$12 = 3 \cdot 2 + 2 \cdot 3$ is all).

What Next?

It is typical of this business that one answer leads to n more questions. For example, how many fault-free tilings does a rectangle have? What if we can use two sizes of tiles instead of just one? What about these same questions in 3 (or more) dimensions? What if we require each fault-line to be broken by at least 2 tiles? At least n? We have reached the frontier of our current knowledge on this topic. We encourage the interested reader to explore this fascinating byway of geometry and discover for himself the gems which must surely lie waiting to be discovered.

* Another more general question is: How many different fault-free tilings does a p by q rectangle have? We don't confront this question here.

Dissections into Equilateral Triangles

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I am delighted to have the opportunity to contribute to this Collection honouring Martin Gardner. I once wrote a paper for his column in *Scientific American*, about dissections of rectangles into squares [7]. Perhaps another article on dissections would be appropriate here.

The afore-mentioned paper was concerned with "squared rectangles"; that is, dissections of rectangles into squares. A dissection of a rectangle or square into unequal squares is a *perfect* rectangle or square respectively. The paper was based on earlier work of Brooks, Smith, Stone and Tutte published in the *Duke Mathematical Journal* in 1940 [2]. This earlier paper gives an example of a perfect square and describes a method (exploiting symmetrical subgraphs) for constructing others. There is also a note on possible generalizations. One of these is the study of dissections of equilateral triangles into other equilateral triangles. There are two later papers on this generalization by myself, ([6] and [8]), and one by Brooks, Smith, Stone and Tutte [3].

I have no cause to complain about the reception of papers on squared rectangles by mathematicians and other scientists. They are all pleased that there should be a connection between Kirchoff's Laws of electrical flow and the dissection problems of pure geometry. But there seems to be no corresponding interest in the triangular problem. Yet, in my opinion the theory of triangulated triangles is a simple and elegant generalization of those theories of squared rectangles. In the remainder of this paper I will try to justify the preceding statement, in the hope of awakening the interest that I feel these dissections deserve.

When we are discussing dissections into equilateral triangles it does not really matter whether the original figure is a triangle or a parallelogram (with angles of 60° and 120°). A dissection of a triangle can be changed into that of a parallelogram by deleting one constituent triangle (at a vertex) and adding another. A dissection of a parallelogram can be changed into that of a triangle by adding new constituent triangles on two adjacent sides.

It is the dissected parallelogram that is the natural analogue of the squared rectangle. Suppose we start with a squared rectangle R and shear it so that it becomes a dissection of a parallelogram into rhombuses. Each rhombus can be dissected along an appropriate diagonal to form two congruent equilateral triangles. The squared rectangle R is thus transformed into a triangulated parallelogram P. Because of this observation we can regard the theory of squared rectangles as a special case of the theory of triangulated parallelograms.

Suppose that R is a perfect rectangle. Does it make sense to say that P is perfect also? Given any triangulated triangle or parallelogram we can fix one of its sides as "horizontal". Then any constituent triangle T has one horizontal side. Let us say that T is positively or negatively oriented according to its position above or below its horizontal side. Let us define the "size" of T as plus or minus the length of its side, depending on whether T is positively or negatively oriented. Then we say that a triangulation is "perfect" if no two of its constituent triangles have the same size. With this definition we can say that the perfect rectangle R shears into a perfect parallelogram P. For the two congruent triangles of P, arising from any square of R, have opposite orientations and therefore have different sizes.

Figure 1 shows a perfect parallelogram that cannot be obtained by shearing a perfect rectangle. In [6] I have satisfied myself that this parallelogram has the least number of constituent triangles, namely 13, consistent with perfection. For the sake of comparisons with squared rectangles, I like to say that its order is 13/2. For each square of a rectangle corresponds when sheared to two triangles. The number entered in each triangle of Figure 1 is its size.

At this stage in papers on squared rectangles, there appears a diagram showing the relation between a squared rectangle and its electrical network. Figure 1 fulfills the same purpose in the theory of triangulated parallelograms. But now the term "electrical" must be used in a generalized sense. It refers to a form of "electricity" not yet found in physics, (as

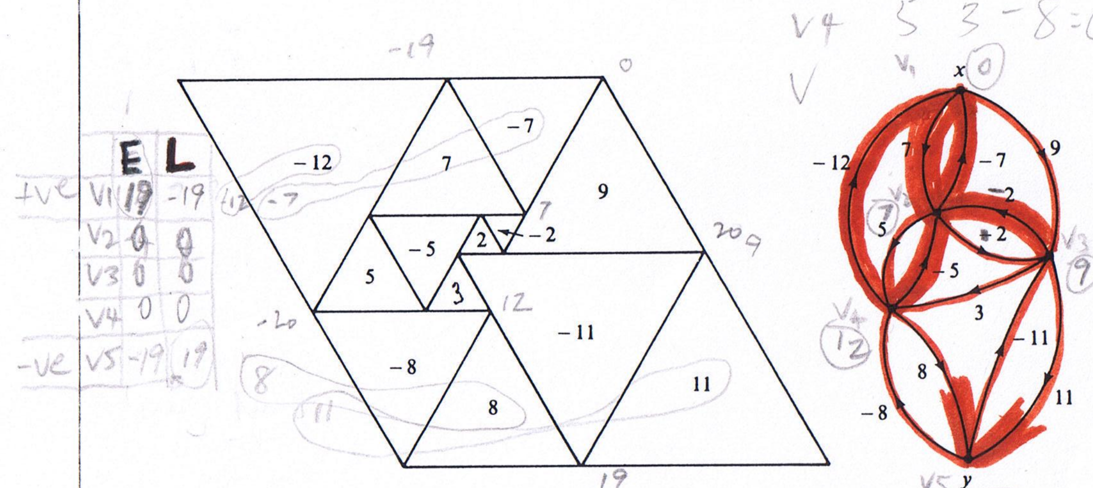

FIGURE 1

The parallelogram P is on the left and the corresponding electrical network is on the right. Each vertex of the network N corresponds to a maximal horizontal segment, made up of sides of constituent triangles in P and is shown on the same level. The network is directed. Each of its darts, or directed edges, corresponds to a constituent triangle of P (the direction of the dart is from the apex to the base of the triangle). The size of each triangle is written against the corresponding dart of N and is called the *current* in that dart. The vertices of N corresponding to the upper and lower horizontal sides of P are called the positive and negative *poles* of N respectively.

We observe that N is a balanced digraph. (*Digraph* is short for *directed graph.*) *Balanced* means that at each vertex the number of incoming darts is equal to the number of outgoing ones. It is not difficult to see that N is planar (for any P), and that it can be drawn in the plane so that the order of the darts around each vertex is the order of the corresponding triangles around the horizontal segment of P. In such a drawing, incoming and outgoing darts must alternate around each vertex. We describe a plane figure with this property as an *alternating map*.

At each vertex v of N, the sum of the currents in the darts directed to v is called the *current leaving N at v*. Its negative is the *current entering N at v*. Thus, the current entering the positive pole is equal to the current leaving the negative pole, and each of these numbers measures the length of a horizontal side of P. But the current entering or leaving N at any non-polar vertex is zero, and the current leaving is the sum of the sizes of the triangles of P having their bases in the corresponding horizontal segment. This rule of currents at non-polar vertices is our analogue of Kirchoff's First Law.

Transferring this number to N we call it the *potential* of the corresponding vertex. Now the vertices of N incident with a dart D are called the *head* and *tail* of D, with the direction of the dart being from tail to head. We can assert the usual form of Kirchhoff's Second Law, as follows. The total fall of potential around any circuit C of N, regardless of the directions of the darts, is zero. We note that the current in a dart D is equal to the fall of potential from tail to head. We can thus apply the Second Law by saying that the sum of the currents in any circuit C is zero, provided that we agree that currents in darts going against the chosen direction of the circuit are to be replaced by their negatives.

The modified Kirchhoff Laws can be used here like the conventional ones in the theory of squared rectangles. Having drawn an alternating map M, with at least four darts at each vertex, we can choose positive and negative poles, incident with a common face. We can solve the equations for a set of currents. We can then reverse the construction of Figure 1 to get a corresponding triangulated parallelogram. With luck this will be perfect. The complete theoretical justification of this procedure may be difficult, but the principle is clear.

It is now possible to deduce properties of triangulations from those of alternating maps. Thus, we can infer from the Euler Polyhedron Formula that the alternating map corresponding to a triangulated parallelogram must have either a 2-sided face or a non-polar vertex incident with only four darts. In either case two of the constituent triangles of the parallelogram must be congruent. This result leads to the assertion in [2], couched in a different terminology, that *there is no perfect equilateral triangle*.

Consider a squared rectangle R with its ordinary electrical network G. Let it be sheared to form a triangulated parallelogram P, and let the corresponding N be derived. We find that N can be obtained from G by replacing each edge with two oppositely directed darts. In the corresponding alternating map these two darts bound a digon. The modified Kirchhoff Laws for N are easily seen to be equivalent to the ordinary ones for G, on the assumption of unit resistances. The squared rectangle, then, is still a special case in the triangular theory.

Precedents set by the studies of squared rectangles would lead us to expect that someone, at this stage, would have constructed extensive catalogues of triangulated parallelograms. Alas, it is not so.

A detailed discussion of the Kirchhoff equations for an undirected graph G would be out of place here. Let us note that the theory can be based on the *Kirchhoff matrix* $K(G)$ of the graph concerned. We suppose the vertices of G to be enumerated as $v_1, v_2, \ldots, v_n$. Then $K(G)$ has n rows and n columns. Its j^{th} diagonal entry is the number of edges joining v_j to other vertices. The non-diagonal entry in the i^{th} row and j^{th} column is minus the number of edges joining v_i to v_j. Thus, $K(G)$ is a symmetrical matrix and its elements sum to zero in each row and column.

The Matrix-Tree Theorem asserts that if we strike out from $K(G)$ the j^{th} row and column then the determinant of the resulting matrix $K_j(G)$ is the number $T(G)$ of spanning trees of G. Let us agree that the resistance of each edge is to be unity, and let us choose two vertices as positive and negative *poles*. It is convenient to take $T(G)$ as the magnitude of the current entering G at the positive pole and leaving at the negative, for then the currents in the various edges are integers. These currents constitute the *full flow* in G with respect to the chosen poles. The potential differences in this flow can be expressed as the determinants of certain submatrices of $K(G)$, appropriately multiplied by $+1$ or -1. In particular, the potential difference between the poles is the determinant of the submatrix obtained from $K(G)$ by striking out the two rows and two columns corresponding to the poles.

All this generalizes very nicely to a theory of unsymmetrical electricity in directed graphs. Let G be any directed graph, not necessarily balanced. We define $K(G)$ much as before. The j^{th} diagonal entry is the number of darts directed to v_j from other vertices. The non-diagonal entry in the i^{th} row and j^{th} column is minus the number of darts directed from v_i to v_j. We note that this $K(G)$ is not necessarily symmetrical. Its elements sum to zero in the rows but not necessarily in the columns.

We can define $K_j(G)$ as before, and its determinant is found to be a number of spanning trees of G. But not all the spanning trees of G are to be counted—only those in which each edge is directed away from v_j. We call these structures the *out-arborescences* of G on v_j. A spanning tree of G in which each edge is directed towards v_j is an *in-arborescence* of G on v_j. (In the undirected case, the value of det $K_j(G)$ is independent of j, but in the directed case this is not necessarily true.)

Let us choose a positive pole v_p and a negative pole v_q, and use the modified Kirchhoff Laws to calculate a corresponding distribution of currents. The current entering at v_p is not necessarily equal to the current

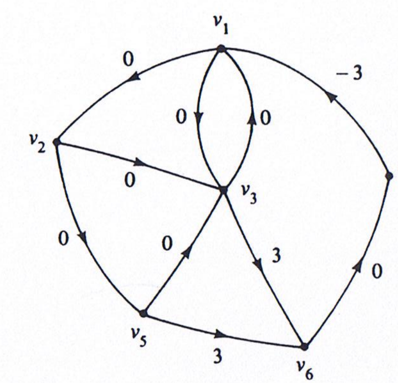

FIGURE 2

leaving at v_q. However, we can find a full flow in which current det $K_q(G)$ enters at v_p and current det $K_p(G)$ leaves v_q. Determinants of appropriate submatrices of $K(G)$ fix the currents and potential differences in this full flow. These currents and potential differences are found to be integers. The rule for the potential difference between the poles is the same as before.

As an example, let us take the graph of Figure 2, with v_1 as positive pole and v_6 as negative. The currents of the corresponding full flow are indicated.

We easily verify the matrix $K(G)$ to be as follows.

$$\begin{bmatrix} 2 & 0 & -1 & -1 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 3 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & -1 \\ 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 2 \end{bmatrix}$$

Let us write $T_j(G)$ for the number of out-arborescences of G on v_j. Either by inspection of Figure 2 or by evaluating det $K_1(G)$ and det $K_6(G)$ we find that $T_1(G) = 6$ and $T_6(G) = 3$. Hence the flow depicted in Figure 2 is full, for a current of 3 enters at v_1 and a current of 6 leaves at v_6. The fact that the currents in the darts from v_1 to v_2 and from v_2 to v_5 are zero, is an immediate consequence of the modified First Law.

What happens if we reverse the direction of every edge, to get a new digraph G'? As far as the non-diagonal elements are concerned we replace $K(G)$ by its transpose. But diagonal elements may change. If G is the graph of Figure 2 then $K(G')$ is the following matrix.

$$\begin{bmatrix} 2 & -1 & -1 & 0 & 0 & 0 \\ 0 & 2 & -1 & 0 & -1 & 0 \\ -1 & 0 & 2 & 0 & 0 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 2 & -1 \\ 0 & 0 & 0 & -1 & 0 & 1 \end{bmatrix}$$

The digraph G' is shown in Figure 3, again with a full flow.

We have $T_1(G') = 8$ and $T_6(G') = 9$. There is an evident 1 – 1 correspondence between the out-arborescences of G' on v_j and the in-arborescences of G on v_j.

When we restrict our theory to balanced digraphs we avoid many complications. For if N is a balanced digraph, the elements of $K(N)$ sum to zero in the columns as well as in the rows. Hence we can show, by elementary determinant theory, that det $K_j(N)$ is independent of j. The

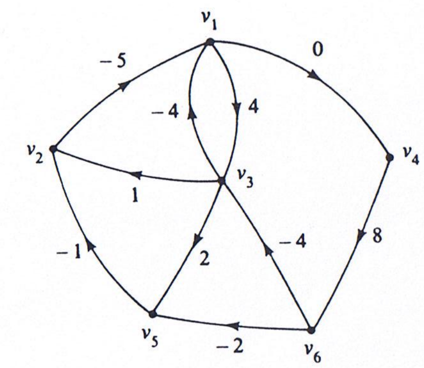

FIGURE 3

We therefore refer to this number simply as the tree-number $T(N)$ of N. As an example we can take the balanced digraph of Figure 1. If the vertices are numbered from above to below in the diagram, then

$$K(N) = \begin{bmatrix} 2 & -1 & 0 & -1 & 0 \\ -1 & 3 & -1 & -1 & 0 \\ -1 & -1 & 3 & 0 & -1 \\ 0 & -1 & -1 & 3 & -1 \\ 0 & 0 & -1 & -1 & 2 \end{bmatrix}.$$

It follows that in the full flow, the fall of potential between the poles is

$$\begin{vmatrix} 3 & -1 & -1 \\ -1 & 3 & 0 \\ -1 & -1 & 3 \end{vmatrix} = 20.$$

We deduce that the flow shown in Figure 1 is full. Accordingly $T(N)$, being equal to the current entering at the positive pole, is 19.

Given a balanced digraph N we can reverse all its darts to obtain a digraph N'. Clearly N' is balanced. Moreover if N defines an alternating map in the plane, then so does N'. But now $K(N')$ is simply the transpose of $K(N)$. Hence $T(N') = T(N)$. Hence, on each vertex of N the numbers of in-arborescences and out-arborescences are equal.

Let two vertices be chosen to be positive and negative poles in both N and N'. Consider the two full flows. The current entering at the positive pole is the same in each, by equality of tree-numbers. Moreover the fall of potential between the two poles is the same in each. For it is given in N by the determinant of a submatrix symmetrically related to the main diagonal, and in N' by the determinant of the transpose of this matrix.

If N corresponds to a triangulated parallelogram, then so does N'. The parallelograms, P and P' respectively, have the same size and shape, as a result of the preceding paragraph. Unfortunately this effect is not wellillustrated by the digraph of Figure 1. Reversing its darts merely gives an isomorphic digraph; moreover each pole is invariant under the isomorphism. If N corresponds to a squared rectangle R, then P and P' correspond to two ways of slicing R. In shearings, we can move the upper horizontal side either to the left or to the right with respect to the lower one.

On page 478 of [6] we are shown two perfect triangulated parallelograms which are the P and P' corresponding to an alternating map N. Each parallelogram has horizontal side 3441 and slanting side 2999. Each is dissected into 36 constituent triangles, which are all different sizes. Each has two congruent constituent triangles of side 129. But apart from this, there is no repetition of triangle-size between the two dissections.

The leaky or asymmetrical electricity relevant to the theory of triangulated parallelograms is discussed in more detail in [3], [6] and [9]. In the most general theory, the conductance (reciprocal of resistance) of a dart is not restricted to 1, but is an indeterminate over the integers.

A squared rectangle R has sides in two perpendicular directions, and either of these directions can be taken as horizontal. The two choices lead to two electrical networks G and G_1. These, however, are simply related. If each is completed by the adjunction of a new edge joining the poles then the two completed networks, or *c-nets*, are plane duals. Since the two corresponding systems of electrical equations give rise to the same pattern of squares, it seems reasonable to expect the determinants det $K_i(G)$ and det $K_j(G_1)$ to be equal. This is known to be the case; it can be shown that dual-plane-connected graphs have equal tree-numbers.

Is there anything corresponding to a pair of dual *c*-nets in the theory of triangular dissections? There is indeed, but to exhibit it we must pass from triangulated parallelograms to triangulated triangles. As an example we can adjoin two new constituent triangles to the triangulated parallelogram P of Figure 1 to obtain the triangulated triangle of Figure 4.

The corresponding modification of the electrical network of Figure 1 is obvious. To represent the new triangle of size 20 we need a new dart on

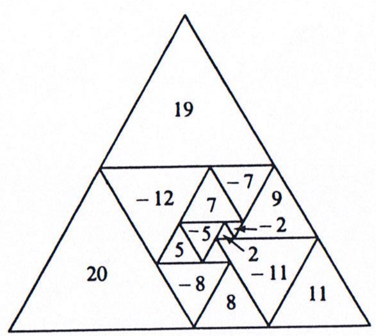

FIGURE 4

the left side, directed from the old positive pole X to the old negative pole Y. For the triangle of size 19 we need a new vertex Z and a new dart from Z to X. We can say that Z represents the apex of the triangulated triangle, even though this apex is not strictly a horizontal segment.

The currents in the new electrical network obey the modified Kirchhoff Laws with Z as positive pole and Y as negative pole. The number of out-arborescences from Z can be shown to be 39; the side-length of the dissected triangle. The number of out-arborescences from Y (or from any vertex but Z) is obviously zero. This corresponds to the fact that the current entering the network at Z is zero.

It is, in some ways, convenient to make Z coincide with Y in the electrical diagram, so that the diagram exhibits a new alternating map M. The dart originally incident with Z can be called the *polar dart*, and it can be distinguished in the diagram by a cross-bar. Figure 5 shows an electrical diagram of this kind corresponding to the triangulation of Figure 4. The vertices correspond to the maximal horizontal segments of the triangulation.

Figures 4 and 5 exhibit one example of a general procedure. Given any alternating map M we can mark in it a polar dart D, and we may then expect to derive a corresponding triangulated triangle. To form the corresponding electrical network we detach D from its tail-vertex Y and give it a new tail Z incident with no other dart. After this, Z is taken as the positive pole, and Y is taken as the negative pole. The calculated currents are interpreted as triangle-sizes. (Sometimes we find a zero current, and then complications arise.)

Returning to Figure 4 we observe that the sides of its constituent triangles make up three families of parallel segments, and that any one of these families could be chosen as horizontal. We might, for example, consider maximal segments parallel to the left slanting side of the dissected triangle, as shown in Figure 4, and make these segments correspond to the vertices of another electrical network. This is shown, with a marked polar

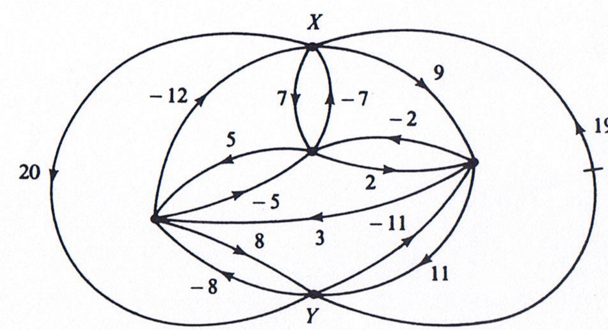

FIGURE 5

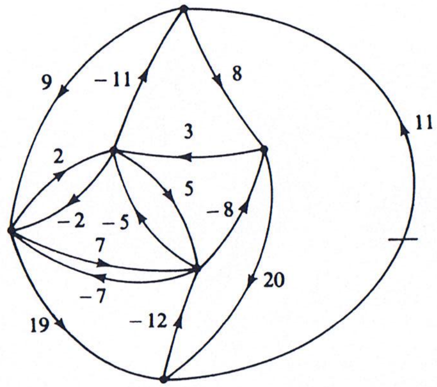

FIGURE 6

edge, in Figure 6. The remaining family of maximal segments, corresponding to the right slanting side, gives rise to the electrical network of Figure 7.

In Figures 5, 6 and 7 we find three different electrical networks, all carrying the same set of currents. Such triads of alternating maps can be regarded as analogues of the pairs of dual maps in the theory of squared rectangles. Our analogue of duality is called *triality* in [6] and *trinity* in [8]. The three maps making up a triad are said in [8] to be *trine* to one another.

Given any alternating map we can define its trine maps directly, without constructing a related triangulated triangle. It is shown in [8] how a triad M_1, M_2, M_3 of alternating maps can be derived from a bicubic map M, and how the triad associated with any triangulated triangle has its corresponding M.

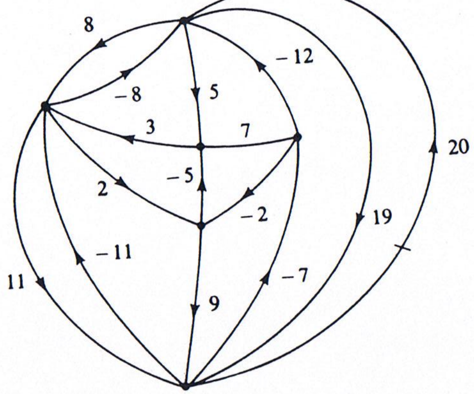

FIGURE 7

In a bicubic map, the vertices can be distinguished as black an white, each edge joining a white vertex to a black one. Moreover, the face can be 3-coloured (say in red, green and blue) so that each edge separate faces of two different colours. Apart from permutations of the three colours the 3-colouring is unique. We can form an alternating map M_1 of th corresponding triad as follows. We give it one vertex in each red face of M We form its darts from the edges of M, separating green and blue faces Each of these is directed from its black end to its white, and each i extended at each end through the adjacent red face of M to the vertex o M_1 contained in that face. The other members of the triad are define similarly, each with a permutation of the three face-colours. (In [8] th definition of a triad is given, not directly in terms of the bicubic map M bu in terms of its dual map. This dual is an Eulerian triangulation of th sphere.)

Suppose M_1 is derived from an undirected planar map N b replacing each edge by two oppositely directed darts. It is then found tha one other member of the triad (say M_2) is similarly derived from the dua map of N. The third member of the triad is an oriented form of the media map of N. This observation justifies our assertion that trinity is a tru generalization of duality.

Since dual plane maps have equal tree-numbers we may expect trin alternating maps to have equal tree-numbers too, and this has been prove to be the case. (Proofs are given in [6] and [8]. A simpler proof wa discovered recently by K. A. Berman [1].)

The common tree-number of three trine alternating maps M_1, M and M_3 can be regarded as a property of the associated bicubic map M. I [3] it is interpreted directly in terms of the structure of M, as a count o matchings, not trees.

It seems appropriate to conclude the exposition here. Any read interested in further study can proceed to the papers on our list o references. However, a warning should be given regarding one min point. In a triangular dissection it may happen that six constitue triangles meet at a single point. Such a point is called a *cross*. It correspon to a cross in the theory of squared rectangles; that is, to a point at whic four constituent squares meet. In neither theory is it usual to encounter cross in an interesting example unless it has been forced to occur by som symmetry in the construction. But when crosses do occur, it is necessary t introduce new conventions so that we may still have pairs of dual c-nets o triads of trine-alternating maps. In short we subdivide some of the maxima segments at the crosses. It is the resulting subsegments that correspond t the vertices of the associated electrical networks. Each cross, in th triangular case, is the intersection of three maximal segments. The rule that exactly two of them (the choice is arbitrary) are to be subdivide there.

Addendum

Writing the above revived my interest in triangulated parallelograms, and I set out to calculate one or two more. I calculated two, and they seem to be sufficiently interesting to record here. They are perfect triangulations—each of the order $11\frac{1}{2}$. Each has horizontal side 401 and slanting side 264. There is no case of a constituent triangle in one being congruent to a constituent triangle in the other. This is the only case known to me of two perfect triangulations of the same parallelogram with no constituent size repeated from one dissection to the other.

The electrical networks of the two parallelograms are related by a reversal of darts. These networks, with their currents, are shown in Figures 8 and 9. I have verified that the flows shown are full. In each case X is the

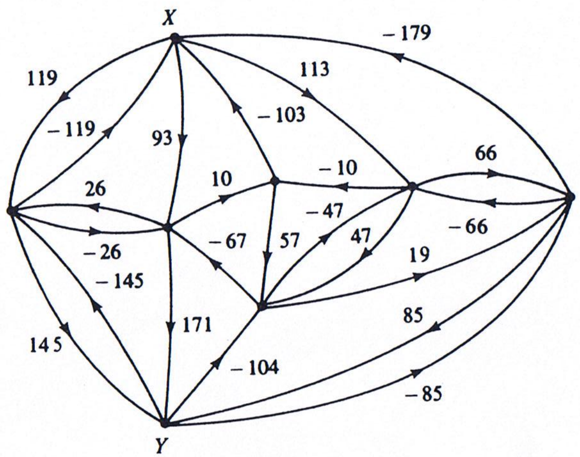

FIGURE 8

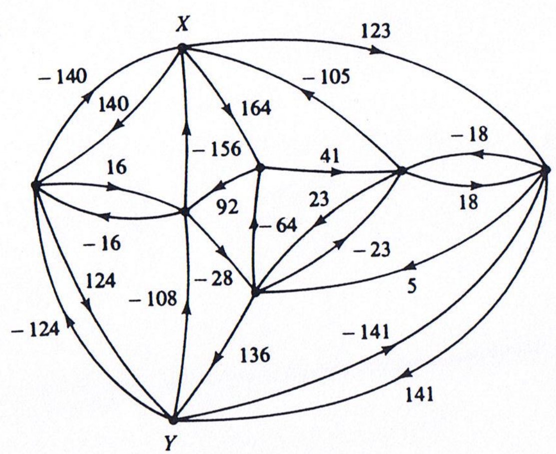

FIGURE 9

positive pole and Y is the negative pole. A current of 401 enters at X and leaves at Y, and the fall of potential from X to Y is 264. The reader should by now have no difficulty in deriving the actual triangulations.

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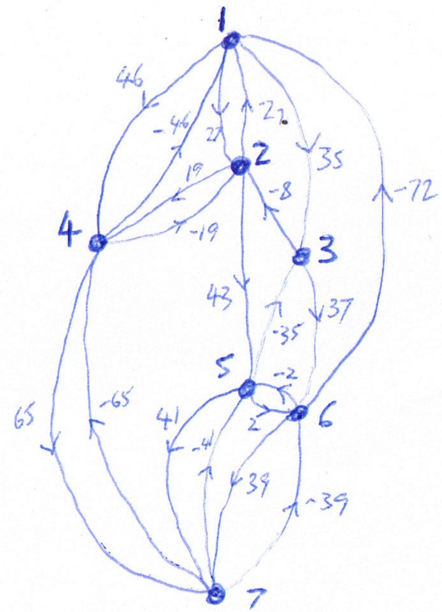

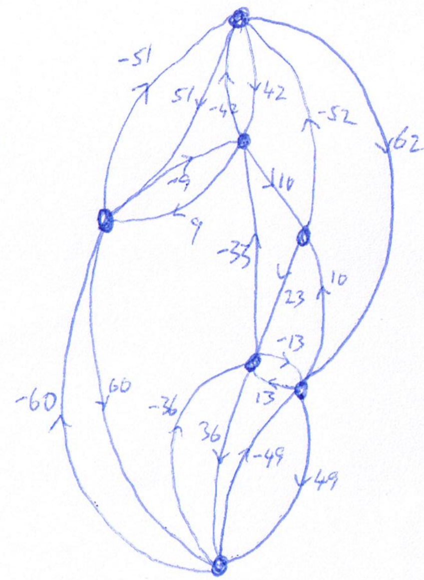


```
KirchhoffMatrix =
     1  2  3  4  5  6  7
1 {{ 3,-1, 0,-1, 0,-1, 0},
2  {-1, 3,-1,-1, 0, 0, 0},
3  {-1, 0, 2, 0,-1, 0, 0},
4  {-1,-1, 0, 3, 0, 0,-1},
5  { 0,-1, 0, 0, 3,-1,-1},
6  { 0, 0,-1, 0,-1, 3,-1},
7  { 0, 0, 0,-1,-1,-1, 3}};

CVMatrix =

{{ 1,-1, 1,-1, 1, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 { 0, 0,-1, 1, 0, 0, 1,-1, 1,-1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0},
 { 0, 0, 0, 0,-1, 0, 0, 0, 0, 1, 0, 0,-1, 1, 0, 0, 0, 0, 0, 0},
 {-1, 1, 0, 0, 0, 0,-1, 1, 0, 0, 1,-1, 0, 0, 0, 0, 0, 0, 0, 0},
 { 0, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0, 0, 1, 0, 1,-1, 1,-1, 0, 0},
 { 0, 0, 0, 0, 0,-1, 0, 0, 0, 0, 0, 0, 0,-1, 0, 0,-1, 1, 1,-1},
 { 0, 0, 0, 0, 0, 0, 0, 0, 0, 0,-1, 1, 0, 0,-1, 1, 0, 0,-1, 1}};
KRowSize = Part[ Dimensions[KirchhoffMatrix],1];
ReducedKirchhoffMatrix = KirchhoffMatrix[[ Range[1,KRowSize-1],Range[1,KRowSize-1] ]];
CVRowSize = Part[ Dimensions[CVMatrix],1];
CVColSize = Part[ Dimensions[CVMatrix],2];
ReducedCVMatrix = CVMatrix[[ Range[1,CVRowSize-1],Range[1,CVColSize] ]];
Complexity = Det[ ReducedKirchhoffMatrix]
FullCurrents = Complexity Inverse[ ReducedKirchhoffMatrix].ReducedCVMatrix;
ReductionFactors = Apply[ GCD, FullCurrents, {1}]
ReducedSides = FullCurrents / ReductionFactors
```

```
145

{1, 1, 1, 5, 1, 1}

{{51, -51, 42, -42, 52, 62, 9, -9, 33, -10, 60, -60, -23, 10, 36, -36, -13, 13, 49, -49},
```

Courtesy J. A. Bond

W. T. Tutte