again but looking at 160 as the Divisor we discover Element 20 gives remainder 80, 38 gives 4, 40 gives 0 and 16 gives 96. 159 gives 1.

But if we look at Divisor 80 the remainders become 0, 4, 0, 16 and 1 and one more pair is found, leaving 3 oddments.



[12] 160 x159

showing Identical

Pairings for
319 (Semi Perimeter)

******

R4.10. ANALYSIS OF A SOLUTION - IDENTICAL PAIRS

1. In considering the Solution [10] 130 x 79 when the divisor 209 (SP) which has factors 11 x 19 is used, then the following Identical Pairs exist -

45 & 12 (diff. 33) 31 & 3 (diff. 38) 34 and 23 (diff. 11) 79 and 35 (diff. 44).

Left overs are 44 (div. by 11), 38 (div. by 19) and 11 (div. by 11).

2. Now we will analyze with Divisor 130 = 2 x 5 x 13

Pairs 23 & 3 (diff. 20), 41 & 11 (diff. 30), 38 & 20 (diff. 26), 44 & 34 (diff. 10).

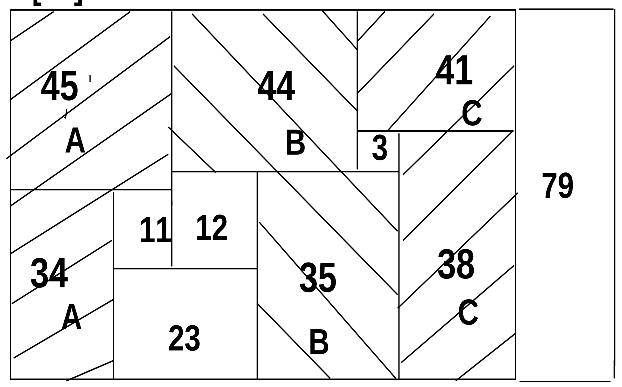
Leftovers 35 (divisible by 5) and 45 (divisible by 5)

We observe that the pairs have differences which have some common factor with the divisor concerned.

Also the Leftovers are multiples of certain factors in the Divisor.

3. Now Divisor 79 which is prime.

[10] 130 x 79



As seen in R3.6, the Elements at A, B, and C do have Compulsory Identical Pairs for this Divisor. But for the others to pair up they need differences of 79 (79 being prime). Clearly impossible.

Typical with other solutions, if the SP Divisor gives a largely Identical Pairing, the Dominant Side (130 in this case) will also. However the Recessive side (*not* always the smaller side) have no identical pairings apart from the pairings as in AA BB... above.

In some solutions both Recessive & Dominant sides have largely Identical Pairing, while the SP does not.

R4.11. SQUARED-SQUARES AND IDENTICAL PAIRS

Since no Even Divisor with a single factor of 2 can ever have Identical Pairs this raises an interesting point for Squared-Squares. For Squared-Squares if the Side is Odd and the Order is Odd, Fully Identical Pairs are impossible, as the Divisor needs to be the Semi-Perimeter and therefore twice an Odd number.

But this is not necessarily true if the Order is Even. If the Side is Even then the SP must be divisible by 4 at least. Are IP'S possible for these?

R4.11.1

ORDER	DIMENSIONS	SEMI-PERIM	QTY OF EVEN ELEMENTS
1. EVEN	ODD x ODD	EVEN	ODD
2. ODD	ODD x ODD	EVEN	EVEN
3. EVEN	EVEN x EVEN	EVEN	EVEN
4. ODD	EVEN x EVEN	EVEN	ODD
5. EVEN	ODD x EVEN	ODD	EVEN
6. EVEN	EVEN x ODD	ODD	EVEN
7. ODD	ODD x EVEN	ODD	ODD
8. ODD	EVEN x ODD	ODD	ODD

a. ODD ELEMENT squared div. by ODD gives EITHER ODD OR EVEN REMAINDER

b. ODD ELEMENT squared div. by EVEN gives an ODD REMAINDER
c. EVEN ELEMENT squared div. by EVEN gives an EVEN REMAINDER
d. EVEN ELEMENT squared div. by ODD gives EITHER ODD OR EVEN REMAINDER

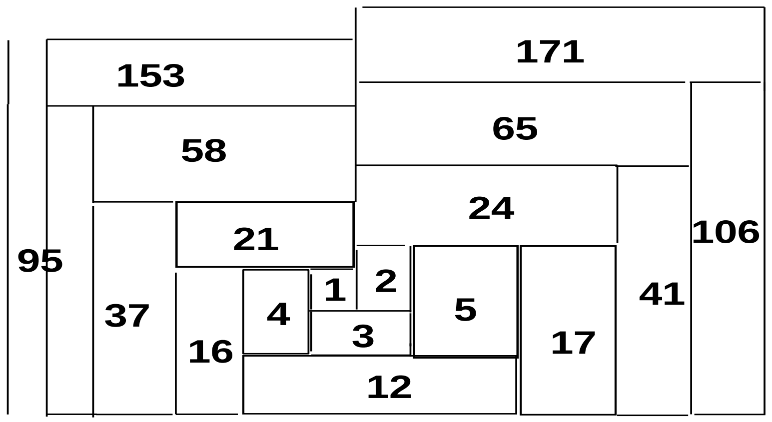
R4.12. USING EVEN DIVISORS

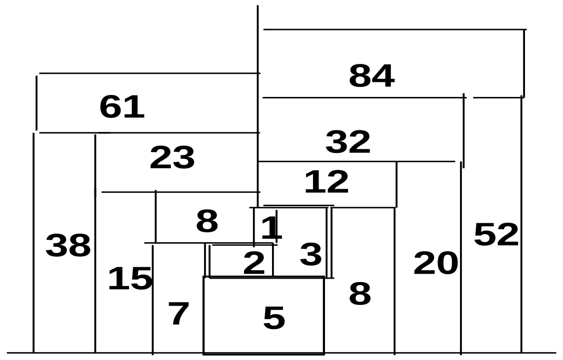
The quantity of Even Elements
For IDENTICAL pairs to exist ODD needs to pair with ODD
or EVEN with EVEN
For RECIPROCAL pairs to exist ODD needs to pair with ODD
or EVEN with EVEN

USING ODD DIVISORS

For IDENTICAL pairs to exist ODD needs to pair with ODD
or EVEN with EVEN
For RECIPROCAL pairs to exist ODD need to pair with EVEN
or EVEN with ODD

R4.13





In Above 2 the starting Elements are 3 and 2 where 3 x 3 - 2 x 2 = 5. The Elements not part of pairs are 1 and 5.
9 - 4 = 5 x 1 64 - 49 = 5 x 3 144 - 64 = 5 x 16 400 - 225 = 175 = 5 x 35 1024 - 529 = 495 = 5 x 99
2704 - 1444 = 1260 = 5 x 252 7056 - 3721 = 3335 = 5 x 667 and so on... with Coefficients of 1 3 16 35 99 252 667...

R4.14. IDENTICAL PAIRS AND SQUARED-SQUARES

TEMPORARY SECTION - PAIRINGS
FOR ODD ORDERS AN ADDITIONAL "END" ELEMENT HAS TO BE ADDED

WHERE AN "END" ELEMENT IS USED IN EVEN ORDERS IT WILL SOMETIMES MATCH ONE OF THE ELEMENTS WITH NO APPARENT PAIRING.
NOTE THAT <u>EXCEPT</u> WHERE THE DIVISOR IS A SINGLE PRIME, OR 2 x PRIME, 2 x A SET OF PRIMES, IDENTICAL PAIRS OCCUR IN THEORY, BUT OFTEN SOME PAIRS DO NOT. ALL SUCH DIVISORS ARE BOTH FACTORIAL AND THE DIFFERENCE BETWEEN TWO SQUARES

IPs IDENTICAL PAIRS RPs RECIPROCAL PAIRS D = DIVISOR DM for side M DN for Side n DS for Semi-Perimeter

<u>CERTAIN ELEMENTS FAIL TO HAVE Ips PROPERTY IF -</u>
1. The Element concerned has a prime factor of 5 or more which clashes with the Divisor concerned. (Does not apply to Adjacent Elements see below). e.g. Element <u>10</u> or 13 for DIVISOR 130 (2 x <u>5</u> x 13)

<u>IPs *DO NOT EXIST* IF</u>
1. Divisor concerned is PRIME, or 2 x PRIME, or 2 x A number of Primes (>2)
2. The Divisor cannot be expressed in the format of x squared - y squared or difference between two squares. This is an alternate way of expressing 1 above as since PRIME or PRIME x 2 or PRIMES x 2 cannot be expressed in this way.

<u>A FEW IPs IN PART EXIST IF</u>
1. For DM & DN Divisors, where 2 Adjacent Elements total the Divisor they form Identical Pairs (however the rest of the Elements may not give pairs).
(Sides index contains at least one 2).
2. In theory whenever Divisor is other than p or 2p or 2 x several primes, IPs are possible, but not necessarily totally throughout.

<u>MAY BE, IPs *MAY POSSIBLY EXIST* IF</u>
1. The Divisor can be written in the format of the difference between two squares.
2. The quantity of Element 2 is other than 1. (e.g. None or 4, 8, 16, 32 etc.)

<u>YES IPs *DO EXIST* IF</u>
1. The Divisor is suitable, and none of the Elements have clashing factors with the Divisor which becomes increasingly likely as Orders increase.

<u>NO, RPs DO NOT EXIST IF</u>
1. Any part of D concerned has a prime factor <u>not</u> divisible by 4 when diminished by one, unless the factor only divisible by 2 appears twice (or 4 times)

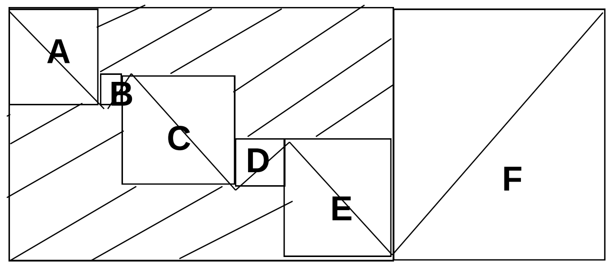
e.g. No factors of 3,7,11,19,23... unless appearing twice or 4 times
2. The D cannot be expressed as the sum of two squares.
3. The Sides Index is 222* or 22** and the Order is EVEN.

MAY BE, RECIPROCAL PAIRS *EXIST IN SOME CASES* IF
1. Prime (p-1 being divisible by 8) x 2
2. The Divisor is P1 x P2, both being divisible by 4 when diminished by 1.
3. The SD can be written in the format of the difference of two squared numbers r
4. The SD can be written in the form x squared + y squared.

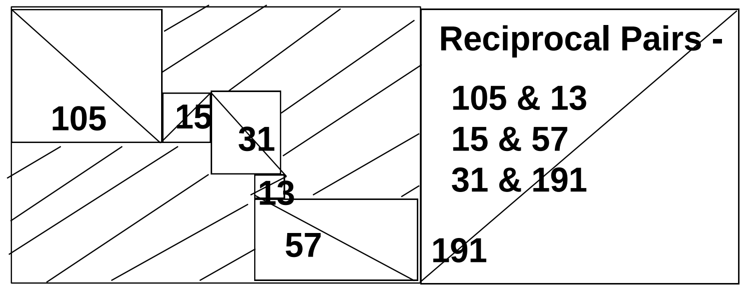
YES, RECIPROCAL PAIRS EXIST IF
1. The Divisor is a single prime p where p - 1 is divisible by 8, provided in the case of Even Orders there are at least 3 Elements along top border.

R5. A RELATIONSHIP BETWEEN TRAILS AND RECIPROCAL PAIRS

(Trails are dealt with in Section C q.v.)



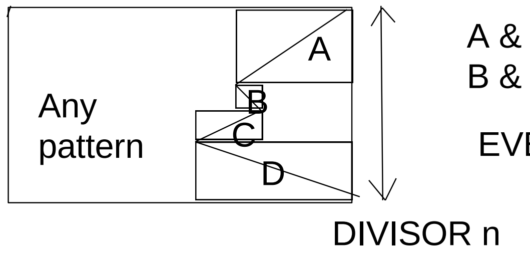
(Shaded areas in both diagrams have to be Even Elements ONLY)



From [11] 195 x 191

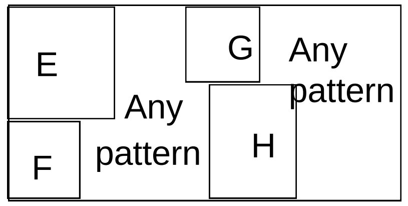
Reciprocal Pairs exist in the following circumstances-
1. Order is Odd.
2. The Divisor is the Semi-Perimeter number. *
3. A to F are all ODD Elements
4. The Solution is Odd x Odd.
5. The Trail is type C5 with no Circuit Trails i.e. 2 Corner Trails only. The RP’s fail in Type C7 Trails!
6. Pairings may be made involving A to F only (both numbers ODD)
In Above the Pairs are A & D, B & E, C & F but many solutions have different pairings.
7. * Not true in some Solutions, but still true for some lower Divisor being a factor of the SP in such cases

Reciprocal Pairs always exist in the following circumstances-



A & C Reciprocal Pairs
B & D Reciprocal Pairs

EVEN ORDERS ONLY



E & F
G & H
are Reciprocal
Pairs.

In Above (left) A squared - C squared = B x n and D squared - B squared = C x n - without exception (A to D may be Odd or Even). In the cases of ABCD all ODD Elements and the rest EVEN it cannot be assumed that the rest always Pair Up.
However whenever there are two Elements between top & bottom the pairs will also be Reciprocal (Above Right).

R6. EXTENSIONS TO THE RECIPROCAL PAIRS SYSTEM

Refer to "Add-Lines" mentioned in Section X.
[11] 177 x 176 with SP of 353 has Reciprocal Elements throughout and pairs as follows
99 & 78 77 & 57 43 & 41 34 & 16 25 & 9 21 (& 176).
But suppose we look at all the Add-Lines in the Solution – the results are!

22 (is Elements 43 - 21)	484 (= 22 squared)	131 (Remainder 484 div 353)
135 (is Elements 78 + 57)	18225 (= 135 squared)	222 + 131 = 353
120 (is Elements 99 + 21 or 77 +43)	14400	280
98 (is Elements 57 +41)	9604	73 + 280 = 353
7 (is Elements 16 – 9)	49	49
59 (is Elements 34 + 25)	3481	304 + 49 = 353
52 (is Elements 43 + 9)	2704	233
66 (is Elements 25 +41)	4356	120 + 233 = 353
64 (is Elements 21 + 43)	4096	213
136 (is Elements 77 + 34 +25)	18496	140 + 213 = 353
36 (is Elements 57 – 21)	1296	237
100 (is Elements 34 + 25 +41)	10000	116 + 237 = 353
111 (is Elements 77 + 34)	12321	319
73 (is Elements 57 + 16 or 21 + 43 + 9)	5329	34 + 319 = 353

So the fourteen Add-Lines completely pair up just like the Elements do! This is always true for any Odd Ordered Solution which has Reciprocal Elements. Remainders are always taken from the Semi-perimeter. 353 = 177 x 176 in the above case.
In the first pair 484 + 18225 = 353 x 59, & remaining pairs – 353 x 68, 353 x 10, 353 x 20, 353 x 64, 353 x 32 & 353 x 50
59 + 68 + 10 + 20 + 64 + 32 + 50 = 303. (303 appears to have no significance!) So all the imaginary Elements total 353 x 303

R6.1. USING EVEN ORDERED RECIPROCAL SOLUTIONS

The Writer expected to find the above to be true for Even Orders providing the Side (Not Semi-Perimeter) is used.
He discovered some pair up & the rest not!
Take for Example-
[12] 377 x 231
Reciprocal Pairs exist for

1. 143 20449 Rem. 91
   169 28561 Rem. 286 + 91= 377
2. 129 16641 Rem. 53
   47 2209 Rem. 324 + 53 = 377
3. 108 11664 Rem. 354
   241 55081 Rem. 23 + 354 = 377
4. 156 24336 Rem. 208
   13 169 Rem. 169 + 208 = 377
5. 128 16384 Rem. 173
   88 7744 Rem. 204 + 173 = 377

Others - 41 rem. 173* 103 rem. 53* 221 rem. 208* 282 rem. 354* (total not div by 377)

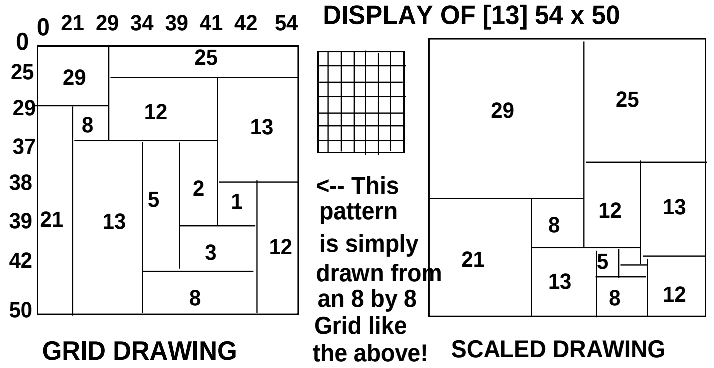
- Duplicated Numbers (see above), so are Identical Pairs
  259 rem. 352 264 rem. 328
  Total remainders = 1468 not div by 377.
  Result – 5 Reciprocal Pairs, 4 Identical Pairs, 6 fail reciprocal test.

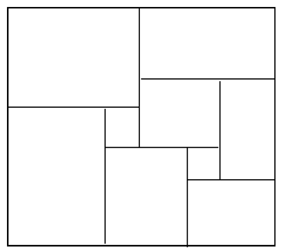
Another Test on [12] 353 x 255
Results in just two Reciprocal Pairs namely.

1. 151 Rem. 209
   12 Rem. 144 +209 = 353
2. 163 Rem. 94
   139 Rem. 259 + 94 = 353
   If 255 is considered there is third pair
   255 Rem. 73

120 Rem. 280 + 73 = 353
The other Remainders total 1791 which is not div by 353.

<u>VARIOUS FORMATS FOR   SQUARED-RECTANGLES</u>
<u>(ELEMENTS CAN BE INDICATED AS A BOX, A LINE, A POINT OR A NUMBER)</u>
<u>A. RECTANGLES (OR SQUARE) (Convention - Largest Element at top left and drawn Horizontally)</u>
RECORDED BY DRAWING
1. RANDOM DRAWING ONLY (ELEMENTS NOT CALCULATED OR SHOWN). BELOW3
NB Resulting pattern has to be modified when zeros and/or negative Elements arise.
2. DRAWN TO EXACT SCALE BELOW 2
3. GRID DRAWING BELOW 1





```
|106  |96_________________|
|  |20|12|7____|9  |48|
|  |  |  |5  |2|___|  |
|_______|  |  |  |11  |  |
|96|10|  |___|___|  |  |
|..|___|___|8|9..|_____|..|
|  |38  |____|3_|8|  |
|  |  |7|2|___|  |  |
|  |  |_|5____|_|  |
|  |  |20_____  _|___|
|  |_________|10___|58  |
|___|48______________|________|
```

[24] 202 x 202 IMPERFECT
CLOSEDGRID SYSTEM shown here is in the Writer's opinion the best way to represent a Squared-Square because
1. The Format is very Compressed (i.e. any closer, and it would be incorrectly drawn)

2. It can be shown tiny (Above 1!) yet still readable - impossible in SCALED SYSTEM.
3. It emphasizes the position of Rows and Columns
4. It avoids need for very small areas and very small numbers
5. It shows Slides where they exist in dotted lines . . . . The above shows an example. Ditto with Crosses.
(With the FULLGRID SYSTEM 3 & 8 would be shown a line further down) M

4. GRID NUMBERS ONLY 29 - 25|- - 12 - - 13|21 13 - 5 2| - - - - - 1 12|- - - - 3|- - - 8
Or a29c25 c12f13 a21b8 b13d5e2 f1g12 e3 d8 (a to f denoting verticals from the left)

RECORDED BY TEXT

1. BOUWKAMP FORMULA e.g. [13] 54 x 50 (29,25) (12,13) (21,8) (13,5,2) (1,12) (3) (8)
With many possible formats of which I like
[13] 54 x 50 +29+25,12-13,+21.8,-13.5.2,1+12,3,-8 where + denotes Corners & - Edges
or shown in layers -
+29+25
.12-13
+21.8
-13.5.2
.1+12
.3
-8

RECORDED BY MATRIX

1. 0,25,29,37,38,39,42,50 Horizontal layers proceeding downwards & 0,21,29,34,39,41,42,54 Vertical layers proceeding left to right which can then be analyzed in different ways in order to obtain each of the Element numbers <u>plus</u> potential Element Add-ons <u>plus</u> Elements which cannot be Added

<u>B. DIAGRAMS</u> (There are always TWO - one vertical and one horizontal pattern for any SD) Elements indicated by LINES not boxes

SMITH DIAGRAM (UNFIXED)

1. NORMAL S D's (WITH FIXED POLES AND ELEMENT NUMBERS)
2. NORMAL S D (FIXED POLES BUT NO ELEMENTS SHOWN)
3. NETWORK S D (POLES AND ELEMENTS NOT SHOWN)

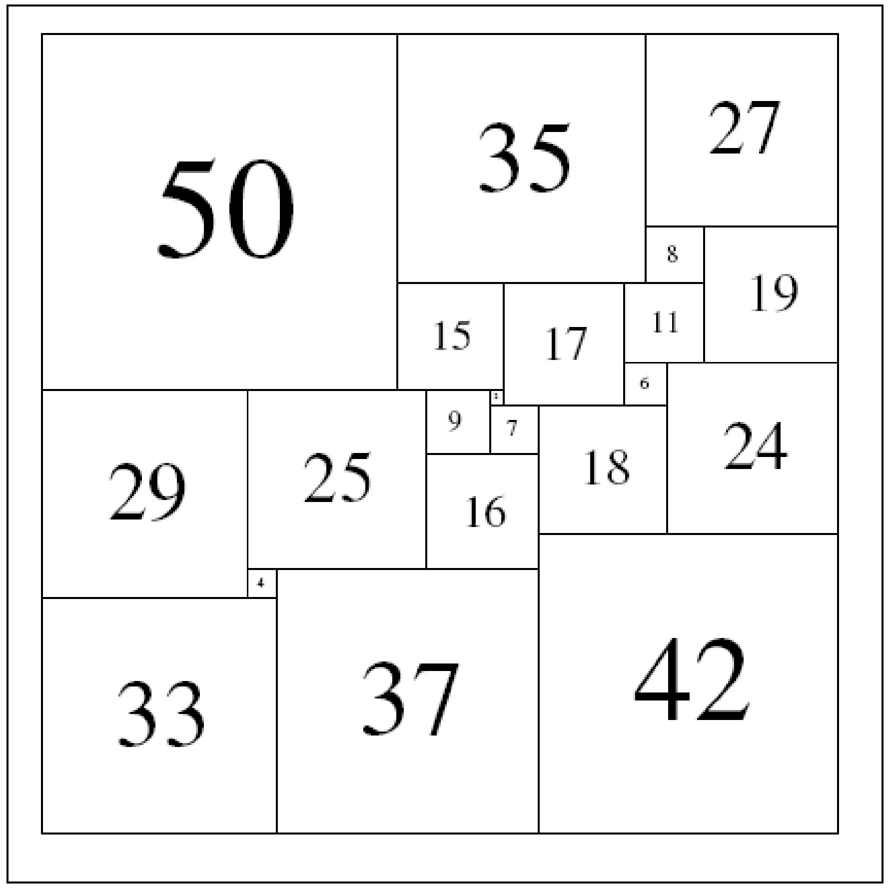
SMITH DIAGRAMS (FIXED)

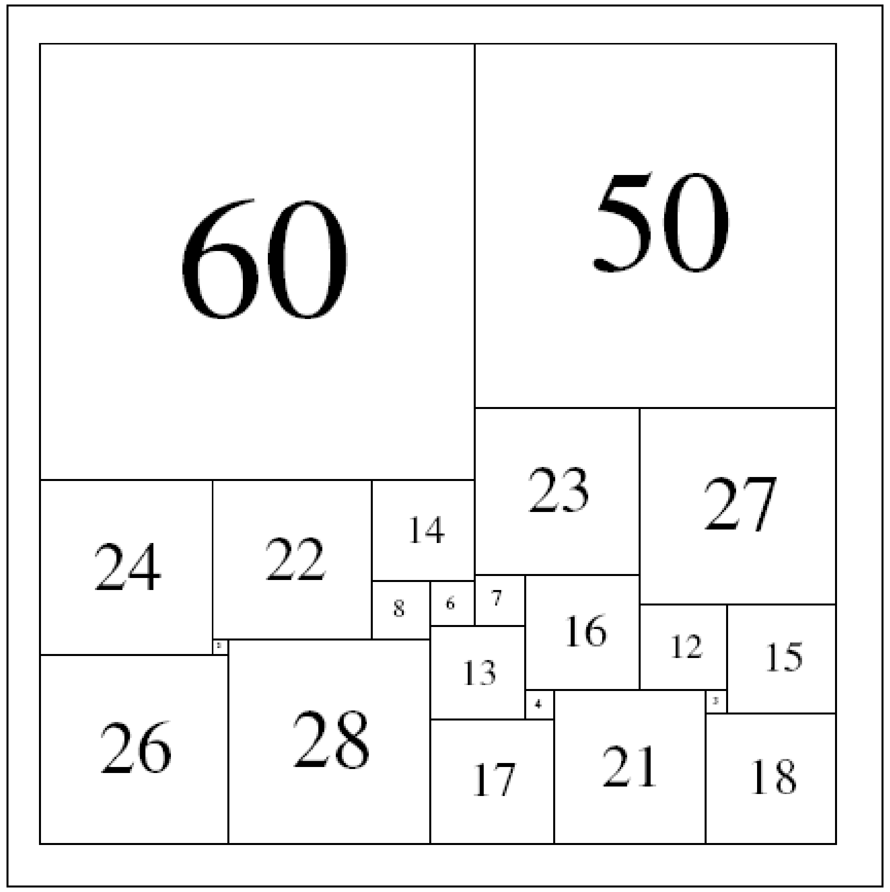
1. SQUARE TYPE
2. ROUND TYPE
3. X & Y TYPE (for xy and xyz solutions only)

MOSS DIAGRAMS

1. MOSS TYPE (based on points for each Element rather than lines)

13		66	x	48		
		3		2		
257		16				23
		9				7
		12	49			
		0				
			30	19		
				11	24	
					5	
			41			
225	32					
	19					
	3					





Face Count in row and Vertex Count in column (or vice-versa)

4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
4	1													
5		1	1											
6		1	2	2	2									
7		2	8	11	8	5								
8		2	11	42	74	76	38	14						
9		8	74	296	633	768	558	219	50					
10		5	76	633	2 635	6 134	8 822	7 916	4 442	1 404	233			
11		38	768	6 134	25 626	64 439	104 213	112 082	79 773	36 528	9 714	1 249		
12		14	558	8 822	64 439	268 394	709 302	1 263 032	1 556 952	1 338 853	789 749	306 470		
13		219	7 916	104 213	709 302	2 937 495	8 085 725	15 535 572	21 395 274	21 317 178	15 287 112			
14		50	4 442	112 082	1 263 032	8 085 725	33 310 550	94 713 809	193 794 051	292 182 191	328 192 346			
15		1 404	79 773	1 556 952	15 535 572	94 713 809	388 431 688	1 134 914 458	2 447 709 924	3 981 512 855				
16		233	36 528	1 338 853	21 395 274	193 794 051	1 134 914 458	4 637 550 072	13 865 916 560	31 277 856 206				
17		9 714	789 749	21 317 178	292 182 191	2 447 709 924	13 865 916 560	56 493 493 990	172 301 697 581					
18		1 249	306 470	15 287 112	328 192 346	3 981 512 855	31 277 856 206	172 301 697 581	700 335 433 295					
19		70 454	7 706 577	274 542 869	4 939 809 506	54 271 705 726	404 008 232 288	2 173 270 387 051						
20		7 595	2 599 554	168 992 630	4 686 995 652	73 247 405 678	741 171 341 224	5 270 785 332 349						

21 527 235 74 424 566 3 380 569 040 77 220 397 213 1 075 323 264 149 10 150 757 285 258
22 49 566 22 229 616 1 823 658 612 63 443 012 728 1 240 159 791 730 15 683 069 986 564
23 4 037 671 713 331 098 40 232 230 880 1 136 847 700 529 19 547 663 107 721
24 339 722 191 283 058 19 322 611 431 8 237 88 552 428 19 682 306 885 581
25 31 477 887 6 799 902 944 466 224 664 031 15 962 912 975 720
26 2 406 841 1 654 924 768 201 829 738 768 10 348 108 651 919
27 249 026 400 64 563 924 319 5 288 847 843 415
28 17 490 241 14 386 939 428 2 084 335 836 704
29 1 994 599 707 611 239 308 239
30 129 664 753 125 619 037 674
31 1 6147 744 792
32 977 526 957
Total (column) graphs 1 2 7 34 257 2 606 32 300 440 564 6 384 634 96 262 938 1 496 225 352 23 833 988 129 387 591 510 244 6 415 851 530 241 107 854 282 197 058
The column node graph total is OEIS sequence A000944.
The edge graph total is OEIS sequence A002840.
The number of edges is the sum of the row and columns indices minus 2 (Euler's Polyhedral formula; v + f = e + 2). Gérard P. Michon & Stuart E. Anderson 2000-2001, 2011

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(1) SPSSs Order 21 112 TO 112 <u>**NUMBER OF KNOWN SIMPLE PERFECT SQUARED-SQUARES KNOWN AT 12.10.2014**</u>
(8) SPSSs Order 22 110 TO 192 RANGE OF SIZES (FROM & TO)
(12) SPSSs Order 23 110 TO 332
(26) SPSSs Order 24 120 TO 479
(160) SPSSs Order 25 147 TO 661
(441) SPSSs Order 26 212 TO 625
(1152) SPSSs Order 27 180 TO 1179
(3001) SPSSs Order 28 201 TO 1544
(7901) SPSSs Order 29 221 TO 2134
(20566) SPSSs Order 30 201 TO 2710

(54541) SPSSs Order 31 (144161) SPSSs Order 32 (378197) SPSSs Order 33 (21329) SPSSs Order 34 (25978) SPSSs Order 35
(48308) SPSSs Order 36 (100202) SPSSs Order 37 (205862) SPSSs Order 38 (414504) SPSSs Order 39 (803483) SPSSs Order 40
(1506848) SPSSs Order 41 (2663241) SPSSs Order 42 (4242554) SPSSs Order 43 (5100736) SPSSs Order 44 (1739) SPSSs Order 45 543
(2159) SPSSs Order 46 601 (2494) SPSSs Order 47 691 (2932) SPSSs Order 48 621 (3290) SPSSs Order 49 779
(3868) SPSSs Order 50 788 (4088) SPSSs Order 51 853 (4660) SPSSs Order 52 1211 BLM
(4996) SPSSs Order 53 824 (5375) SPSSs Order 54 971 (5194) SPSSs Order 55 740 BLM (5426) SPSSs Order 56 929
(5434) SPSSs Order 57 985 (5561) SPSSs Order 58 1100 (5218) SPSSs Order 59 1060 (5088) SPSSs Order 60 1097
(4821) SPSSs Order 61 1043 (4590) SPSSs Order 62 1115 (4182) SPSSs Order 63 1171 (3896) SPSSs Order 64 1263
(3528) SPSSs Order 65 1365 (3062) SPSSs Order 66 1174 (2858) SPSSs Order 67 1335 (2444) SPSSs Order 68 1394
(2125) SPSSs Order 69 1310 (1823) SPSSs Order 70 4153 high (1476) SPSSs Order 71 1243 (1256) SPSSs Order 72 5054 high
(1123) SPSSs Order 73 5078 high (908) SPSSs Order 74 4559 high (680) SPSSs Order 75 1412
(592) SPSSs Order 76 5275 high (447) SPSSs Order 77 5319 high (367) SPSSs Order 78 5735 high
(267) SPSSs Order 79 5499 high (279) SPSSs Order 80 4029 high (181) SPSSs Order 81 Much lower
(123) SPSSs Order 82 possible from (112) SPSSs Order 83 here down. (95) SPSSs Order 84
(72) SPSSs Order 85 (51) SPSSs Order 86 (38) SPSSs Order 87 (34) SPSSs Order 88
(28) SPSSs Order 89 (15) SPSSs Order 90 (9) SPSSs Order 91 (1) SPSSs Order 92
(6) SPSSs Order 93 (6) SPSSs Order 94 (4) SPSSs Order 95 (3) SPSSs Order 96
(2) SPSSs Order 97 (1) SPSSs Order 98 (2) SPSSs Order 99 (4) SPSSs Order 100
(3) SPSSs Order 101 (2) SPSSs Order 102 (1) SPSSs Order 103 (2) SPSSs Order 104
(2) SPSSs Order 105 (1) SPSSs Order 106 (1) SPSSs Order 107 (1) SPSSs Order 108
(1) SPSSs Order 109 (1) SPSSs Order 110 (1) SPSSs Order 111 (1) SPSSs Order 112
(1) SPSSs Order 113 (1) SPSSs Order 114 (1) SPSSs Order 115 (2) SPSSs Order 116
(1) SPSSs Order 117 (1) SPSSs Order 118 (2) SPSSs Order 119 (1) SPSSs Order 120
(2) SPSSs Order 121 (1) SPSSs Order 122 (2) SPSSs Order 123 (1) SPSSs Order 124
(2) SPSSs Order 125 (1) SPSSs Order 126 (1) SPSSs Order 127 (2) SPSSs Order 128
(2) SPSSs Order 129 (1) SPSSs Order 130 (1) SPSSs Order 133 (1) SPSSs Order 140
(1) SPSSs Order 169 (1) SPSSs Order 208 (1) SPSSs Order 216 (1) SPSSs Order 220
(1) SPSSs Order 236

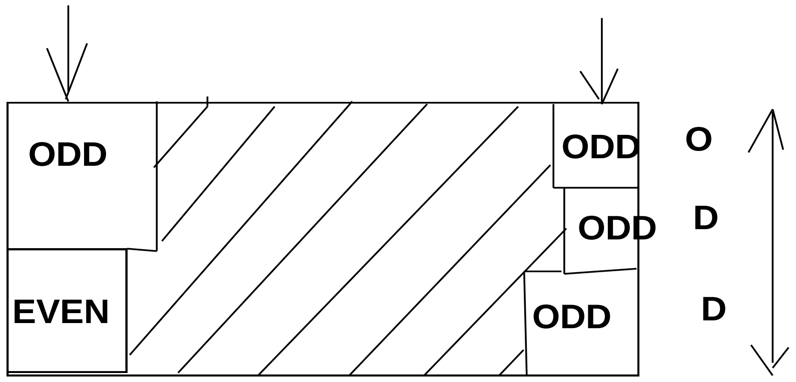
S. TRAILING OF ELEMENTS

DISTRIBUTIONS AND TOTALS OF ODD ELEMENTS

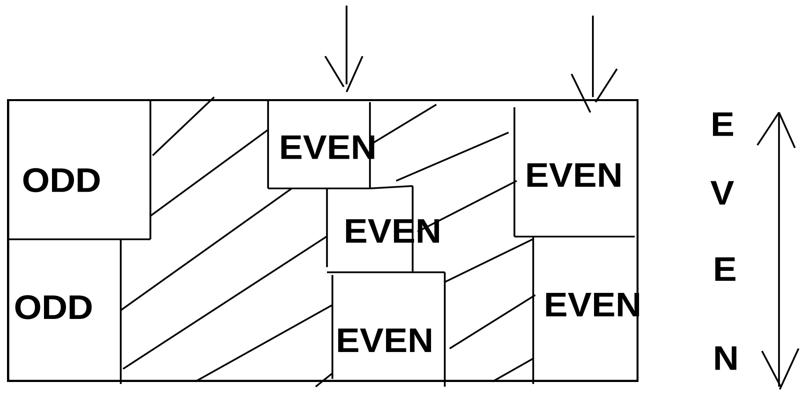
THIS SECTION NEEDS CAREFUL REVISION AND RENUMBERING

S1. DISTRIBUTION OF ODD ELEMENTS

An elementary observation first



Where n is ODD

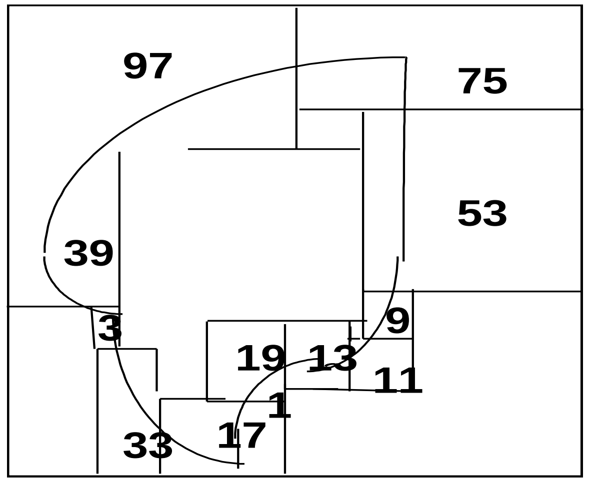


Where n is EVEN

1. In any Rectangle n is Odd or Even. It is clear than where n is ODD then at any vertical drawn at random will cross an Odd quantity of Elements, 1,3,5,7... etc. This means that Odd Elements must form a jagged LINE across the whole of the Solution from left to right.
If the Solution is Odd x Odd then clearly at least one ODD Element occurs at any point between the top and bottom also.
2. Likewise where n is EVEN the number of Odd Elements any random vertical line will cross is 0, 2, 4, 6, 8 etc...
This means that IF the solution has any vertical points which contain only Even Elements (2 or more) then Odd Elements will not exist across the whole of the Solution from left to right. Otherwise the solutions will have at least 2 Odd Elements from left to right throughout and as we shall see often form a RING.

S2. A RING OF ODD ELEMENTS

In some EVEN x EVEN Solutions the Odd Elements are adjacent to one another, an example being



[22] 172 x 172

EVEN ELEMENTS ARE OMITTED

Sometimes, as above where the Solution is EVEN x EVEN, the Odd Elements form a RING either Regular or Irregular (as Above).

Others where the Solution is ODD x ODD form a LINE of numbers In such cases we could draw straight lines over all Odd Elements in turn starting at the edge and not having to lift the pen. Although it is a single line it stretches from Top to Bottom and from Right to Left.

However some have Odds forming BITS which do not join.

I have taken a Solution at Random [26] 356 x 356. This has 135 55 21 43 31 3 39 51 1 111 61 71 113 127 11 105 (135) and forms a Ring.

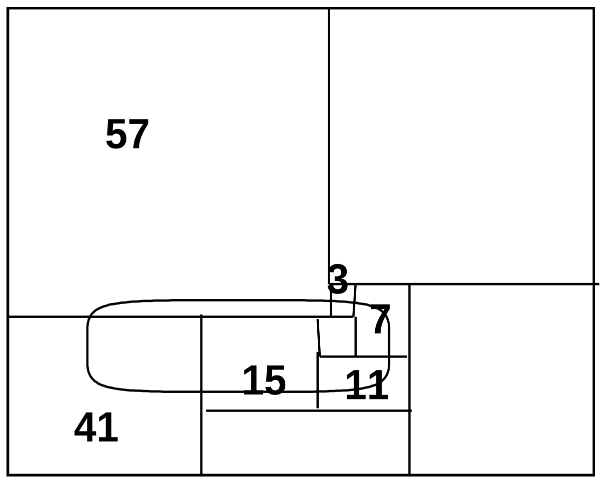
Another at Random [25] 275 x 275 has 81 13 107 33 41 17 25 35 7 31 3 37 71 53 51 49 55 which forms a TRAIL since 55 and 81 do not border each other.

Another at Random [21] 112 x 112 has 15 35 27 19 11 17 7 9 25 37 33 29 which forms a TRAIL

S3. ODD ELEMENTS FORMING A RING

The Solution [11] 112 x 81 has 43 29 19 9 1 41 33 5 (43) a RING.

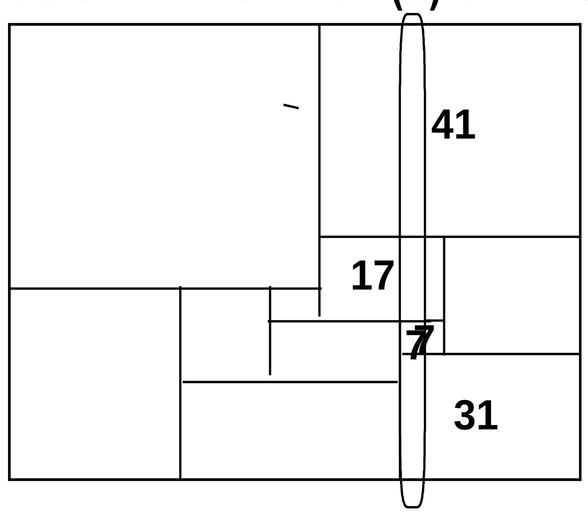
The Solution [9] 33 x 32 has 15 7 1 9 A TRAIL whereas [9] 69 x 61 has 33 5 7 9 25 a TRAIL. Solution [10] has 25 17 15 13 11 3 again a TRAIL. [10] 105 x 104 has 45 19 7 33 a TRAIL. [10] 111 x 98 has 3 7 11 15 41 57 (3) a RING as Below -



<[10] 111 x 98

[11] 97 x 96 >

(Even Elements not shown)



In this case an oblong line can be traced, such that all Elements falling within it are Odd!

Although some Solutions contain some blocks of Elements which are entirely Even (particularly xyz Solutions rather than xy ones), the overall occurrence of Even Elements tends to be fragmented into several unconnected Areas.

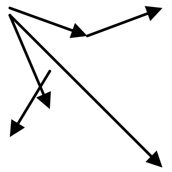
In [11] 97 x 96 the Odds (only 4) also appear in a Line.

The study is complicated by the fact that trail lines are not usually confined to one single set pattern.

S4. TRAILING OF ELEMENTS – CORNER & CIRCUIT TRAILS

TRAILS are a set of continuous STROKES drawn diagonally across each Element of a Solution.
Using one set of STROKES only results in the Elements
1. Having no STROKES (i.e. Missed) or,
2. Have one STROKE \ or / or,
3. Have two STROKES X; In this case Elements Crossed with 2 STROKES (X) from the *same* SET always have EVEN values.

In any Solution, a SET of STROKES is possible from the North-west CORNER. It always ends at one of the three remaining Corners.



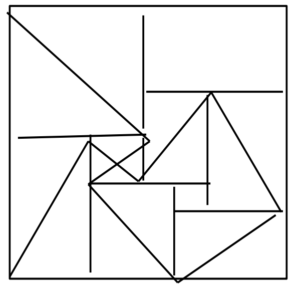
1. To A - HORIZONTAL North-East Corner, or
2. To B - VERTICAL South-West Corner, or
3. To C - DIAGONAL South-East Corner.
These are the first of two CORNER TRAILS. The second CORNER TRAIL links the two remaining CORNERS which are always found to link up - e.g. If 1st TRAIL is NW to NE the 2nd Trail is SW to SE.

All TRAIL SETS will be STROKE CODED* A B or C and followed by two numbers which denote the number of strokes linking the Corners. (* See also TRAIL CODES later in Book)
e.g. C07,17 means 7 STROKES link NW with SE & 17 STROKES link NE with SW, both DIAGONAL.

1. **In some cases these two Corner Trails together result in every Element Crossed by two STOKES “X”. Where true, the two numbers total twice the Order of the Solution.**
2. **This is always the case where the REDUCTION INDEX IS ODD (1, 3 ,5 ,7 ,9 ...x).**

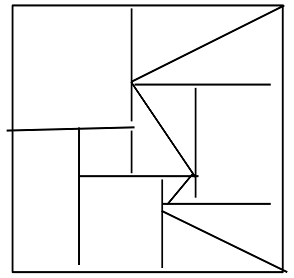
2. In the remaining cases the two Corner Trails result in some Elements having ONE STROKES or possibly NO STROKES. In such cases the REDUCTION INDEX IS EVEN (2,4,6,8 ...x) and at LEAST ONE further TRAIL always occurs.

CORNER TRAIL 1 CORNER TRAIL 2 CIRCUIT TRAIL

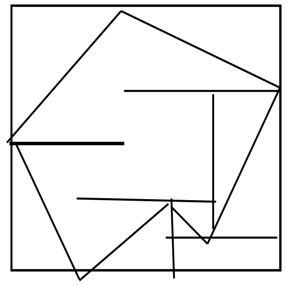


B

8 STROKES



4 STROKES *



6 STROKES **

The Solution above is [9]33 x 22 and may be coded B08,04,06
The Reduction Index is 2 i.e. EVEN as per 2 above. The Corner Trails are formed vertically so Coded "B". Other Trails where they occur will be called CIRCUIT TRAILS and form an unending series. Some Solutions have several CIRCUIT TRAILS - to prevent confusion they are shown in increasing order of size, e.g. A16, 12, 6, 8, 8,12.

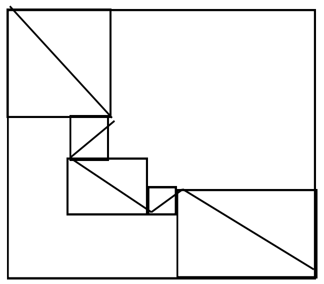
S5. NUMBER OF TRAILS PER SOLUTION

There must be at least 2 i.e. 2 CORNER TRAILS only.
Sometimes 3 4 or higher. Any given size Order will dictate a limit to the number of CIRCUIT TRAILS possible - otherwise in theory there is no limit.

S6. NUMBERS OF STROKES

1. All CIRCUIT TRAILS (where they exist) contain an EVEN number of STROKES.
2. All 'A' and 'B' CORNER TRAILS also contain an EVEN number of STROKES.
3. All 'C' CORNER TRAILS contain an ODD number of STROKES.

1. The least STROKES for a CORNER TRAIL is 4 - as shown above for Codes A4... or B4...
2. C CORNER TRAILS require at least 5. Code C5... In the case of Squares-Squares the 2^{nd} & 4^{th} of these are the same value & the Squared-Square automatically Imperfect.



C

3. The least STROKES for a CIRCUIT TRAIL is 6 - as shown above.

<u>DIMENSIONS OF RECTANGLE</u> 2 CORNER TRAILS ONLY.
1. Type Ax,y EVEN x ODD (RI EVEN)
2. Type Bx,y ODD x EVEN (RI EVEN)
3. Type Cx,y ODD x ODD (RI ODD).
<u>DIMENSIONS OF RECTANGLE</u> (WHERE CIRCUIT TRAILS OCCUR)
1. Ax,y,z ... Some EVEN x ODD, others EVEN x EVEN
2. Bx,y,z ... Some ODD x EVEN, others EVEN x EVEN
3. Cx,y,z ... Some ODD x ODD, others EVEN x EVEN

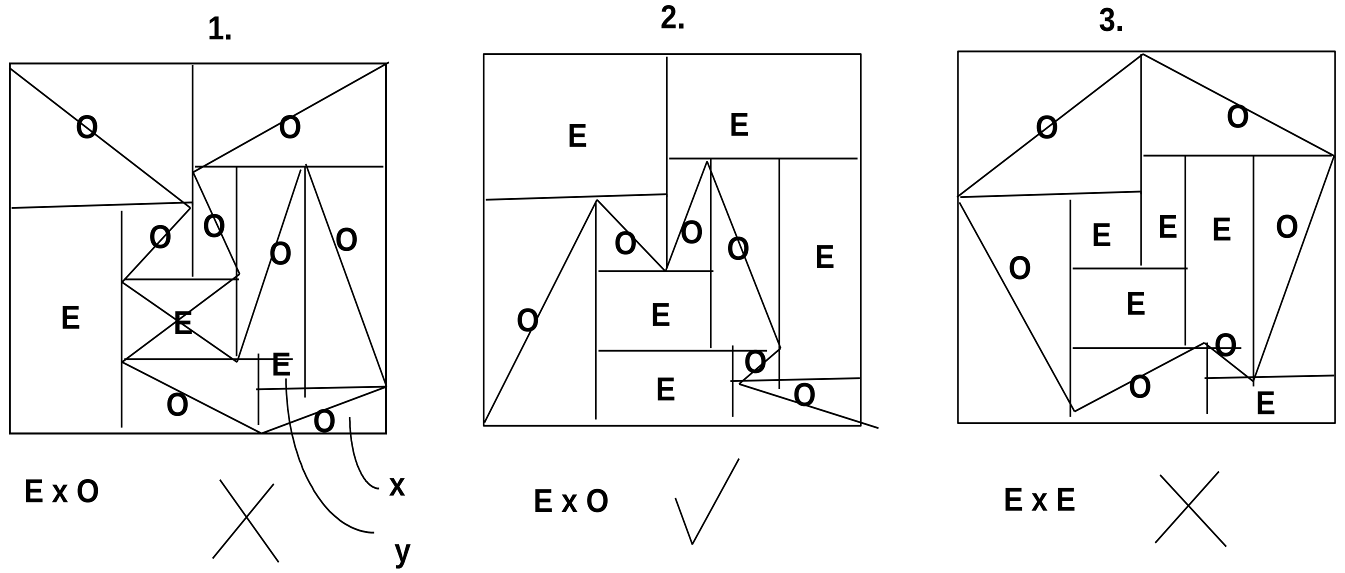
<u>EVEN x EVEN CASES FOR Ax,y,z & Bx,y,z OR Cx,y,z having a SINGLE CIRCUIT TRAIL</u>
Where THREE TRAILS apply AND result is an EVEN x EVEN Solution, then all Odd Elements are found from the Circuit Trail only. Any Element struck twice, or not at all, is EVEN.
Simply put, <u>All SINGLE Strike Elements within the Circuit Trail are ODD</u>.
(NB. Ignore either Corner Trail when ascertain Odd Elements)

<u>S7. 'BLIND' SOLUTION PATTERNS</u>

By drawing a random SR pattern, it is sometimes possible to ascertain information about the Solution without calculating it.
1. Codes Ax.y and Bx.y cannot apply to SQUARES only RECTANGLES as Solutions are either EVEN x ODD or ODD x EVEN as stated above). (Squares with other Codes are possible).
2. With Solutions with 2 Corner Trails only, using just one Trail we can ascertain whether each Element is ODD or EVEN, and also whether the Dimensions are ODD x EVEN or whatever.
<u>Once again A or B or Cx.y.z... Solutions indicate the Reduction Index is EVEN. A or B or Cx.y Solutions indicate the RI is ODD</u>

3. Where there are <u>2 CORNER TRAILS ONLY</u> The Trail from NW Corner will indicate that the Elements are ODD, where the Element is struck ONCE, and EVEN if struck TWICE (X) or not at all.
But the other Corner Trail will obtain the same result!
Put simply, <u>ONE STROKE denotes ODD ELEMENTS otherwise the Element is EVEN</u>.

4. Where Circuit Trails occur obtaining correct information is usually trickier. e.g.



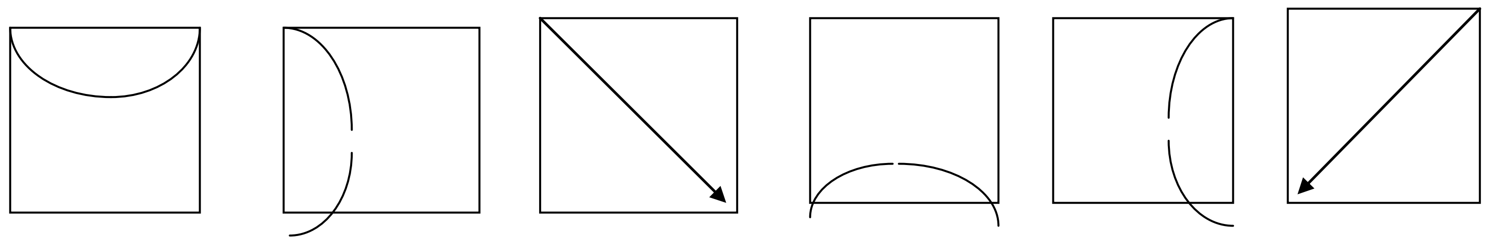
[11] 98 X 95

CODE A10.6.6

Solution is actually [11] 98 x 95

Where a pattern is drawn at random like that above where there are 3 Trails in A10,06,06 one pattern of which must contain all Odd Elements, only one can be the valid pattern. Yet we have three and mathematically, examination suggests each is correct! This seems a paradox. However by using x and y in bottom left corner, there are three possibilities before calculating, E & O, O & O or O & E. In this Blind Solution we can ascertain that it is Even x something, and that one Element is EVEN. That's it!

S8. TRAIL CODES & THEIR RECIPROCALS



A1 (Horizontal,) A2 (Vertical) A3 (Diagonal) B1 (Horizontal) B2 (Vertical) B3 (Diagonal)
Where both Corner Trails are combined the Codes are shown as A1B (Horizontal), A2B (Vertical) or A3B (Diagonal).
Circuit Trails are Coded C, D, E, F... etc. C being the one nearest to the top and left, D the next nearest top & left, and so on.
Every Solution can be given a TRAIL CODE. For example some Solutions it is A1 and by Trailing from NW to NE Corner all Elements with ONE STROKE resulting are ODD (so those with NIL or TWO STROKES are EVEN).

But for every Solution TWO TRAIL CODES apply, one being the Reciprocal of the other. E.g. A1 & B1 A2 & B2 or A3 & B3 (where two Trails only) one usually preferable to the other.
If say there are 4 Trails the Trail Code & reciprocal together include ABC & D with A & B suffixed with 1 or 2 or 3.
Listed below are the various Combinations ONE of which indicates all ODD Elements by Single Strokes. (The letters in brackets are the equivalent Reciprocals) -

TWO TRAILS A (B) 1 combination
THREE TRAILS A (BC), B (AC) or C (AB) 3 combinations
FOUR TRAILS A (BCD), B (ACD), C (ABD, D (ABC)
AB (CD), AC (BD) & AD (BC) 7 combinations
FIVE TRAILS A (BCDE), B (ACDE), C (ABDE), D (ABCE), E (ABCD)
AB (CDE), AC (BDE), AD (BCE), AE (BCD), BC (ADE), BD (ACE), BE (ACD), CD (ABE), CE (ABD) & DE (ABC)
15 combinations (+15 reciprocal)
SIX TRAILS 31 combinations ….. and so on –
Number of Combinations is 2 to a power minus one.

The correct TRAIL CODE for ODDS in a Solution can be ascertained as follows: (Less complicated than this suggests!)

1. **Search for Odd Elements adjacent to each other – having chosen a likely group draw the entire trail.**
   If all Odds are accounted for then stop.
   Otherwise –
   If there is an Even Element Stroked just once then by Stroking it the other way trace out the Trail in both directions until complete. If all Odds are accounted for then stop. Otherwise -
2. **Choose an Odd Element already Trailed. Trace out with your finger the opposing Trail (which is incorrect) until a new Odd Element is found (i.e. not lined). Then by reversing the direction on that Odd Element, draw out a Correct second Trail. If all Odds now found stop. Otherwise repeat this process until finished.**
3. **Record which Trails are Corner & which Circuit ones. E.g. C or BDE or whatever.**
4. **If process fails then in 1. Reverse direction of the first Odd Element chosen.**

Note - Unfortunately with many BLIND PATTERNS including Circuit Trails it is impossible to ascertain exactly whether Elements will be Odd or Even. Worse still where there are several patterns of O&E's, each appearing to be valid (the Dimensions always agreeing with the Elements) only ONE pattern is actually right (as we cannot have 2 or more Solutions for any single pattern!).

S9. TWO TRAILS ONLY HAVING NO CIRCUIT TRAIL

STROKE CODES One of Ax.y Bx.y or Cx.y
TRAIL CODES One of A1 (Recip. B1), A2 (Or B2), A3 (Or B3)
Simply ODDS (& EVENS) are always determined from the 1st Corner Trail.
Likewise, the Reciprocal Trail is equally valid.

S10. THREE TRAILS HAVING ONE CIRCUIT TRAIL

STROKE CODES One of Ax.y.z Bx.y.z or Cx.y.z
TRAIL CODES One of A1 (or B1C) A2 (or B2C) A3 (or B3C) C (or A1B) C (or A2B) C (or A3B)
1. E x E DIMENSIONS FOR Ax.y.z Bx.y.z Cx.y.z
The one CIRCUIT TRAIL conveniently shows all ODD ELEMENTS by Single Strikes (0 and 2 strikes EVEN).
By the Reciprocal Rule, amalgamation of the Two Corner Trails also shows all ODD ELEMENTS by Single Strikes (0 and 2 strikes EVEN).
2. E x O DIMENSIONS FOR Ax.y.z
In some, 1st Corner Trail alone. Otherwise 2nd Corner Trail (HORIZONTAL)*
3. O x E DIMENSIONS FOR Bx.y.z
In some, 1st Corner Trail alone. Otherwise 2nd Corner Trail (VERTICAL)*
4. O x O DIMENSIONS FOR Cx.y.z
In some, 1st Corner Trail alone. Otherwise 2nd Corner Trail (DIAGONAL)*
*** Which one applies can be found from actual Solutions, but not possible from 'Blind' patterns!**

S11. FOUR TRAIL SET CASES

Complicated but -
1. E x E DIMENSIONS FOR Ax.y.z.a Bx.y,z.a Cx.y.z.a
O & E Elements determined from one of these three -
In some, The Addition of both Circuit Trails, or
In others, The 1st Circuit Trail alone, or
In remainder, The 2nd Circuit Trail alone.
2. OTHER DIMENSIONS
In some cases an addition of a Corner Trail to a Circuit Trail. To find the correct combination requires care.

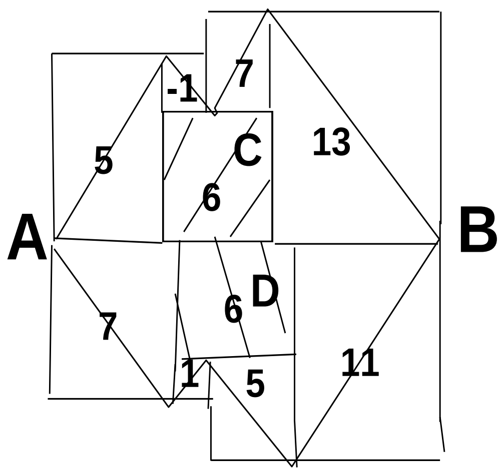
S11.1. A CODE INDICATING THE NUMBER OF ODD ELEMENTS, AND CORRESPONDING RECIPROCAL CODE

Suppose we have a code A10,08,10,06 with Four Trails.
The four numbers are given codes a=10 b=8 c=10 and d = 10. (a b c d e …)
Suppose that all the Odds are contained in the third Trail c and that there are 8 of them, then it is coded 08-10c.
This means that out of 10 strokes in the third trail, 8 are Odds.
In practice this means in the Circuit Trail c, one is crossed twice indicating an Even Element and eight crossed once all of which are Odd Elements.
10+8+10+6 = 34 = twice the Order 17.

Its RECIPROCAL CODE contains the remaining letters of a to d, and the balance from 34 strokes. It may often be a clumsy way of defining the Odd Numbers but that does not matter.
08-10c has the RECIPROCAL CODE 08-24abd. NB. 8 is still the number of Odd Elements. 10 + 24 = 34 strokes. There are 24 strokes in total for the trails a b and d together. But only 8 out of the 24 strokes give Odd Elements.
Now suppose we have a simpler Code C05,43. The Odds Code is 05-05a. This indicates there are 5 Elements from NW to SE Corner and all five happen to be ODD. The Order is 24 i.e. 5+43 divided by 2.
Note that the number of different Elements involved will often be less than the code numbers shown e.g. in 24abd above only 16 Elements are involved 8 Odd (x1 stroke) and 8 Even (x2 strokes).

S12. PENTADS ARE RELATED TO SOME CIRCUIT TRAILS

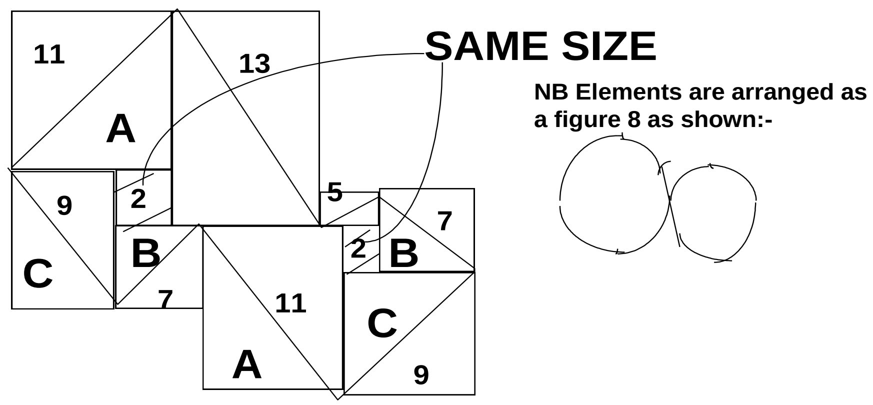
In this section we look at solutions which contain a trail of 8 Odd Elements in a Ring. Within the 8 Odds may be several Even Elements but here we look at Solutions with just two Even Elements within.
The two Elements are always the same size. Consider two Pentads back-to-back:



(The Above patterns forms part of Imperfect Squared-Squares [22] 116 x 116 & others).
Note that one or both Pentads can have a negative Element as has deliberately shown above with a -1. (The Odds Trail still joins up but changes a bit in shape).
In Pentads, Elements C and D both have to be ¼ of line CD, being 6. So C equals D whatever the Elements. But if C & D happen to be ODD then the Odds Trail changes from a continuous ring into a single jagged line.

S13. ANOTHER CIRCUIT TRAIL OF 8 ODD ELEMENTS

Consider the pattern below:

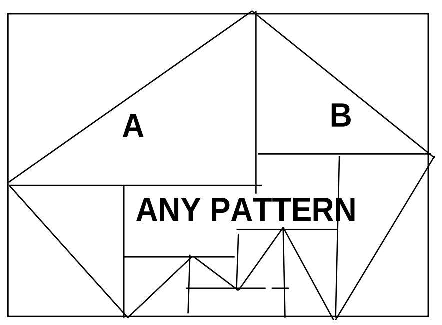


Again there are two Even Elements here shown as 2. In this type of construction there are also three other matching pairs at A B & C. Only 13 is distinct. 7 9 11 13 and 5 7 9 11 form arithmetic progressions. Again, if this is part of any solution, that solution will always be Imperfect.

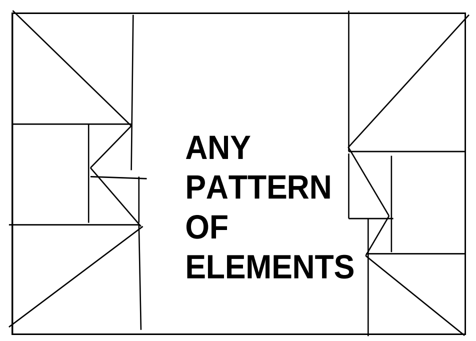
In EVEN x EVEN Solutions the Trail does not use the Corner Points but is a ring of Odd Elements usually (but not always) touch each of the four sides - see C14.2 for an example.
What is being said is that in ANY Solution a continuous line can be drawn diagonally dissecting every ODD Element whilst not diagonally dissecting any EVEN Element. In some cases however two parallel lines will need to be drawn to make sense of the construction!

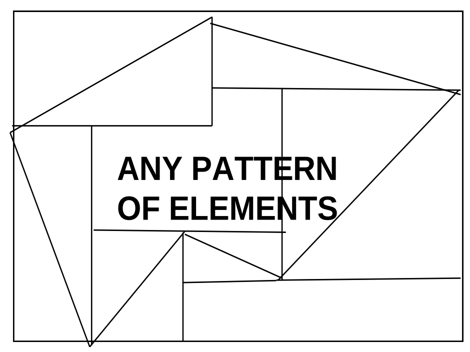
S14. CIRCUIT TRAILS

Consider the patterns below - the first, sides 2225 clearly has a Ring Pattern. Notice the Pentad between the bottom corner Elements. This means that any Solution with bordering Elements of this type

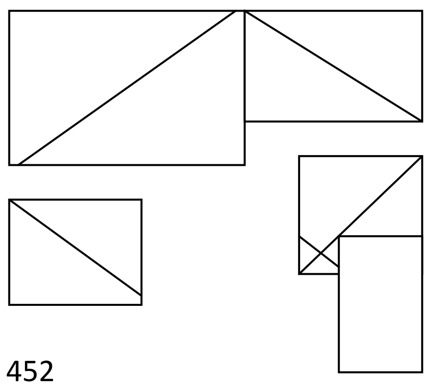


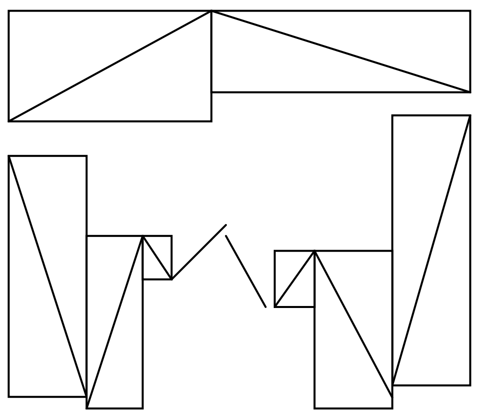
2225

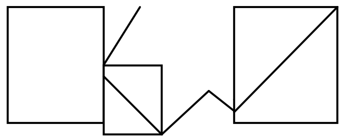




2233

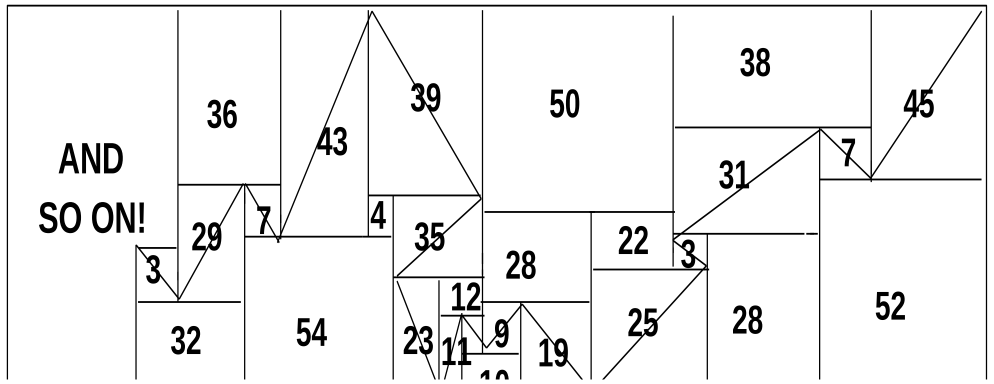






Inspection reveals (subject to further checking) that <u>all of the Elements Above dissected once, are all ODD</u>.
This does not necessary mean that every other Element is automatically EVEN. Also some patterns will contain more Circuit Trails which may have some further ODD Elements.

In ENZ solutions such a Trail can be drawn across the entire Solution from left to right! Illustrated below is a bit of Trial & error work (N.B. no actual solution from this). Curious isn't it?



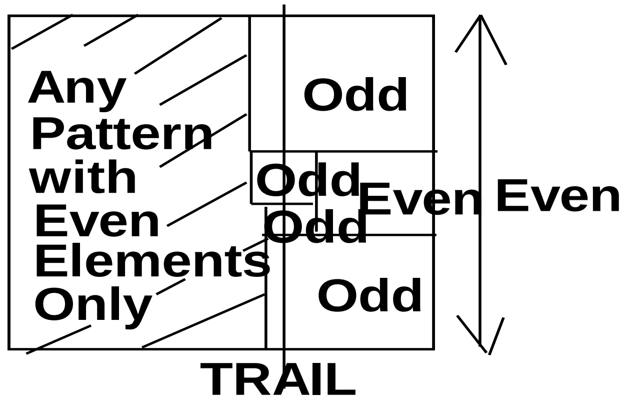
S15. FURTHER CONSIDERATIONS OF TRAILS

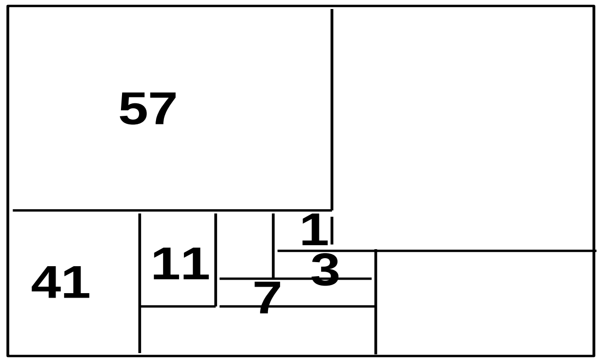
In calculated Solutions complicated by several Circuit Trails and well as 2 Corner Trails, the Odd Elements are determined from some combination of Trails or sometimes from a single Trail. But because of the Reciprocal Rule, addition of all remaining Trails also gives a pattern where Single Strikes point out the Odd Elements.
From this statement Odds can be determined by starting to use any particular Trail, and so this means we can always use the North-east Corner Trail to start the process.
If needed, Solutions could be Coded by just indicating which Trails (if any) need to be added to the North-east Trail to form the Odd Elements pattern.
One way of drawing the Trails needed to form the Odd Elements pattern could be-
1. Draw the Corner Trail from North-east first. Have all Odd Elements been accounted for by a single Strike? If not then -
2. Look for an Even Element struck through once and form that Trail which joins the crossing of that Even Element.
3. Repeat 2. above as many times as necessary.
4. Suppose we find 3 Trails out of a total of 4. It is much simpler to draw the 4th Trail only which will contain all the Odd Elements. If what we have found is complicated - drawing the remaining Trails separately man be simpler.

S16. TOTALLING OF ALL ODD ELEMENTS

There are many solutions of the type below with a Pentad at one end - with Odd Elements in the corners, and a series of Elements all divisible by two at the other. In such cases there are only four odd Elements in the entire Solution. These may always be cut by one straight trail line as shown.

It is easily seen that the total of these four Elements is equal to a single Dimension of the Rectangle - either greater or smaller Reduced Dimension.





[12] 113 x 98 showing ODD Elements only.

On observing some other Solutions it is found that two lines of Odd Numbers may be drawn in parallel fashion which each lot of Odd Elements conveniently tota1ling the same Reduced Dimension. So far so good!

Look at above 2 where 57 + 41 = 98, the lower Dimension. But the 1 3 7 and 11 obviously do not total 98.

Here the nice relationship is spoilt - but is it? Actually not entirely as 11 = 1 + 3 + 7, and putting the two together we have 57 + 41 + 11 - 1 - 3 - 7 = 98 or, if you prefer, 57 + 41 + 1 + 3 + 7 - 11 = 98 equaling a Single Dimension!

S17. REGARDING SOME ELEMENTS AS NEGATIVE IN THE TOTAL

Looking at many Solutions I have found (so far) that the Odd Elements can be manipulated to total the Single Dimension in some cases and the Double Dimension in others. It is believed that no 'leftover' Odd Elements ever exist.

Possibly larger Order solutions might total 3 4 5 6 . . . times the Dimension.

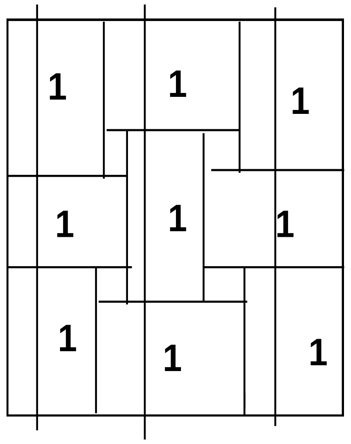
To do this it is often necessary to regard one or more Elements as NEGATIVE for this to work, though not always.

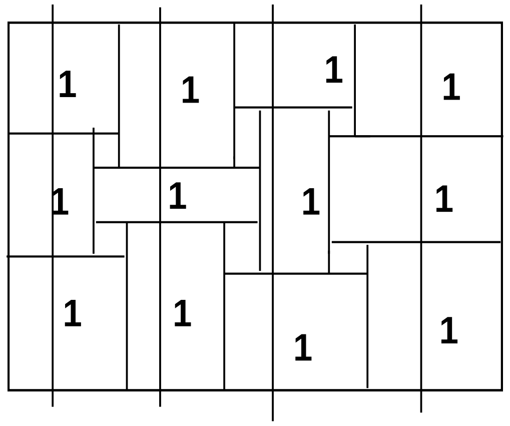
The true positive total of all Odd Elements often is found to be one of the following.

1. Exactly Dimension times ONE.
2. Over Dimension times ONE but less than Dimension times TWO.
3. Exactly Dimension times TWO.
4. Over Dimension times TWO but less than Dimension times THREE.

S18. TOTAL OF ODD ELEMENTS CAN BE GREATER THAN TWICE A DIMENSION

This can be easily proved by the following series of Invalid Zero Solutions -





And so on with
4 x 4 ones
5 x 4 ones
5 x 5 ones
6 x 5 ones .. ..

Above 1 has odd Elements totaling 9 being 3 x Dimension of 3. Above 2 has odds totaling 12 being 4 x Dimension of 3. And the series is ad infinitum. This is also true of some Valid solutions.

Note that in all this series that the lines can also be drawn Horizontally. In Above 2 12 = 3 times Dimension of 4.
In [11] 97 x 96 Above the Odds total is clearly 96 alone and 97 is clearly not obtainable.

It is found that the Odds Total may relate to (1) The Upper Dimension (alone) or (2) The Lower Dimension (alone) or (3) Both.

S19. LOOKING AT TOTALS OF EVEN ELEMENTS

Are there any parallels concerning the EVEN Elements only? Taking some random examples -

[12] 368 x 265 has even Elements of 136 34 16 and 126 which total 316. Neither 368 or 265 or zero can be obtained by making any of these negative.
[11] 195 x 191 has even Elements of 90 44 86 34 18 52 which total 326 from which neither 191 or 195 can be obtained.

Nothing of interest found.

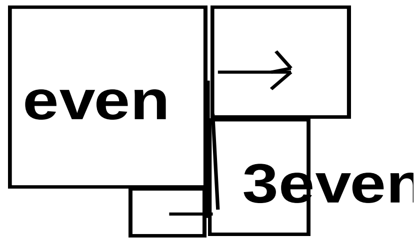
S20. QUESTIONS STILL TO BE RESOLVED

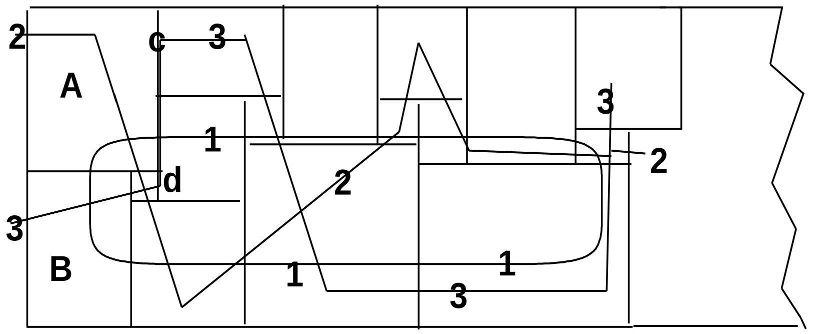
1. Is the total (including negatives) always some integer times one of the Dimension? So far this is so in all solutions examined.
2. Which Dimension is involved in individual cases - the larger (m) or smaller (n)? Which can be displayed either way?
What rules determines which of these?
3. In Squared-Squares is it true that Odd Sided solutions have a Single Dimension and Evens a Double Dimension?
Also are results affected by whether the Order is Odd or Even, in conjunction with whether the Dimensions are both Odd or Even or Mixed?
4. Are there more practical uses for these theories?

S21. XY SOLUTIONS AND ODD ELEMENT TRAILS

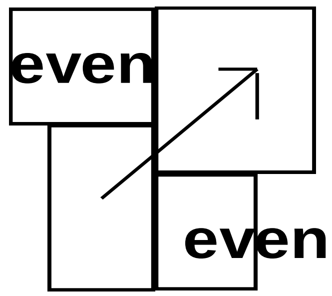
The following only applies to constructions of the type XY alone
It is interesting to draw a ‘Trial and Error’ set of Elements inserting hypothetical values and observing the patterns created by the Odd Elements. The simplest construction of all is to commence with a Diad (A & B below) and form an ENZ pattern.

Although A and B could theoretically be Evens this would lead to all Elements being even and therefore the Elements could be reduced to at least half size. This leaves the three possibilities shown below, 1 2 or 3.

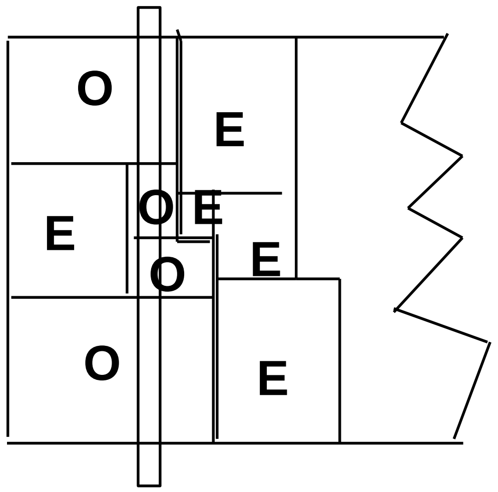


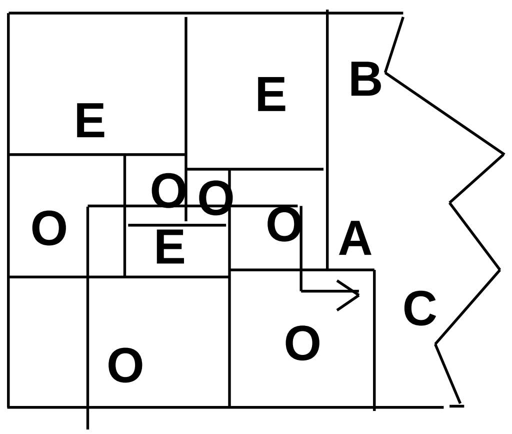


RANDOM ENZ SOLUTION
A & B MAY BE
1. ODD and ODDor
2. ODD and EVEN or
3. EVEN and ODD



In the above if two Odds are used a circular trail line is formed.
This will often be a circular trail with many ins and outs rather than the neat Trail 1 above. If A is Odd and B Even then the Trail Line marked 2 is formed.
If A is Even and B Odd then Trail Line 3 is formed. But note that at CD the line runs along the border line not through <u>any</u> Elements! Have I cheating then! Not so. Look at above 2 which shows it is a question of how the squares are drawn i.e. The trail is always continuous providing we sometimes have to distort the patterns! But xy type solutions are of course not restricted to the above type. Now look at the effect of a Pentad start -





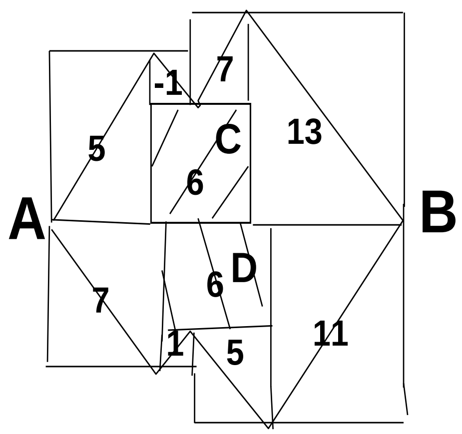
Again the red Elements Above can be Odd & Odd, Odd & Even or Even & Odd. But look at above 1 with two Odds!

The Reader can test this: whatever Elements are added to the right - without introducing a further algebraic Unknown.
Every subsequent element is automatically Even - whatever pattern is employed! The result theoretically and in practice is a Rectangle with only four Odd Elements totaling the smaller Dimension n.

In Above 2 using Odd and Even, it is found that when Elements are added, there will often come a time when all subsequent Elements are Even. For example Elements added at A then B and C are all Even and all further ones also. Up to this point the Odds will form a Ring of some sort.
If Even and Odd is used instead then the same type of pattern is clearly the case.
Though obvious to say - if the smaller Dimension is an Odd number then a barrier leading to Even Elements only thereafter will never happen.
(With Pentads however the downward Dimension is always Even)

S22. ODD ELEMENTS TRAILS AND PENTADS ARE RELATED

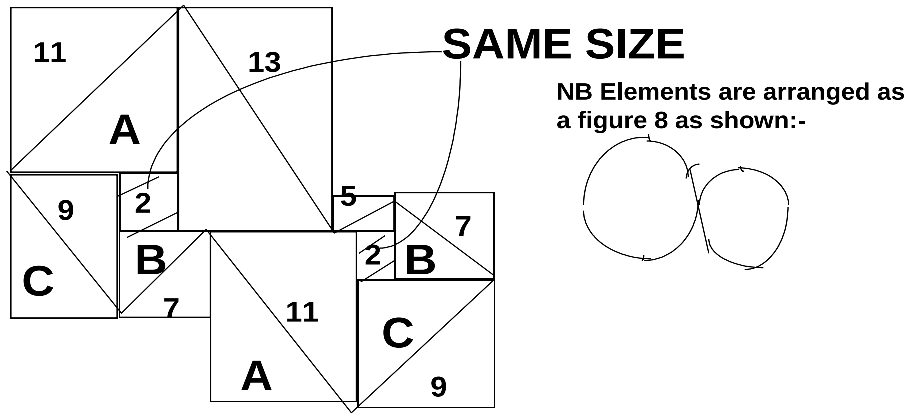
In this section we look at solutions which contain a trail of 8 Odd Elements in a Ring. Within the 8 Odds may be several Even Elements but here we look at Solutions with just two Even Elements within.
The two Elements are always the same size. Consider two different Pentads placed back-to-back:



(The Above patterns forms part of Imperfect Squared-Squares [22] 116 x 116 and several others] .
Note that one or both Pentads can have a negative Element as has deliberately shown above with a -1. This pattern needs to be have -1 5 & 6 adjusted with the Element -1 showing as positive 1.
(The Odds Trail still joins up but changes a bit in shape).
In Pentads, Elements C and D both have to be ¼ of line CD, being 6. So C equals D whatever the Elements. But if C & D happen to be ODD then the Odds Trail changes from a continuous ring into a single jagged line.

S23. ANOTHER TRAIL OF 8 ODD ELEMENTS

Consider the pattern below:



Again there are two Even Elements here shown as 2. In this type of construction there are also three other matching pairs at A B & C. Only 13 are distinct. 7 9 11 13 and 5 7 9 11 form arithmetic progressions.
Again, if this is part of any solution, that solution will always be Imperfect.

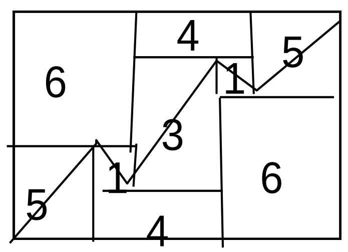
S24. OTHER TRAILS WITH 8 ODD ELEMENTS IN A RING AND 2 INTERNAL EVEN ELEMENTS

Constructions with 8 Odd Elements in a Ring may have any number of Internal Even Elements.
However when there are just two then these Even Elements are always identical, making the Solution Imperfect.

S25. ODD ELEMENTS IN SOME SOLUTIONS HAVE A TRAIL LINE

In many solutions it is possible to trail all Odd Elements in turn without taking the pen off the paper! However there are some twists to this.

Solutions which are Odd x Odd must contain an Odd quantity of Odd Elements - a minimum of 5 - and should start at one corner and proceed to the corner diagonally opposite.



TRAIL LINE
FOR [9} 15 x 11

1. The Solution shown is very straightforward - 5 1 3 1 5 containing all positive numbers (totaling 15 = m).
2. Often however negatives occur e.g. turn this solution 90 degrees and we have 5 -1 3 -1 & 5 (totaling 11 = n).
3. Sometimes some fiddling needs to be done! Look at this [11] 199 x 178 i.e. ODD x EVEN where the trail runs from one corner to a corner adjacent to it.

In EVEN x EVEN Solutions the Trail does not use the Corner Points but is a ring of Odd Elements usually (but not always) touch each of the four sides - see C14.2 for an example.
What is being said is that in ANY Solution a continuous line can be drawn diagonally dissecting every ODD Element whilst not diagonally dissecting any EVEN Element. In some cases however two parallel lines will need to be drawn to make sense of the construction!

S26. TRAILS - SOME RULES

1. Each Trail forms one fixed set of strokes assuming no Crossover(s) or Invalid Solutions exist.
2. Trails are of two types - Corner Trails and Circuit Trails.
3. Trails starting at a Corner Point always reach another Corner Point, so
4. There are Two Corner Trails in every Solution namely,
1. From NW corner to some other corner.
2. One adjoining the two remaining Corners.
5. Diagonal Corner Trails apply to ODD x ODD or EVEN x EVEN Solutions (in its most reduced form).
6. Single Corner Trails sometimes hit the same Element twice - creating a cross ‘X’.
7. Different Corner Trails often hit the same Element - creating a cross ‘X’.
8. The 2 Corner Trails together will either
1. Put an ‘X’ against EVERY ELEMENT
or 2. Leave some Elements without an ‘X’ i.e. ‘ /’ or ’\’ or ‘ ‘.
9. Where some Elements are not Stroked twice by the both Corner Trails together, then a CIRCUIT Trail (or more than one) applies. All remaining Trails will form continuous CIRCUITS and never touch any corner.

S27. SOLUTIONS BORDERED BY ODD ELEMENTS ONLY

Below [13] 663 x 482 is shown. This is bordered by ODD Elements 257 169 237 245 193 & 225. Sides 2323.

```
|
A___________________________________C
|257       |169_______________|237|
|          |120|49____________|   |
|          |   |30  |19_______|___|
|          |   |____|11_|245      |
|__________|   |41      |         |
|225|32_|____|__________|         |
 D__|193_______________|__________|B
```

A Corner Trail from A to B passes all Odd Elements, but crosses through Evens 32 & 120 and leaves 30 untouched.

A Corner Trail from C to D passes all Odd Elements, but crosses through Even 30 and leaves 32 & 120 untouched.

Below [15] 200 x 166 is shown. This is bordered by ODD Elements 101 99 67 35 33 & 65.
This Solution has a Ring Trail - Starting “/” trace out 101 99 67 35 3 [8] 13 15 17 19 [8] 5 33 & 65. 8 the only even number appears twice i.e. crossed in both directions “x”

```
_________________________________
|101                     |99___|
|________________________|32|67|
|65|17___|19             |  |  |
|  |15|2_|_______________|  |  |
|  |  |13     |8_________|  |  |
|  |  |       |5  |3_|___|  |  |
|  |__|_______|___|35       |  |
|__|33____________|_________|__|
```

Up to end of Order 15 there are only 4 similar solutions of sides S2224 - [15] 200 x 146, [15] 412 x 344, [15] 452 x 392 and [15] 760 x 634.

```
______________________________________
385                         |375____|
____________________________|116|259|
249|57__________|79         |   |   |
   |35 |22_____|____________|   |   |
   |___|13_|61       |27    |   |   |
   |48     |         |______|___|   |
   |_______|_________|143       |   |
___|109______________|__________|___|
```

Of these four only [15] 760 x 634 (above) has a totally ODDS Trail - Starting ”/” trace out the Trail Ring 385 375 259 143 27 79 57 35 13 61 109 249.

S28. COMPARING Cx,y TRAILS WITH SEMI-PERIMETERS IN ORDERS 9 TO 14

(Note: This Section refers to full size Dimensions and Semi-Perimeters).

The following concerns solutions with 2 Corner Trails only of type Cx,y which join diagonally opposite Corners.

The dimensions are found to be Odd x Odd, with an Even Semi-Perimeter. The Reduction Index is always Odd (1, 3, 5, 7 etc.)

In the following the reverse Code also applies (e.g. C05,13 to C13,05, C07,21 to C21,07 etc.): -

The numbers shown are Semi-Perimeters.

1. For Order 9 Code C05,13 130 (divisible by 2 but not by 4).

2. For Order 10 No solutions of type Cx,y.

3. For Order 11. C05,17 386 (divisible by 2 but not by 4).
C09,13 336 368 (both divisible by 4).
C07,15 & C11,11 no solutions.

4. For Order 12. C05,19 638 (div by 2 not 4).
C07,17 560 (div by 4).
C09,15 608 (div by 4).
C11,13 No Solutions.

5. For Order 13. C05,21 986 1066 1082 1122 1166 1170 (all div by 2 but not by 4)
C07,19 928 992 1040 1088 (all div by 4)
C09,17 962 (div by but 2 not 4). 1008 (Div by 4). Mixed
C11,15 1058 (div by 2 but not 4).
C13,13 1176 (div by 4)

6. For Order 14. C05,23 1682 1722 1734 1770 1802 1818 1834 1930 1938 1942 2002 2014 (div by 2 but not 4).
C07,21 1488 1648 1680 1712 1776 1792 1828 1840 1852 1872 1932 1968 (all div by 4).
C09,19 1570 (div by 2 not 4) 1600 1632 1696 1760 (all div by 4). Mixed
C11,17 1666 1746 1762 1778 1822 1890 1918 (all div by 2 but not 4).
C13,15 1602 1810 1828 1852. (some div by 2 not 4, other div by 4). Mixed

Solutions with a given Semi-perimeter have differing Codes. THE ABOVE MAY BE INCORRECT – CHECK IT!

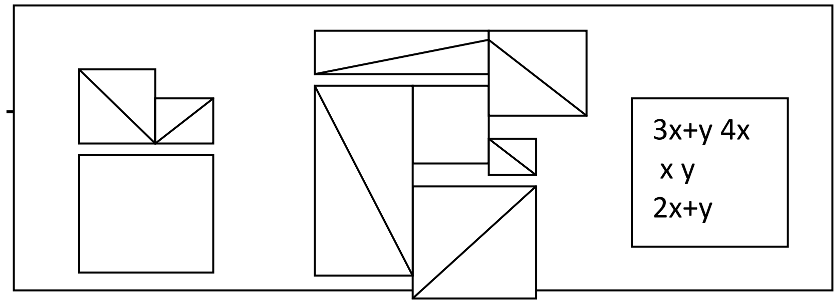
Whereas some groups are all divisible by 4, others are never divisible by 4, and some have both in C09's & C13's!
It appears that C05,y cases give Semi-Perimeters always double an Odd number (Not divisible by 4). In the case of Squares this will hold true. In the case of Squared-Squares, Semi-Perimeters of this type can only be SINGLY EVEN. C09,y cases DO exist and are not restricted to being DOUBLY EVEN SP's as the above might suggest. This means that there is no consistency regarding SINGLY EVEN or DOUBLY EVEN. Both will exist for most Codes.

S29. COMPARING TRAILS WITH DIMENSION m THIS NEEDS CHECKING!

Bearing in mind the last section there is a close relationship with Code A solutions that have 2 Corner Trails only.
As Code C05,13 Order 9 gives Semi-Perimeter 130, it is found Code A06,14 Order 10 gives Solutions with Upper Dimension 130!
So with A12,18 cases we will have solutions with Upper Dimensions of 1666 or 1746 or 1762 etc. as listed for C11,17 above, but for Order 15 instead of 14.
In testing, some new values are found for the list above. However it does not follow that Valid solutions must exist for both Orders for each value. So a SP value for an Odd Order may not occur as a Dimension in the Even Order above it - or vice versa. In theory however the list for C07,19 (as Semi-Perimeters) and A08,20 (as Dimensions for the Order above) are the same. Ditto with C05,21 and A06,22 and so on..

S30. XY SEQUENCIES AND CONTINUOUS TRAILS

"XY SEQUENCIES" refers to the construction of Rectangles by random means using adjacent Elements x and y and building up a pattern of values in x and y. By making the pattern into a Rectangle an equation is taken of the form Px = Qy from which x = Q and y = P (often subject to cancelling down) from which the Elements are fully calculated.
Is there a fixed construction of x & y giving a Continuous Trail?



A line can be shown for x and y in above 1, but not for x + y,
But if x is ignored and the line started at y (Above 2) the line can continue in x + y, 2x + y, 3x + y and 4x as shown!
In fact it is possible to continue indefinitely in a clockwise fashion. As the pattern of Trails is a fixed one so is the XY Sequence.

S31. CONTINUOUS X AND Y TRAIL SEQUENCE

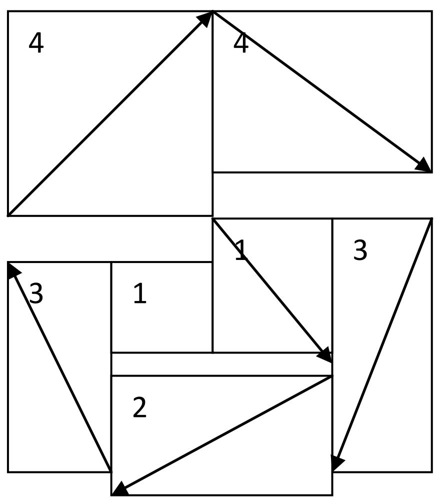
The Trail Sequence in terms of x and y are fixed for each position, as follows:

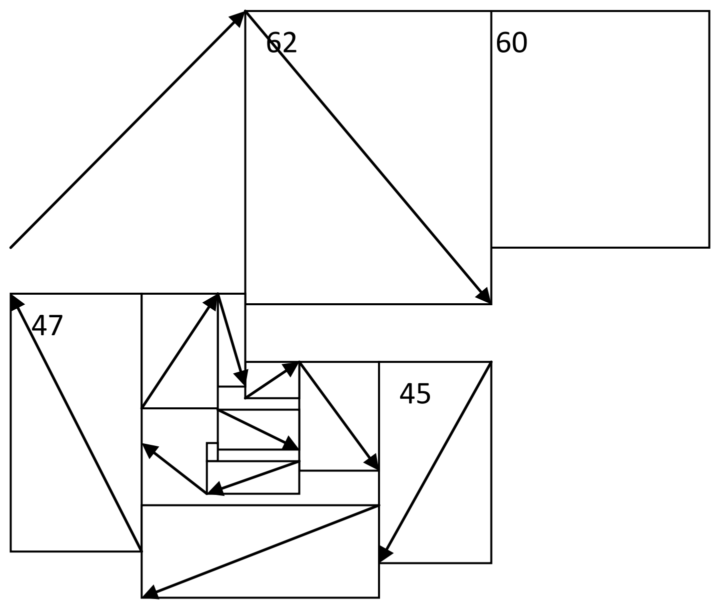
1	x + 0y	9	0x + 5y	17	24x + 47y	25		33	
2	0x + y Trail start	10	5x + 7y	18	71x + 57y	26		34	
3	x + y	11	12x + 8y	19	128x + 55y	27		35	
4	2x + y	12	20x + 7y	20	-183 – 33y	28		36	
5	3x + y	13	27x + 3y	21		29		37	
6	4x + 0y	14	-30x + 5y	22		30		38	
7	4x – y	15	-25x + 17y	23		31		39	
8	-3x + 2y	16	-8x +32y	24		32		40	

(NB Where Elements calculate to a negative value, the shape can be adjusted to ensure positive values).

S32. ACTUAL RECTANGLES FOUND FROM CONTINUOUS X AND Y TRAIL SEQUENCE

Solutions are found for Order 7, 13, 19, 25 etc in increments of 6 and the first few are as follows
[7] 24 x 21 which reduces to [7[8 x 7 and is Invalid. Shown below -
[13] 610 x 535 which reduced to [13] 122 x 10 as below -
[19]





S33. CONTINUATION OF THE TRAIL LINE

From the 1st solution, the Trail Line naturally extends in two directions until it reaches two corners – the NE & NW Corners. A second trail goes from SE to SW Corners – 4 Elements long.
From the 2nd solution, the Trail Line extends in two directions reaching the NE & NW Corners. A second trail goes from SE to SW Corners – 6 Elements long.

S34. QUANTITY OF ODD ELEMENTS WITHIN SQUARED-SQUARES

This subject is linked to Trails, and to understand this we need to consider the various Codes of Corner & Circuit Trails.
First look at Squares which have reduced sizes of Odd times Odd, and are made up of two Corner Trails only.
Take C17,27 cases (Trail NW to SE having 17 Element strikes and NE to SW, 27, the Order being (17 + 27) div,2 = 19). There are 17 examples each found to have 13 ODDS. C13,33 C15,29 & C17,25 cases are all found to have 13 ODDS too, Does each Code therefore have a fixed number of ODDS. It seems so but no!
For example, C17,29 give four with 13 ODDS & one with 9 ODDS ([23] 177 x 177d)!
THE QUANTITY OF ODDS IS NOT FIXED FOR ANY GIVEN CODE, though often the same values are repeated many times.
In testing many Codes the Writer found the number of ODDS to be 9, 13 or 17 in number.
In fact it is found that the series is (for Odd Numbers) 5, 9, 13, 17, 21, 25 … which means that no Squared-Square ever has 7, 11, 15….etc ODD Elements!

We now consider Squares with Even x Even Solutions where the quantity of Odd Elements is EVEN, and Circuit Trails always apply.
Ax,y,z* or Bx,y,z* Cases always give EVEN x EVEN Solutions and an EVEN number of ODD Elements.
Cx,y,z* Cases sometimes give EVEN x EVEN Solutions but often are ODD x ODD.
In these cases the number of ODDS found were 8, 12 & 16.
The Series for Even Numbers is 4,8,12,16,20,24 …. Which means no Squared-Square ever has 2 or 6 or 10 or 14…. ODD Elements!

- ***zz here is the total number of strikes in all Circuit Trails, and not in just one Circuit Trail.***

S34.1. LIMIT TO QUANTITY OF ODD ELEMENT IN Cxx,yy CODES – ONLY APPLIES TO SQUARED-SQUARES

In these either the x or y series on their own reveal all ODD Elements which exist. If x < y then the maximum number of Odds we might expect is x. But it will be less where some Elements are crossed twice, and so are Even Elements.
Codes C01,y, C03,y are not possible but C05,y is. In C05,y the maximum number of Odds might be 5, but could there be 3 ODDS? No! The series is 5,9,13 etc. So C05,yy cases must have 5 ODD Elements.
What about C07,y? There cannot be 7 ODD Elements. In fact Coding C07,yy does NOT EXIST! C07,yy,zz does however.
So, C09,y can have 9 or 5 Odd Elements, C11,y can also have 9 or 5 Odd Elements (not 11). C13,y can have 13 or 9 Odd Elements.
So the lower number limits the maximum quantity of Odd Elements.

S34.2. CODE C05,yy

It is found that in the series of five Elements that the 2nd & 4th are always the same, for example in [9] 15 x 11 the numbers are 5 1 3 1 5.
This means that any such C05,yy solution (square or rectangle) is always IMPERFECT.

S35. INFO FOUND FROM BLIND SR PATTERNS

In any given random pattern –

1. Trail Lines can be drawn, and counted. If there are more than two (i.e. Circuit Trails) the Reduction Index is Even otherwise, Odd.
2. xy Sectors can be drawn, and counted. Sometimes there are single Elements bordering two Sectors – called Shared Elements. Sometimes there are Single Elements remaining – called Extra Elements.
3. Usually but possibly not always it is possible to find Elements which can be the Unknowns x and y, and any further z a b c…. all occur within one Single Trail Line. That Line may be a Corner or a Circuit Trail. (NB in some cases which this appears untrue, a different choice of x y* etc will all fall on a Single Trail Line).
4. Within any single Trail all Elements crossed twice "X" are Even values.
5. Where only Corner Trails apply, 3. Is always true and from one of the Trails each Element crossed once only (/ or \) is Odd in value. All others [X] or [ ] are Even Value.
6. In some cases (e.g. 5. Above) it is possible to establish whether the Reduced Dimensions are Odd x Odd, Even x Odd or Odd x Even.

Generally the more Trails that exist, the less info one is likely to find from Blind Patterns.

- Choices of x y z etc for calculating by Algebra are by no means fixed, and this complicates the issue.
- Determining which Elements could be calculated from x and y only will fill One Sector. Choosing a possible position for z will result in one or more Sectors being filled, and so on using a b c.. until the whole rectangle is filled. The Sectors bordered by x y, x, a etc are often fewer than the original Sectors.

TO BE CONTINUED

CODING z a & b always even val. Applies to Rectangles R or Squares S.	x y even or odd	Red'd Dims format	RI is always	Range of possible Reductions	Full Dims Format	Min. Values of x y	Min. Values z a b	Qty of Odd Elements "QO"	Max Qty of Odd Elements "QO"	Sqd Index is
R- Ax,y	E	E x O	Odd	1,3,5,7..odd	E x O	Min 4	Na	4,6,8,10…	(Smaller	n/a
R- Bx,y	E	O x E	Odd	1,3,5,7,,odd	O x E	Min 4	Na	4,6,8,10…	(Value Of	n/a
RS Cx,y	O	O x O	Odd	1,3,5,7..odd	O x O	Min 5	Na	5,7,9,11…	(x & y	1,3,5,7..odd
RS Ax,y,z	E	E x O	Even	2,4,6,8..even	E x E	Min 4	8	4,6,8,10…	{Not	1,3,5,7..odd
RS Ax,y,z	E	E x E	Even	2,4,6,8..even	E x E	Min 4	8	4,6,8,10…	{Above	1,3,5,7..odd
RS Bx,y,z	E	O x E	Even	2,4,6,8..even	E x E	Min 4	8	4,6,8,10…	{The	1,3,5,7..odd
RS Bx,y,z	E	E x E	Even	2,4,6,8..even	E x E	Min 4	8	4,6,8,10…	{Order	1,3,5,7..odd
RS Cx,y,z	O	O x O	Even	2,4,6,8..even	E x E	Min 5	6	5,7,9,11…	{Less	2,4,6,8,10..
RS Cx,y,z	O	E x E	Even	2,4,6,8..even	E x E	Min 5	6	5,7,9,11…	{four	2,4,6,8,10..

RS Ax,y,z, a	E	E x E	Even	4,8,12..even	E x E	Min 4	6	4,6,8,10…	{Not	2,4,6,8,10..
RS Bx,y,z, a	E	E x E	Even	4,8,12..even	E x E	Min 4	6	4,6,8,10…	{Above	2,4,6,8,10..
RS Cx,y,z, a	O	E x E	Even	4,8,12..even	E x E	Min 5	6	5,7,9,11…	{The	4,8,12,16..
RS Cx,y,z, a	O	O x O	Even	4,8,12..even	E x E	Min 5	6	5,7,9,11…	{Order	4.8.12,16..
RS Ax,y,z, a, b	E		Even	8,16,24..even	E x E	Min 4	6	4,6,8,10…	{Less	4,8,12,16..
RS Bx,y,z, a, b	E		Even	8,16,24..even	E x E	Min 4	6	4,6,8,10…	{four	4.8.12,16..
RS Cx,y,z, a, b	O	E x E	Even	8,16,24..even	E x E	Min 4	6	5,7,9,11…	" "	8,16,24,32..
RS Cx,y,z, a, b	O	O xO	Even	8,16,24..even	E x E	Min 4	6	5,7,9,11…	" "	8,16,24,32..

T. GRID PATTERNS

GENERAL

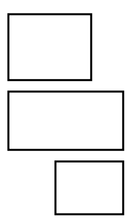
The following study is made through frivolous and possibly not terribly logical.

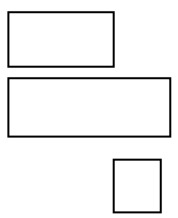
T1. FIXED PATTERN FOR GRID

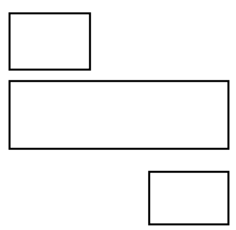
Although many types of GRID Systems can be proposed the best types are two – namely -
For Invalid and or Adjacent Solutions the larger format which may show rows or column 0 apart. And,
For Valid Solutions the smaller format is preferable. Where SLIDES and/or CROSSES exist these show as single vertical and/or horizontal line.

1. The Fixed Pattern is created strictly as follows using | for vertical lines and __ for horizontal lines. Each Element value must be preceded by vertical line "|" e.g. " |47" not " | 47".
2. Each vertical line must be drawn strictly in numerical order distance from the left. The numerical distance of one Vertical Line must not be less than the one to its right. (e.g. A vertical line of distance 23 must appear left of one of 27). Any errors found must be adjusted. Each Pattern requires careful checking. It is easy to get it wrong !

Consider the three diagrams below with the middle Element of say, size 10







a. In the first 2 10 and 3 would be wrongly placed, but 7 10 and 6 would be correct.
b. In the second 4 10 and 6 would be correctly placed (with a vertical slide occurring). But wrong if the top & bottom Elements do not total 10
c. In the third 2 10 and 7 would be correctly placed but 4 10 and 6 or 6 10 and 7 would not.

3. The intention is to keep the Grid pattern the smallest possible whilst retaining rule 2 above. This means keeping the horizontal distances the smallest possible whilst keeping numerical Distances from the left in strict order.
***4.* Where Elements could show in various vertical positions, use the left-most position (unless this infringes Rule 2).**
5. ":" is shown where vertical SLIDE (Slide Lines) occurs and "…" is shown where horizontal SLIDE occurs.
CROSSES indicated by #. The parts of the lines must be merged.
6. In Computer script lines can be conveniently omitted at the top bottom & left, but best put in at the right. (As "|"). Other borders can easily be added in bulk to many solutions.

T1.1. COMPACTED GRID USEFUL FOR SHOWING SOLUTIONS IN COMPUTER DATABASES OR SPREADSHEETS.

Although it is possible to show pictures of Solutions (drawn to scale) in Computer programs the memory usage is each can be many thousands of bytes!

The Compacted Grid System is far more economic in memory – any Squared-Square from Order 13 to 24 inclusive needs less than 400 bytes being one byte per typed unit plus one for each Carriage Return (end of each line). Some of these use under 200 bytes! Also the Solution can be shown a fraction of the size of the scale drawings whilst being much easier to read!

T2. WIDE VARIATION IN SIZES OF GRIDS

Despite two Solutions being Similar in Order & Dimensions there is often considerable variation in the Size of the Cramped Grid.

END OF BOOK
