Tilings

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Introduction. We will discuss tiling of various plane regions by rectangles and squares and also briefly discuss what happens in higher dimensions. Our analysis follows many of the ideas introduced in 1940 in [1]. We have updated and expanded these methods to use more modern techniques. First we need to start with a definition.

Definition 0.1. A tiling of a region R is a dissection of R into finitely many connected subsets whose union is R and whose interiors are pairwise disjoint.

1   General resistor networks

We begin with the basic definitions.

Definition 1.1. A *graph with boundary* is a triple $\Gamma = (V, \partial V, E)$, where the members of the set V are called *vertices*, the members of the set ∂V are called *boundary vertices* or *poles*, members of the set $V \setminus \partial V$ are called *interior vertices*, and the members of the set E are called *edges*. We must have $\partial V \subset V$ and E is a subset of the set of unordered pairs of vertices.

Definition 1.2. A *resistor network* is a pair (Γ, γ), where Γ is a graph with boundary and $\gamma : E \to \mathbf{R}^+$ is a positive real valued function on the edges of Γ. γ is the *conductivity function* and if e is an edge, $\gamma(e)$ is called the *conductivity* of e.

Definition 1.3. A function $f : V \to \mathbf{R}^+$, where V is the set of vertices of a network N, is called $\gamma - harmonic$ if for each $p \in V \setminus \partial V$, we have

$$I_p(f) = \sum_{q \in N(p)} \gamma(p,q) \left(f(p) - f(q) \right) = 0$$

Here $N(p)$ is the set of all vertices which are connected to p by an edge, and $\gamma(p,q)$ denotes the conductivity of the edge joining p and q.

Instead of saying f is a function on the vertices of a network N, we will simply say that f is a function on N itself. Notice that this condition can be re-written as

$$f(p) = \frac{\sum_{q \in N(p)} \gamma(p,q) f(q)}{\sum_{q \in N(p)} \gamma(p,q)}$$

This formula can be used to compute the unique γ-harmonic function f with given boundary values. Of course, we need the following well-known

Theorem 1.1. *If* $f : N \to \mathbf{R}$ *is a γ-harmonic function, then f assumes its maximum and minimum on the boundary nodes of N. If f assumes a minimum or a maximum at an interior node, then f is constant.*

Theorem 1.2. *Given any resistor network N, if we let ∂N be the set of boundary vertices, say $\partial N = v_1, \ldots, v_n$, and we are given n real numbers $a_1, \ldots, a_n$, then there exists a unique γ-harmonic function f on N such that $f(v_i) = a_i, \ \forall \ 1 \le i \le n$. This function f is called the solution to the Dirichlet problem on N with boundary data $a_1, \ldots, a_n$.*

For any resistor network N, we can define the Kirchhoff matrix as follows: $K = (\kappa_{i,j})_{1 \le i,j \le M}$ where $M = |V|$. For $i \ne j$ we have $\kappa_{i,j} = 0$ if ij is not an edge, and $\kappa_{i,j} = -\gamma(i,j)$ if ij is an edge. We also have $\kappa_{i,i} = \sum_{j \in N(i)} \gamma(i,j)$.

Some properties of the Kirchhoff matrix for a connected network.

    1. K is symmetric, *i.e.* $K^t = K$.

    2. K is positive semi-definite, *i.e.* given any vector $v \in \mathbf{R}^n$, we have $v^t K v \ge 0$ and $v^t K v = 0 \Leftrightarrow v$ is constant, *i.e.* all its components are the same.

3. Any principal sub-matrix of K is positive definite. A principal sub-matrix is one obtained by deleting the same rows and columns. For example, a matrix obtained by removing the rows 2,3 and 5 and columns 2,3 and 5 would be a principal sub-matrix.

4. If we let $K_{r,s}$ denote the matrix obtained from K by deleting row r and column s, then $(-1)^{r+s} \det(K_{r,s}) > 0$ and this quantity is independent of r and s. This common value is called the *complexity* of the network N and is denoted by C.

Theorem 1.3., the Matrix Tree Theorem   *If the conductivity γ of a network N is identically equal to 1, then the complexity C is equal to the number of spanning trees of the network N.*

Given any γ-harmonic function f on a network N, we define two quantities, the *potential drop* and the *current* as follows. If $e = (p, q)$ is an edge, the potential drop across e due to f is $f(p) - f(q)$. Notice that this quantity depends on the direction we go along the edge. The current across e due to f is given by Kirchhoff's Law, and it is $\gamma(e)(f(p) - f(q))$. Current flows from the node with higher potential toward the node with lower potential.

Number the vertices of the network so that the boundary vertices are vertices 1 through n and the interior vertices are numbered from $n + 1$ through M. Then we can partition K as follows:

$$K = \begin{bmatrix} A & B \\ B^t & C \end{bmatrix}$$

A consists of rows 1 through n and columns 1 through n, B consists of rows 1 through n and columns $n + 1$ through M, B^t consists of rows $n + 1$ through M and columns 1 through n, and C consists of rows $n + 1$ through M and columns $n + 1$ through M.

Notice that Theorem 1.2 guarantees the existence of a unique vector of boundary currents given a vector of boundary potentials. It is clear that the map which takes boundary potentials to boundary currents is linear and therefore can be expressed as a matrix. This matrix is called the Lambda matrix.

Definition 1.4. The *Lambda matrix* Λ of a network N is the matrix which has the following property. Given a boundary potential vector $v = [v_1, \ldots, v_n]^t$, we have $\Lambda v = I$, where I is a vector whose entries are the induced boundary currents.

Theorem 1.3. $\Lambda = A - BC^{-1}B^t$, *where* A, B *and* C *are the matrices obtained by partitioning the Kirchhoff matrix* K *as above.*

Some properties of the Lambda matrix for a connected network

  1. Λ is symmetric.

  2. The row sums of Λ are all 0.

2   The network associated to a tiling of a rectangle by rectangles

Let R be a rectangle of height h and width w, and suppose it is tiled by finitely many rectangles $R_1, \ldots, R_n$ of heights $h_1, \ldots, h_n$ and widths $w_1, \ldots, w_n$ respectively. It is clear that the sides of the rectangles and the sides of R will all be parallel to two perpendicular lines in the plane. Choose one of these lines to be the horizontal axis and the other to be the vertical axis; this gives an orientation of R.

We associate a 2-pole graph Γ to the tiling of R as follows. Associate to each horizontal segment in the tiling a vertex. Connect two vertices if there is a rectangle in the tiling with its top edge on the segment associated to the first vertex and its bottom edge on the segment associate to the other vertex. The two boundary nodes are the vertices corresponding to the top and bottom edges of R itself. We turn Γ into a resistor network by assigning each edge a conductivity equal to the width of the rectangle corresponding to the edge divided by its height. Note that if the rectangle is tiled by squares, all conductivities will be one. Note also that the graph obtained by this process is always planar.

Theorem 2.1. *Let N be the network associated to a tiling. Then the function which associates to each vertex of N the distance of the level corresponding to that vertex from the bottom of R is a γ-harmonic function.*

Proof. We need to compute $I_p(f)$ for each $p \in V \setminus \partial V$, where V is the set of vertices of N. Let $p \in V \setminus \partial V$. Then p corresponds to some horizontal segment s in the interior of R. Let E_1 denote the set of edges with one endpoint being p and the other endpoint being a vertex corresponding to a segment below s; let E_2 be the set of edges with one endpoint being p and the other endpoint being a vertex corresponding to a segment above s. By the definition of f and of γ, we have

$$\sum_{(p,q) \in E_1} \gamma(p,q) \left(f(p) - f(q) \right) = - \sum_{(p,q) \in E_2} \gamma(p,q) \left(f(p) - f(q) \right)$$

This common quantity is, by the definition of γ and of f, the width of the horizontal segment s. Since we have $E_1 \cup E_2 = N(p)$, f satisfies the conditions for a γ-harmonic function set forth in Definition 1.3. $\qquad \square$

The uniqueness of f tells us that *any* γ-harmonic function with boundary values 0 and h will be identically equal to f on all of N. Thus we can compute the levels of the tiling from the network. This is the basis of the method we will use to construct tilings of rectangles given an arbitrary planar 2-pole network.

Theorem 2.2. *If a rectangle is tiled by rectangles all of which are commensurable, i.e. which have sides whose ratios are all rational, then the rectangle itself is commensurable.*

Proof. The sum of the currents flowing out of the boundary node associated to the top of the rectangle is equal to the width w of the rectangle and is also equal to the current flowing into the boundary node associated with the bottom of the rectangle. The the potential drop between the two boundary vertices is equal to the height h of the rectangle. Since all of the rectangles in the tiling are commensurable, the conductivities in the network are all rational. Since there are only two boundary nodes, using the properties of the lambda matrix given above, we see that it will have the form

$$\Lambda = \begin{bmatrix} \lambda_{1,1} & -\lambda_{1,1} \\ -\lambda_{1,1} & \lambda_{1,1} \end{bmatrix}$$

By the properties of Λ, we must have

$$\begin{bmatrix} w \\ -w \end{bmatrix} = \Lambda \begin{bmatrix} h \\ 0 \end{bmatrix}$$

So we have $\lambda_{1,1} = \frac{w}{h}$ and since all conductivities are rational, Λ has all rational entries, so $\frac{w}{h}$ is rational, hence the rectangle is commensurable. $\quad \square$

Corollary 2.3. *If a rectangle is tiled by squares, then the rectangle is commensurable.*

3 Duality for networks

We will introduce the notion of the dual of a circular planar network. We will show how this can be used to construct tilings of rectangles from arbitrary circular planar 2-pole networks.

Definition 3.1. A *circular planar graph* is a graph with boundary which is embedded in a disk in the plane so that all of its boundary vertices lie on the circle bounding the disk. A network is circular planar if the graph associated with it is circular planar.

Note that any circular planar graph must come with an embedding. The embedding is crucial: two identical graphs with different embeddings into the disk are not the same circular planar graph. Suppose N is a circular planar network with boundary vertices $v_1, \ldots, v_n$. We construct the dual network N^* as follows. Say N is embedded in some disk D in the plane. Then N divides D into M regions. Place one dual vertex in each region. If the region is partially bounded by the boundary of the disk, place the vertex in that region on the boundary of the disk, otherwise place the vertex in the interior of the region. So the dual graph also has n boundary vertices and is also circular planar. Now connect the vertices as follows. If two faces are adjacent, connect the vertices corresponding to the vertices through the edge shared by the two faces. So there should be one dual edge for every edge in the original graph, and the dual edges should be perpendicular to the original edges. If e is an edge in the original graph and e^* is its dual edge, then assign conductivity $\gamma^*(e^*) = \frac{1}{\gamma(e)}$ to the dual edge. Thus we have constructed the dual network. The dual network plays a very important role in tilings of rectangles.

Theorem 3.1. *Let R be a rectangle which is tiled by finitely many rectangles $R_1, \ldots, R_n$. Suppose the network associated to the tiling of R is N. Then the network associated to the rotation of R and all of the R_i's by $\frac{\pi}{2}$ is N^*.*

Proof. Postponed to the next section. ☐

4 The correspondence between 2-pole networks and tiled rectangles

Here we will show how to construct a tiled rectangle from a planar 2-pole network whose every edge carries a nonzero current when differing potentials are applied to the poles.

Given a 2-pole network N embedded in a disk D, we will call one pole the North pole, and the other the South pole. The East and West faces are then defined in the obvious manner. Now, setting the potential at the North pole to some positive Y and the potential at the South pole to 0, we can solve for the unique γ-harmonic potential, V, that satisfies the given boundary data.

We now define a "current potential" function J on the faces of N, and will subsequently prove that it is indeed a function. Let J satisfy the following properties:

1. Given adjacent faces F_1 and F_2 that share an edge e, the difference between $J(F_1)$ and $J(F_2)$ is equal to the current flow in e determined by the potential function V.

2. For adjacent faces F_1 and F_2, the sign of $J(F_1) - J(F_2)$ should be determined by a consistent convention. We will adopt the right-hand rule, i.e., if an edge runs North-South and the North potential is greater than the South potential, then the face to the East should have a greater value of J than the face to the West.

3. The value of J on the West face of the bounding disc D should be 0.

To prove that J is indeed a function, we need:

Theorem 4.1. *Given a series of faces $F_1, F_2, \ldots, F_n$ such that F_i is adjacent to F_{i+1} and $F_1 = F_n$, $\sum_{i=1}^{n-1} (J(F_{i+1}) - J(F_i)) = 0$.*

Proof. First we note that a modified version of Kirchhoff's Current Law holds for arbitrary sets of interior vertices in a network. Suppose we have a given set $P_1, \ldots, P_n$ of interior vertices, and we let $e_1, \ldots, e_m$ be all the edges having exactly one endpoint in the P_i. Furthermore, let currents flowing away from a P_i along an e_i have negative sign, and let currents flowing

toward a P_i along an e_i have positive sign. Then the sum of all the currents flowing through the e_i is 0.

We now establish another preliminary result. Suppose some pair of faces, F_1 and F_2, has more than one edge in common. Each shared edge must have the same current, otherwise we would have a violation of the modified Kirchhoff's Current Law for the set of vertices between and including the endpoints of two of the common edges. (See Figure 1.) So no matter which of the common edges we choose, $J(F_2) - J(F_1)$ has the same value.

To prove the theorem at hand, it suffices to look at non-intersecting sequences of adjacent faces, i.e., the case where $i, j \neq 1, n$ and $F_i = F_j \Rightarrow i = j$, since any closed path of adjacent faces can be decomposed into several such loops. Such a sequence of faces will divide the vertices of the graph into two sets: those within the loop or on its inner edge, and all other vertices. (Figure 2).
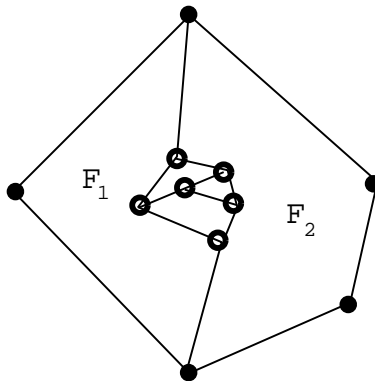

Figure 1: F_1 and F_2 have two common edges. We see the current in these two edges is the same by applying the modified Kirchhoff's Voltage Law on the circled vertices.

Assume that each pair of faces F_1 and F_2, F_2 and F_3, ..., F_{n-1} and F_1 only has one edge in common. (By the preceding result about faces sharing more than one edge, it suffices to consider this case.) If we consider the inner set of vertices as our set $P_1, \ldots, P_n$ of interior vertices, we see that the
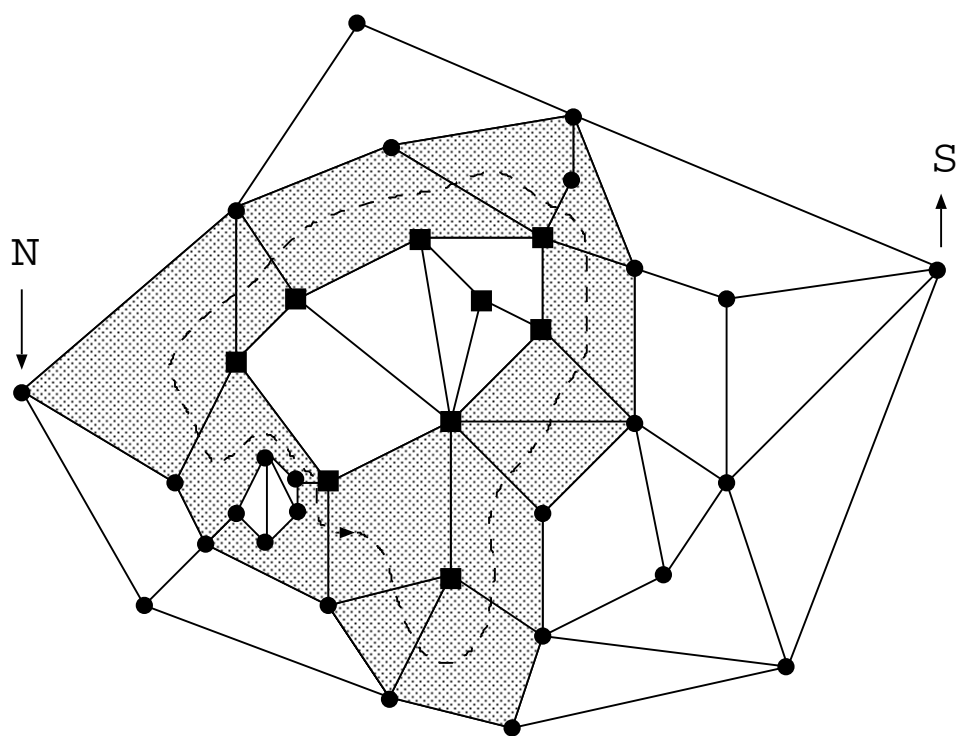
Figure 2: *An example of a closed path of adjacent faces on a 2-pole network. The "inner set" of vertices is marked with squares, whereas its complement is marked with circles. The dashed curve shows one possible path of adjacent faces and common edges.*

edges $e_1, \ldots, e_m$ having exactly one endpoint in the P_i are exactly the set of common edges shared by F_1 and F_2, F_2 and F_3, etc. Since $J(F_{i+1}) - J(F_i)$ is equal to the current across the common edge e_i, we obtain the desired result by application of the modified Kirchhoff's Current Law.

By Kirchhoff's Current Law, it can easily be seen that the theorem holds on a loop of faces going around a single interior vertex. The set of all such loops forms a basis for all closed paths of adjacent faces in the graph. Then the result is immediate, thus giving another proof of the theorem. $\square$

We now know that J is a well-defined function. We will call it the *standard dual potential*. Functions satisfying conditions 1 and 2 of the definition of J shall be called simply *dual potentials*, and in general differ from J by a constant. We may also call a dual potential a *current potential* if we wish to emphasize the property that the potential drop across a shared edge gives the current flowing through that edge.

If we identify the faces of N with the vertices of the dual network N^*, we see that J is a γ^*-harmonic function on the dual, where γ^* is the conductivity on N^*. Kirchhoff's current law on N is equivalent to Kirchhoff's voltage law on N^*, as in the proof of Theorem 4.1, and vice versa.

Remark 4.2. *If J' is a dual potential for a potential V, then $C - V$ is a dual potential for J', where C is an arbitrary additive constant. $Y - V$, where Y is the value of V at the North Pole of N, is the standard dual potential of the standard dual potential J of V. These facts are somewhat analogous to the result that N is the dual of its dual N^*.*

We now have a method of generating tilings from networks. Apply voltages Y and 0 to the North and South poles of the network, respectively. Then solve for the standard dual potential J, calling the dual potentials at the West and East poles of the dual 0 and X, respectively. Let the potential at the ends of each edge in the network represent the y-coordinates of a rectangle, and the current potential across the corresponding dual edge represent the x-coordinates of a rectangle. From the Maximum Principle, we see that all such rectangles must lie within a large rectangle of dimensions X by Y.

It remains to show every point in the rectangle, excluding the boundaries of tiles, is tiled exactly once. We first need some lemmas.

Lemma 4.1. *Every interior node in a 2-pole planar network experiences at most two alternations of ingoing and outgoing currents as the edges terminating at the node are traversed in a circular direction.*

Proof. This follows very easily from the planarity of the network. Suppose we have a node at which we have more than two alternations of current. Then there must be at least two ingoing currents that separate at least two outgoing currents. By Kirchoff's Current Law, we can follow each outgoing current along a chain of decreasing potential; each such path must terminate at the South pole. We can also follow the two ingoing currents along chains of increasing potential to the North pole; however, by the planarity of the network, one of these chains of increasing potential must intersect one of the chains of decreasing potential, yielding a contradiction. $\square$

Lemma 4.2. *On the boundary of each face in N, we can find two vertices P_i, P_j such that all currents on the boundary travel from P_i to P_j.*

Proof. By duality, the face of N corresponds to a vertex with several edges in the dual network N^*. By Lemma 4.1 and the right-hand rule, we obtain the desired result. $\square$

Theorem 4.3. *The tiling corresponding to a 2-pole circular planar network N tiles a rectangle with no overlaps, excepting possibly tile boundaries.*

Proof. We will say an edge e of N comprises $(\lambda, \ )$ if the endpoint potentials V_1^* and V_2^* of the dual edge e^* satisfy $V_1^* < \lambda < V_2^*$. We will also say e comprises $(\ , \mu)$ if the endpoint potentials V_1 and V_2 of e satisfy $V_1 < \mu < V_2$. Furthermore, we will label the poles of the network P_1 and P_2, and the poles of the dual P_1^* and P_2^*. Without loss of generality, $V_1 > V_2$ and $V_1^* > V_2^*$.

Now, given arbitrary λ between V_1^* and V_2^* and μ between V_1 and V_2, where $\lambda \neq V_1^*, V_2^*, \dots$ (the potentials on the vertices of the dual network), and $\mu \neq V_1, V_2, \dots$, we wish to show that exactly one edge e of N comprises (λ, μ).

By the Maximum Principle, we see that no current flows into P_1 from the interior of the network, nor out of P_2 into the interior of the network. Then,

by duality, at each pole of N, there is exactly one edge comprising $(\lambda,)$, as in the diagram. Furthermore, by applying Lemma 4.2 to the dual network N^* and applying the right-hand rule, we see that each interior vertex that has one ingoing edge that comprises $(\lambda,)$ must also have one outgoing edge that comprises $(\lambda,)$. Thus, there is a single path of descending potential from P_1 to P_2 that comprises exactly the edges of N comprising $(\lambda,)$.

But since the potential is steadily decreasing along this path, and the currents along the path are nonzero by the condition on λ, we find exactly one edge on the path that comprises $(, \mu)$. Hence, exactly one edge of N comprises (λ, μ). Since by the algorithm outlined above, each edge e of N corresponds to exactly one element of a tiling, and λ and μ were arbitrarily given, we see that the rectangle is completely tiled without overlap. $\square$

We now restate and prove

Theorem 3.1. *Let R be a rectangle which is tiled by finitely many rectangles $R_1, \ldots, R_n$. Suppose the network associated to the tiling of R is N. Then the network associated to the rotation of R and all of the R_i's by $\frac{\pi}{2}$ is N^*.*

Proof. We have seen that the y-coordinates of an element in the tiling are given by the potential across a certain edge, while its x-coordinates are given by the standard dual potential across its dual edge. Let the rectangle R be given by $[0, X] \times [0, Y]$, and let the potential on N be given by V. Similarly, the standard dual potential of V on N^* shall be given by J.

A counterclockwise rotation of R and all the R_i's by $\frac{\pi}{2}$, followed by a translation to keep the lower left corner of the rectangle at $(0, 0)$, corresponds to the interchange of (x, y) with $(Y - y, x)$ for all (x, y). But from Remark 4.2, we see that $C - V$, where C is an arbitrary additive constant, must be a dual potential for J on N^*. In particular, $Y - V$ must be the standard dual potential of J. Thus J on N^* gives the vertical coordinates of the elements of the tiling, and the standard dual potential $Y - V$ gives the x-coordinates of the elements of the tiling. Thus we see that all the R_i's, as well as R itself, are properly rotated by $\frac{\pi}{2}$. $\square$

5 Perfect and simple tilings

We will begin by using the results of the previous section to demonstrate how to quickly compute low-order tilings of rectangles by squares, and show how these methods can be used to quickly discover very interesting tilings.

Definition 5.1. A tiling of a rectangle is said to have *order n* if there are n tiles used in the tiling.

Now that we know there is a one-to-one correspondence between tilings of rectangles by rectangles (or squares) and 2-pole networks, we can quickly construct all squared rectangles of a given order n by computing all 2-pole networks with n edges and assigning each edge conductivity 1. To speed this process, use Theorem 3.1: dual networks correspond to equivalent tilings, so we needn't compute them. Also, if one interior vertex of the network has only one edge attached to it, clearly the current in that edge will be zero so we can eliminate such networks.

Definition 5.2. A rectangle R which is tiled by squares $S_1, \ldots, S_n$ is called *compound* if there are squares $S_{i_1}, \ldots, S_{i_m}; 1 < m < n$, such that $\bigcup_{j=1}^m S_{i_j}$ is a rectangle. If there are no such squares, the tiling is called *simple*.

Definition 5.3. A rectangle R which is tiled by squares $S_1, \ldots, S_n$ is called *perfect* if no two of the squares have the same dimensions. Otherwise it's called *imperfect*.

Notice that the currents in the network associated to a perfect tiling are all different, while in the network associated to an imperfect tiling, at least two of the currents will be the same. This gives us a good method of computing perfect tilings. First we can eliminate some networks which will lead to imperfect tilings. Given any 2-pole network, we define its *completion* to be the network obtained by adding an edge connecting the poles to the network.

Properties of networks associated to simple and perfect tilings

    1. The network associated to a perfect tiling has the following property: each interior vertex has at least three neighbors.
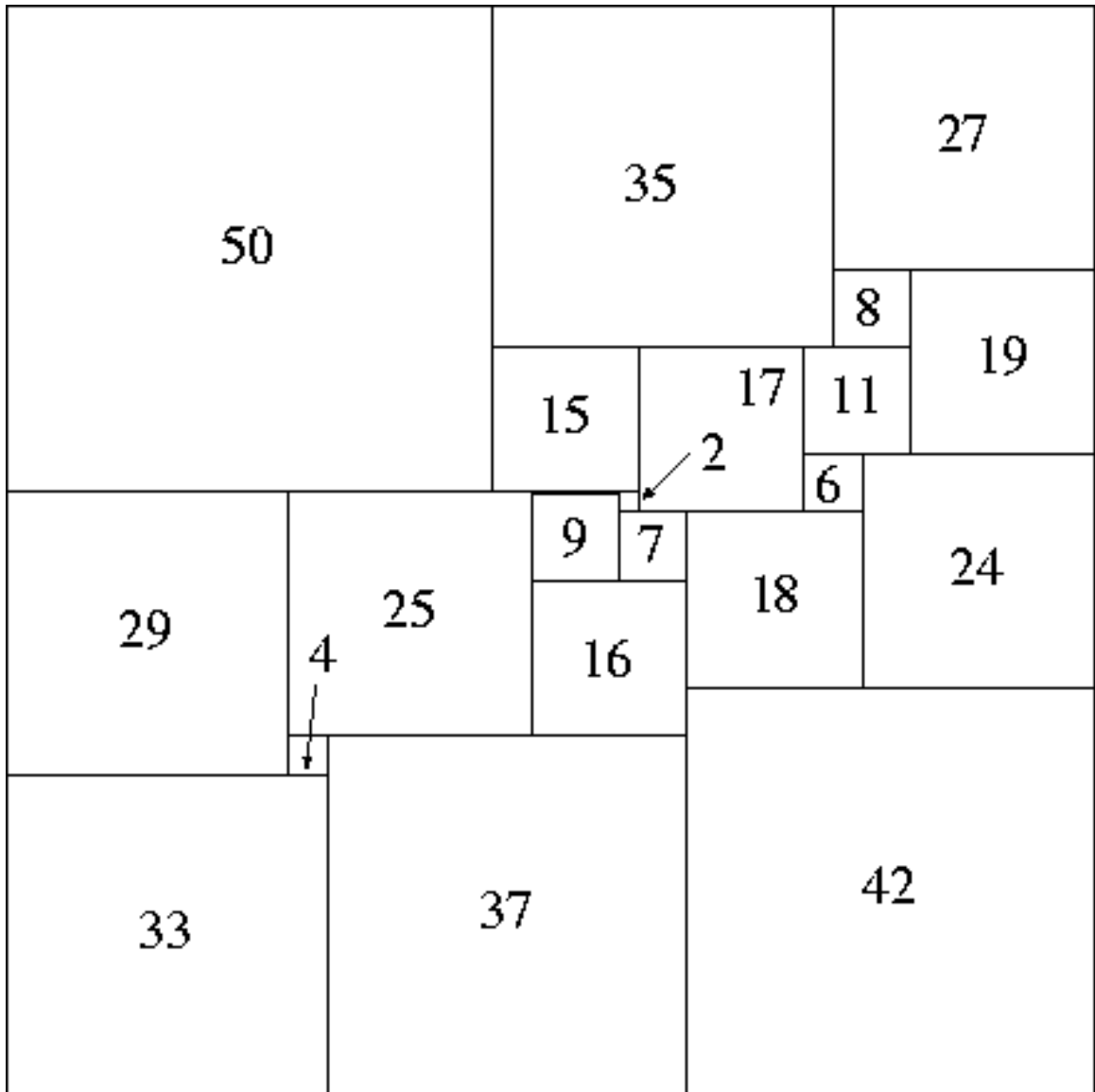
Figure 3: *The smallest simple, perfect squaring of a square.* [4]

Proof. Suppose not. Suppose there is an interior vertex v with neighbors k and j and no other neighbors. Solve the Dirichlet problem on the network to get a γ-harmonic function f. Then $2f(v) = f(k) + f(j)$, or $f(v) - f(k) = f(j) - f(v)$, so the current in edge vk is the same as the current in edge vj, so the tiling is imperfect. $\square$

Remark 5.1. *The tiling is also compound: the two edges vk vj can be made into one edge using the laws for conductors in series; this new edge corresponds to a sub-rectangle that is tiled by the squares corresponding to the edges vk and vj.*

2. The network associated to a perfect or a simple tiling has only one edge between any given vertices.

Proof. Suppose not. Say there are two vertices which are connected by edges e and f. Then by the definition of current, the current in these two edges will be the same. Also, using the laws for conductors in parallel we can make these edges into one new edge with non-unit conductivity; this will correspond to a sub-rectangle which is tiled by the squares corresponding to the edges e and f. $\square$

Theorem 5.2. *Let a rectangle R be tiled perfectly by squares. Then the network associated to its tiling is asymmetric around any axis which either avoids both poles or includes both poles.*

Proof. Suppose first that the network is symmetric about an axis which avoids both poles. Suppose f is the solution for the Dirichlet problem on the network. Then f is symmetric about the axis of symmetry. For, if Kirchhoff's current laws are satisfied on one half of the network, the same currents on the reflected half will also solve Kirchhoff's current laws. Hence, by uniqueness of f, if e is an edge of the network and f is its reflection about the axis of symmetry, we must have that the current in e is the same as the current in f, hence the tiling is imperfect. The same argument works if the network is symmetric about an axis which includes both poles. $\square$

Remark 5.3. *This proof does not work if the axis includes one pole but not the other, for then symmetry is preserved geometrically but not electrically, as one side of the network will have a pole but the reflected side will not.*

We can now compute many different perfect and simple perfect rectangles. It is clear that a perfect rectangle of least order is simple; if it contained a
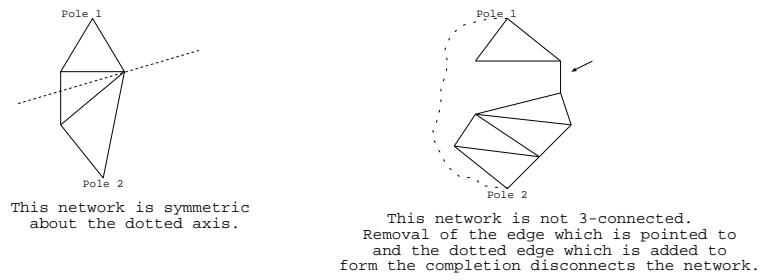
Pole 1

This network is symmetric
about the dotted axis.

Pole 1

This network is not 3-connected.
Removal of the edge which is pointed to
and the dotted edge which is added to
form the completion disconnects the network.

Pole 2

Pole 2

Figure 4: *Here are some networks that yield imperfect tilings.*

Pole 1

This network is compound.
The dotted line separates
the two squared rectangles
which are put together to
form the compound squared
square.

Pole 2
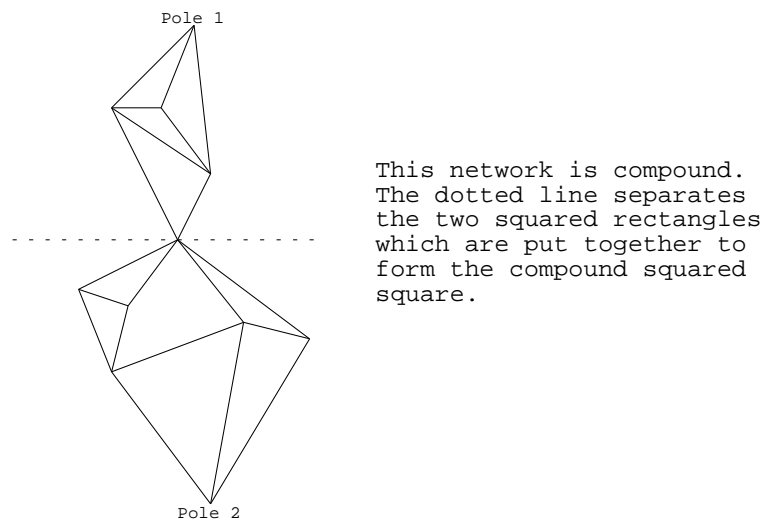
Figure 5: *Here is a network which yields a compound tiling.*

sub-rectangle then the sub-rectangle would also be tiled perfectly and would have lower order. Using Theorems 5.1 and the properties above to eliminate many networks, we find

Theorem 5.4. *There are no perfect rectangles of order less than* 9 *and precisely two of order* 9*.*
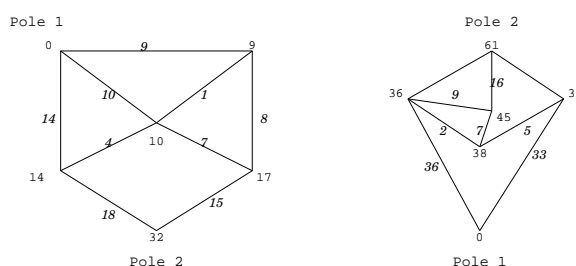

Figure 6: *Here are the networks for the two order* 9 *(simple) perfect tilings. Notice the asymmetry of these networks. Plain text denotes potentials, while text in italics denotes currents.*

Now we will discuss simple tilings and see which networks correspond to simple tilings and which correspond to compound tilings.

Theorem 5.5. *If a network* N *contains a sub-network* M *which has more than one vertex but not all of the vertices and which is connected to the rest of the network at only* 2 *vertices, then the tiling associated to* N *is compound. Otherwise it's simple.*

Proof. Suppose N contains a sub-network M as above. All of M has some effective conductivity which can be determined by treating M as a 2-pole network and looking at its Lambda matrix. Say it has conductivity γ_1. Replace M by a single edge e of conductivity γ_1. Then the flow in N is the same as before, but instead of the squares associated to the edges of M we have one big rectangle. Since the flow is the same, the potentials at the vertices of e are the same as they were before the substitution, so all of the rectangles corresponding to edges in M must have formed a tiling of the new rectangle associated to e. Since e is not the only edge in the new network, the rectangle associated to e is not the whole rectangle which is tiled, so the tiling is compound. Also, it is clear that any compound tiling must have such a sub-network. Hence the proof is complete. $\square$

Remark 5.6. *If N contains a sub-network as above connected to the rest at only one vertex, clearly the current flow in this sub-network will be identically zero, so we can safely eliminate all such networks.*

Theorem 5.7. *Let a rectangle R be tiled perfectly and simply by squares. Then the completion of the network associated to its tiling is 3-connected, that is, if any two edges are removed the network remains connected. The same is true of the dual of the completion.*

Proof. Suppose the completion is not 3-connected. Then it's either 1-connected or 2-connected. If it's 1-connected, then there is an edge e whose removal disconnects the graph. Certainly e cannot be the added edge joining the poles, so it must be one of the edges of the original graph. Since its removal disconnects the completion of the graph, it must be part of some path in the graph which eventually terminates at an interior node. But then, by Kirchhoff's current laws, the current in this path must always be 0, so the current in e is 0. Thus this network cannot be that of a tiling. Now suppose that the network is 2-connected. Then there are two edges e and f whose removal disconnects the graph. If one of these edges is the edge connecting the two poles, then the other one must be an interior edge, so its removal disconnects the original graph. Thus it must be the case that there is a part of the graph connected to the rest at only 2 vertices, so the tiling is compound by Theorem 5.3. Now suppose both e and f are interior edges. Since their removal disconnects the completion, it becomes divided into at most three connected sections. One of these sections does not contain either pole, since the poles are connected in the completion. This section is connected to the rest of the completion at no more than two vertices, so the tiling is compound by Theorem 5.3. That the same holds true of the dual is a trivial consequence of Theorem 3.1. □

6 Medial graphs and their relation to tilings

Suppose $\Gamma = (V, \partial V, E)$ is a circular planar graph with n boundary nodes $v_1, \ldots, v_n$ which occur in clockwise order around a circle C inside of which Γ is embedded.

Definition 6.1. The *medial graph* $\mathcal{M}(\Gamma)$ of a circular planar graph Γ as above depends on the embedding and is defined as follow. For each edge e of Γ, let m_e be its mid-point. Place $2n$ points $w_1, \ldots, w_n$ on C so that

$$w_1 < v_1 < w_2 < w_3 < v_2 < \ldots < w_{2n-1} < v_n < w_{2n} < w_1$$

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in the clockwise circular order around C. Then the vertices of $\mathcal{M}(\Gamma)$ are $\{m_e\}_{e \in E} \cup \{w_i\}_{1 \leq i \leq 2n}$. Two vertices m_e and m_f are connected whenever e and f share a common vertex and are incident to the same face in Γ. Each vertex of the form w_j has one edge emanating from it. w_{2i} is connected to m_e, where e is the edge of the form $v_i r$ so that e comes first after the arc $v_i w_{2i-1}$ on C in the clockwise direction around v_i. w_{2i-1} is connected to m_f, where f is the edge of the form $v_i s$ so that f comes first after the arc $v_i w_{2i-1}$ in the counter-clockwise direction around v_i.

Notice that all vertices of the form m_e are the interior vertices and are all 4-valent, *i.e.* they all have 4 edges emanating from them. The boundary vertices, those of the form w_j, are all 1-valent, *i.e.* they all have 1 edge emanating from them. The medial graph encodes both the original graph and its dual. The medial graph divides the circle C in which it is embedded into a number of regions. These regions can be 2-colored. Suppose we color them black and white. Then if we place one vertex in each black region, making sure to move vertices in regions partially bounded by an arc of C out onto C itself, and connect two vertices if and only if the two regions associated with them share a common vertex, we will obtain the original graph (or the dual graph, depending on the coloration). If we do the same with the white regions, we will obtain the dual of the original graph (or the original graph itself, depending on the coloration). So the two graphs thus obtained are always dual to each other.

Definition 6.2. An edge vw of the medial graph $\mathcal{M}(\Gamma)$ is a *direct extension* of another edge uv if the edges uv and vw separate the other two edges incident to v. A path $v_0 v_1 \ldots v_n$ in $\mathcal{M}(\Gamma)$ is called a *geodesic arc* if each edge $v_i v_{i+1}$ is a direct extension of the edge $v_{i-1} v_i$ $\forall 1 \leq i \leq n-1$. A geodesic arc $v_0 v_1 \ldots v_n$ is called a *geodesic* if either $v_0, v_n \in C$ or if $v_0 = v_n$ and $v_0 v_1$ is a direct extension of $v_{n-1} v_n$.

We will now discuss a method to obtain the medial graph of the network associated to a tiling of a rectangle directly from the tiling, without having to first construct the network. Place one vertex in each square (or rectangle) of the tiling and place one boundary vertex at each corner of the rectangle. Start in the upper left hand corner and continue to the right. When one reaches the end of the rectangle, go down another level and continue to the right until finished connecting vertices. Always obey the following rules:

1. Connect two vertices only when the associated squares share part of an edge.

2. Give priority to horizontal connections over vertical ones.

The vertices in the corners connect only to the squares (rectangles) in the corners. Each interior vertex has four edges emanating from it, one going towards each corner of the square the vertex is in. Connect the edge going towards vertex v of the square to a vertex in a square which shares vertex v with the original square and which also shares part of an edge with it. If there are many such squares, pick the one that will form a horizontal connection, using Rule 2 above. But never allow any vertices to have more than 4 edges incident to them. If we continue the edges in the natural way, by continuing diagonally through the square and connecting vertices as described above, we will obtain the geodesics of the medial graph. Thus not only can we quickly construct the medial graph, but we can discover the geodesics with ease.

7 Multiple-pole networks and tiling other regions

The results above have been discussed only for two-pole networks, but with suitable care, it is possible to generalize them to networks with arbitrarily many boundary nodes.

Consider a simply connected polygon having only 90-degree angles and tiled by a certain number of rectangles. (At this stage, we only require that each region of the polygon be tiled at least once.) We will call an element of the tiling an *interior rectangle* if both its top and bottom edges are completely shared with the edges of other rectangles. An element of the tiling that does not satisfy this property shall be called a *boundary rectangle*.

Letting the height of each rectangle represent the potential drop across an edge in an electrical network, and its width represent the associated current flow, we see that each interior rectangle obeys Kirchhoff's Current Law, and is thence identified with an interior node of a network. Similarly, each boundary rectangle does not obey Kirchhoff's Current Law, and must therefore correspond to a pole of the network. Any of the constructions outlined

in the sections above may be used to construct tilings from multi-pole networks, although certain constraints must be imposed to obtain well-behaved tilings.

It may be easily seen that the shape of the tiled polygon depends not only upon the electrical network specified, but also upon the given boundary data. If inappropriate boundary data are specified, the polygon will intersect itself, yielding an ill-behaved tiling in which some regions are multiply covered. The reasonable question to ask, then, is under which conditions we may expect to obtain well-behaved tilings.

Theorem 7.1. *The tiling associated with a general circular planar network will be a proper tiling if there are only two alternations in the sign of the boundary currents, i.e., if we can decompose the circular boundary into two arcs such that all the nodes on one arc have positive current flow, and all the nodes on the other have negative current flow.*

Proof. First we number the current-carrying poles of N clockwise around the boundary of the disc such that $P_1, P_2, \ldots, P_m$ contain positive (ingoing) currents, and $P_{m+1}, P_{m+2}, \ldots, P_n$ contain negative (outgoing) currents. We then number all other vertices (including the poles carrying zero current) however we please.

Now if $V_1 = V_2$, we identify the two vertices to get a single new vertex (call it P_2') at potential V_2. If currents I_1 and I_2 entered at P_1 and P_2 before, we now let current $I_1 + I_2$ enter at P_2'. If $V_1 > V_2$, add a new edge of conductance $\frac{I_2}{V_1 - V_2}$ joining P_1 and P_2, and let current $I_1 + I_2$ enter at P_1 (henceforth call it P_2'). Similarly, if $V_1 < V_2$, join the vertices with an edge of conductance $\frac{I_1}{V_2 - V_1}$, and let current $I_1 + I_2$ enter at P_2 (henceforth call it P_2'). In this process, the current and potential at all other nodes remains unaffected. Now repeat the process for V_2' and V_3, then V_3' and V_4, etc., until we have a single node V_m' that carries positive current.

Now we follow a similar process for the poles V_{m+1} through V_n, excepting that the currents are outgoing instead of ingoing as in the above paragraph. We then end up with the current flow of the original network N subsumed within the current flow of a 2-pole network. We thus have a rectangle R tiled properly by rectangles. Since some edges were added to the network

in this argument, we must strip away the elements corresponding to those added edges to recover the tiling corresponding to the original network N. We thus have a proper tiling of some sort of polygonal region. □

We expect the converse of the above theorem to be true; more about this will be said in the next section.

The theory of multi-pole tilings may be extended to certain planar networks that are not circular planar. In order to get nondegenerate tilings, "nested circular planar" networks must be considered. Such a network must be embedded in a disc that has zero or more circular holes in it. The poles of the nested circular planar network must lie on the boundary of the surface (whether that be the exterior edge of the disc, or the edges of the holes; see Figure 7). Furthermore, to obtain well-behaved tilings for this case, we require that the sum of the net current flows on the boundary nodes of each bounding circle be zero. The resulting tiled region will not, in general, be simply connected; each hole in the tiled region corresponds to one of the holes in the disc in which the network is embedded.
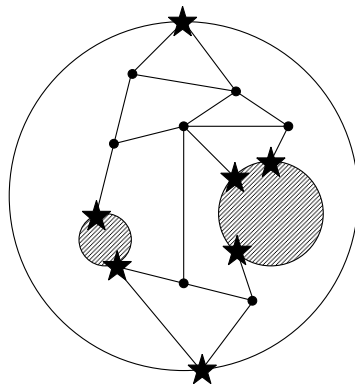
Figure 7: *A nested circular planar graph. Dots mark interior nodes, and stars mark boundary nodes.*

8   Future directions

Here we will discuss some directions, unfinished work, and generalizations. We will begin by discussing the promised converse of Theorem 7.1. We have not been able to prove it, but it appears to be true.

Theorem 8.1. *If we are given a circular planar network N with n boundary nodes so that there are m alternations in the direction of the boundary currents as we go around the circle, then we will obtain a tiled region which has a part that is tiled $m - 1$ times by rectangles.*

One of the ideas we wanted to use to prove this theorem was to consider the medial graph. Once we constructed the medial graph, we'd 2-color the faces. One color would correspond to the original graph and the other color would correspond to the dual. Then we could solve the appropriate Dirichlet problems for the original graph and its dual and place the solutions in the faces of the medial graph. Then by going around the faces of the medial graph which touched the bounding circle, we would be able to draw out the region which was tiled. (See Figure 8.) We would do this by considering a complex-valued function $f : E \to \mathbf{C}$ on the edges of the medial graph which was defined as $f = J + iv$. Since each edge is part of precisely two faces, each of one color, f can be well-defined by the formula above. Further, f satisfies a discrete version of the Cauchy-Riemann equations and is discrete analytic in a suitable sense. For, we have $J_x = v_y$ and $J_y = -v_x$ using the right-hand rule for current flow. If we trace the graph of f when we go around the outer regions of the medial graph, we get a region in the plane which should be the region tiled by the network. If there are m alternations in the directions of current flow at the boundary, there should be a part of this region which is tiled $m - 1$ times. We hoped that in the graph itself, if there were many alterations in the direction of current flow, there would be an vertex in the graph itself which was a saddle point, that is, which had many alterations in the direction of the current flow. This was not true, however; see the figure on the next page.

The above approach was motivated by an analogy to the concept of *winding number* in complex analysis — if the winding number is 1, then the tiling is nondegenerate. Another analogy is to somehow define the concept of an outward-pointing normal vector, and then to make sure that this vector undergoes 1 complete rotation when the boundary of the medial graph is followed in a circular fashion, again yielding a nondegenerate tiling. We postulate that the potential and associated current flow (the differences of the current potential) on a medial graph can be characterized in terms of sign relations between their derivatives and second derivatives, in analogy with the sine and cosine. For our purposes, we would define the derivative of a potential as its increase between two successive colored cells on the
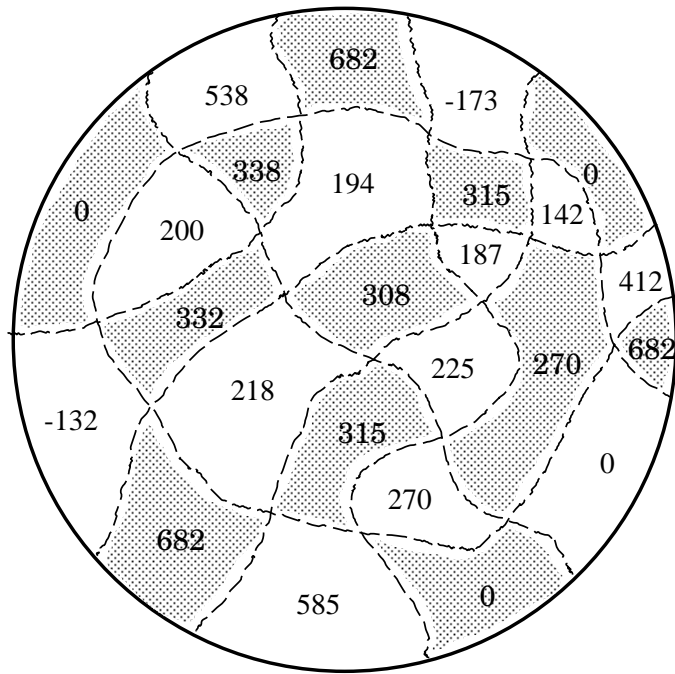
Figure 8: *An example of a potential and its dual placed on the faces of a medial graph. Note that each potential drop is equal to its dual potential drop — if all conductivities were not 1, we would see a corresponding scale factor at each vertex.*
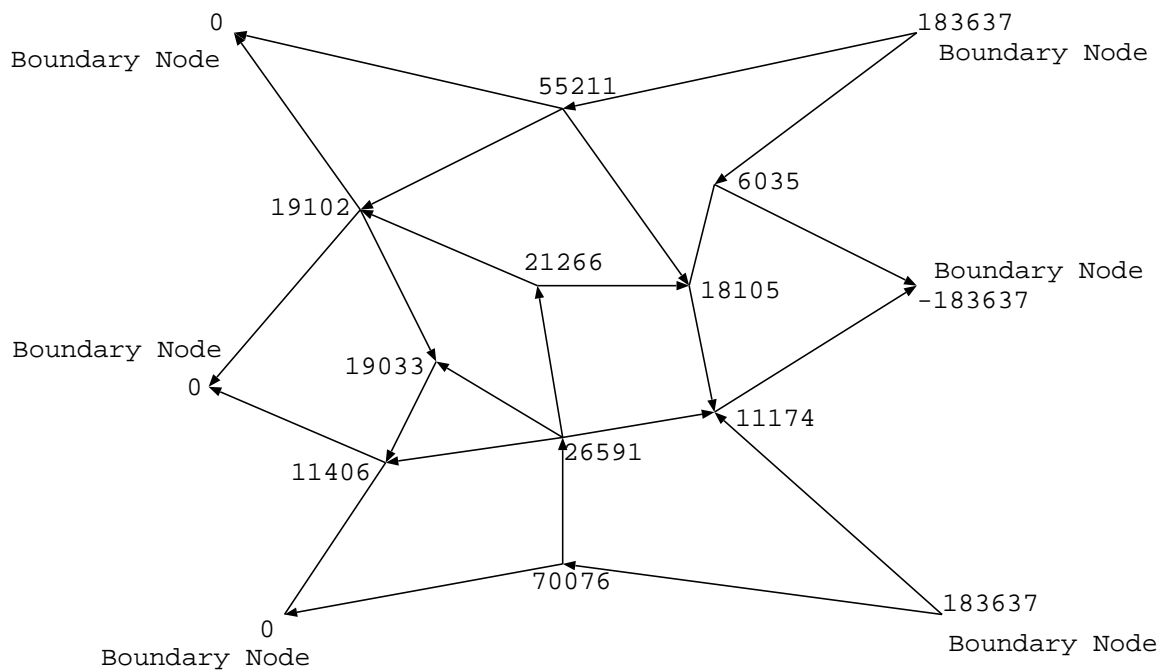
Figure 9: *A counter-example to the existence of saddle points. There are no saddle points in this graph even though there are alterations in the direction of current flow on the boundary nodes. All vertices have only two alterna- tions in current direction. The arrows indicate the direction of current flow; the numbers are the potentials induced by the given boundary potentials.*

boundary, and its second derivative as the increase between two successive derivatives. Although the conjecture seems to hold for the examples we have looked at, not enough work has been done in this area to merit a detailed discussion.

One way of obtaining tilings is to ignore the network and the dual network and only use the medial graph. Once we have a medial graph, we can 2-color it; say it has black cells $B_1, \ldots, B_n$ and white cells $W_1, \ldots, W_m$. Each white cell W_i has potential v_i, while each black cell B_i has dual potential J_i. If we wish to generalize to tilings by rectangles instead of by squares, it will be necessary to assign a "conductivity" to each vertex of the medial graph. By simply using the right-hand rule, we can obtain an equation relating the potentials to the dual potentials at each vertex of the medial graph. Given fixed boundary potentials on either the white cells or the black cells, we can solve the system of equations thus obtained uniquely and obtain the potentials and dual potentials for our medial graph. Then hopefully we could trace out the figure being tiled as above, and tile it by going around each vertex in a circular manner, visiting all the edges and plotting the graph of f to draw the tiles. They would all correspond to squares (or rectangles) since the vertices of the medial graph are all 4-valent, so we'd get a tiling by squares (or rectangles) every time.

We can also obtain an electrical network for higher-dimensional tilings. However, its usefulness has not been determined. To obtain this network, we proceed inductively. We have already established it for 2-dimensional tilings, so we induct on the dimension of the region being tiled. An n-dimensional box tiled by n-dimensional cubes should correspond to an $(n-1)$-dimensional complex embedded in $\mathbf{R}^n$. Every slice of the box parallel to one of its faces gives us an $(n-1)$-dimensional box tiled by $(n-1)$-dimensional cubes. Now inductively, these slices should correspond to $(n-2)$-dimensional complexes embedded in $\mathbf{R}^{n-1}$. If we "piece together" these complexes in some suitable sense, we should get an $(n-1)$-dimensional complex which corresponds to the original tiling of the n-dimensional box. This complex can clearly be embedded in $\mathbf{R}^n$. We have m-dimensional parts of this complex corresponding to $(m+1)$-dimensional parts of the tiling. If we assign width and length to some of the faces of the complex, it should be possible to formulate a definition of harmonicity on it. Then we would be able to solve the Dirichlet problem on the complex and as before, the solution would hopefully correspond to the position and size of the boxes in the tiling. We shoould be able to recover

a tiling from an appropriate complex by choosing an orientation and slicing the complex parallel to the chosen orientation. Then we will obtain a complex of one less dimension, so inductively we can form a tiling. Then we can just piece together all of these tilings to form a tiling of a box. Since we will tile the box by only finitely many cubes, we only have to piece together finitely many different pieces. Thus using this method for computational purposes is feasible.

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